Diameter Preserving Surjections in the Geometry of Matrices

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DIFFERENTIALGEOMETRIE UND GEOMETRISCHE STRUKTUREN

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Rectangular Matrices

Let $M_{m,n}(\mathcal{D})$, $m, n \geq 2$, be the set of all $m \times n$ matrices over a division ring $\mathcal{D}$.

- Two matrices (linear operators) $A, B \in M_{m,n}(\mathcal{D})$ are *adjacent* if $A - B$ is of rank one. (Rank always means left row rank.)
- We consider $M_{m,n}(\mathcal{D})$ as an undirected graph the edges of which are precisely the (unordered) pairs of adjacent matrices.
- Two matrices $A, B \in M_{m,n}(\mathcal{D})$ are at the graph-theoretical distance $k \geq 0$ if, and only if,

$$\mathrm{rank}(A - B) = k.$$

Grassmannians

Let $\mathcal{G}_{m+n,m}(\mathcal{D})$ be the Grassmannian of all m-dimensional subspaces of $\mathcal{D}^{m+n}$, where $m, n \geq 2$.

- Two subspaces $V, W \in \mathcal{G}_{m+n,m}(\mathcal{D})$ are *adjacent* if $\dim(V \cap W) = m - 1$.
- We consider $\mathcal{G}_{m+n,n}(\mathcal{D})$ as an undirected graph the edges of which are precisely the (unordered) pairs of adjacent subspaces.
- Two subspaces $V, W \in \mathcal{G}_{m+n,m}(\mathcal{D})$ are at the graph-theoretical distance $k \geq 0$ if, and only if,

$$\dim(V \cap W) = m - k.$$

Connection

$M_{m,n}(\mathcal{D})$ can be embedded in $\mathcal{G}_{m+n,m}(\mathcal{D})$ as follows:

$$\begin{array}{ccccc} M_{m,n}(\mathcal{D}) & \longrightarrow & M_{m,m+n}(\mathcal{D}) & \longrightarrow & \mathcal{G}_{m+n,m}(\mathcal{D}) \\ A & \longmapsto & (A|I_m) & \longmapsto & \text{left rowspace of } (A|I_m) \end{array}$$

$\mathcal{G}_{m+n,m}(\mathcal{D})$ may be viewed as the *projective space* of $m \times n$ matrices over $\mathcal{D}$.

Hua's Theorem

Fundamental Theorem (1951). *Every bijective map* $\varphi : M_{m,n}(\mathcal{D}) \to M_{m,n}(\mathcal{D}) : A \mapsto A^\varphi$ *preserving adjacency in both directions is of the form*

$$A \mapsto TA^\sigma S + R,$$

where T *is an invertible* $m \times m$ *matrix,* S *is an invertible* $n \times n$ *matrix,* R *is an* $m \times n$ *matrix, and* σ *is an automorphism of the underlying division ring.*

If $m = n$*, then we have the additional possibility that*

$$A \mapsto T(A^\sigma)^t S + R$$

where T, S, R *are as above,* σ *is an anti-isomorphism of* $\mathcal{D}$*, and* A^t *denotes the transpose of* A*.*

The assumptions in Hua's fundamental theorem can be weakened.

W.-l. Huang and Z.-X. Wan (2004), P. Šemrl (2004).

Chow's Theorem

Fundamental Theorem (1947). *Every bijective map* $\varphi : \mathcal{G}_{m+n,n}(\mathcal{D}) \to \mathcal{G}_{m+n,n}(\mathcal{D}) : X \mapsto X^\varphi$ *preserving adjacency in both directions is induced by a semilinear mapping*

$$f : \mathcal{D}^{m+n} \to \mathcal{D}^{m+n} : x \mapsto x^\sigma T \text{ such that } X^\varphi = X^f,$$

where T *is an invertible* $(m+n) \times (m+n)$ *matrix and* σ *is an automorphism of the underlying division ring.*

If $m = n$*, then we have the additional possibility that* φ *is induced by a sesquilinear form*

$$g : \mathcal{D}^{m+n} \times \mathcal{D}^{m+n} \to \mathcal{D} : (x, y) \mapsto xT(y^\sigma)^t \text{ such that } U^\varphi = U^{\perp_g},$$

where T *is as above and* σ *is an anti-isomorphism of* $\mathcal{D}$*.*

The assumptions in Chow's fundamental theorem can be weakened.

W.-l. Huang (1998).

Geometries of Matrices

Similar fundamental theorems (subject to technical restrictions) hold for:

- Spaces of Hermitian matrices ($\mathcal{D}$ a division ring with involution $\bar{\ }$).
- Spaces of symmetric matrices ($\mathcal{D}$ commutative).
- Spaces of alternate matrices ($\mathcal{D}$ commutative)
  (with a different definition of adjacency: $\operatorname{rank} A - B = 2$).
- The associated projective matrix spaces (dual polar spaces).

In all cases the fundamental theorem is essentially a result on isomorphisms of graphs with finite diameter.

Diameter Preservers

Recent work focusses on diameter preservers between matrix spaces and related structures.

P. Abramenko, A. Blunck, D. Kobal, M. Pankov, P. Šemrl, H. Van Maldeghem, H. H.

In this lecture we aim at pointing out the common features.

Main Theorem

Below we shall state five conditions (A1)–(A5) on a graph, one of them being the finiteness its diameter.

Theorem (W.-l. Huang and H. H.). *Let* $\Gamma = (\mathcal{P}, \mathcal{E})$ *and* $\Gamma' = (\mathcal{P}', \mathcal{E}')$ *be two graphs satisfying the conditions* (A1)–(A5)*. If* $\varphi : \mathcal{P} \to \mathcal{P}'$ *is a surjection which satisfies*

$$d(x, y) = \operatorname{diam} \Gamma \quad \Leftrightarrow \quad d(x^\varphi, y^\varphi) = \operatorname{diam} \Gamma' \text{ for all } x, y \in \mathcal{P},$$

then φ *is an isomorphism of graphs. Consequently,* $\operatorname{diam} \Gamma = \operatorname{diam} \Gamma'$*.*

Application

Conditions (A1)–(A5) are satisfied by the graphs on the following spaces ($m, n \geq 2$):

- The graph on the space $M_{m\times n}(\mathcal{D})$ of rectangular matrices provided that $|\mathcal{D}| \neq 2$.
- The graph on the Grassmannian $\mathcal{G}_{m+n,m}(\mathcal{D})$ for any $\mathcal{D}$.
- The graph on the set of Hermitian $n \times n$ matrices over a division ring $\mathcal{D}$ with involution (subject to certain technical restrictions).

By applying the known fundamental theorems, explicit descriptions of these diameter preserving surjections can be given.

Conditions (A1)–(A5)

We focus our attention on graphs $\Gamma = (\mathcal{P}, \mathcal{E})$ satisfying the following conditions:

(A1) Γ is connected and its diameter $\operatorname{diam}\Gamma$ is finite.

(A2) For any points $x, y \in \mathcal{P}$ there is a point $z \in \mathcal{P}$ with

$$d(x,z) = d(x,y) + d(y,z) = \operatorname{diam}\Gamma.$$

(A3) For any points $x, y, z \in \mathcal{P}$ with $d(x,z) = d(y,z) = 1$ and $d(x,y) = 2$ there is a point w satisfying

$$d(x,w) = d(y,w) = 1 \text{ and } d(z,w) = 2.$$

(A4) For any points $x, y, z \in \mathcal{P}$ with $x \neq y$ and $d(x,z) = d(y,z) = \operatorname{diam}\Gamma$ there is a point w with

$$d(z,w) = 1, \ \ d(x,w) = \operatorname{diam}\Gamma - 1, \ \text{ and } \ d(y,w) = \operatorname{diam}\Gamma.$$

(A5) For any adjacent points $a, b \in \mathcal{P}$ there exists a point $p \in \mathcal{P} \setminus \{a, b\}$ such that for all $x \in \mathcal{P}$ the following holds:

$$d(x,p) = \operatorname{diam}\Gamma \ \Rightarrow \ \Big(d(x,a) = \operatorname{diam}\Gamma \ \vee \ d(x,b) = \operatorname{diam}\Gamma\Big).$$

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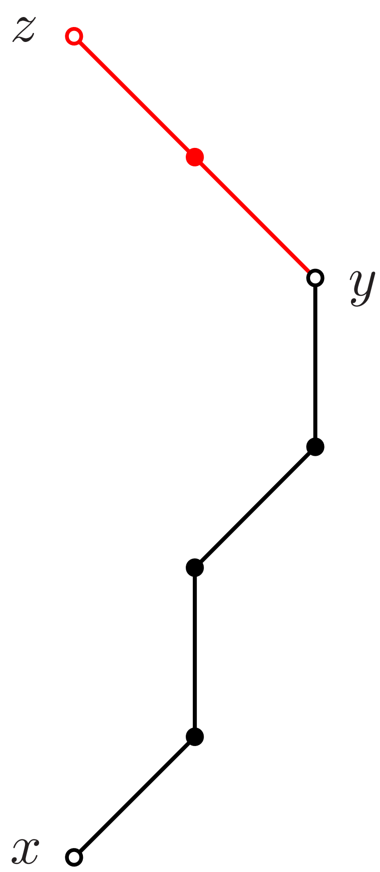
Condition (A2)



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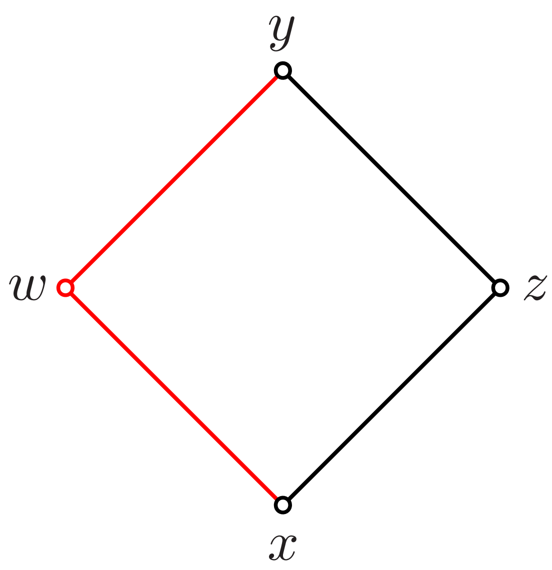

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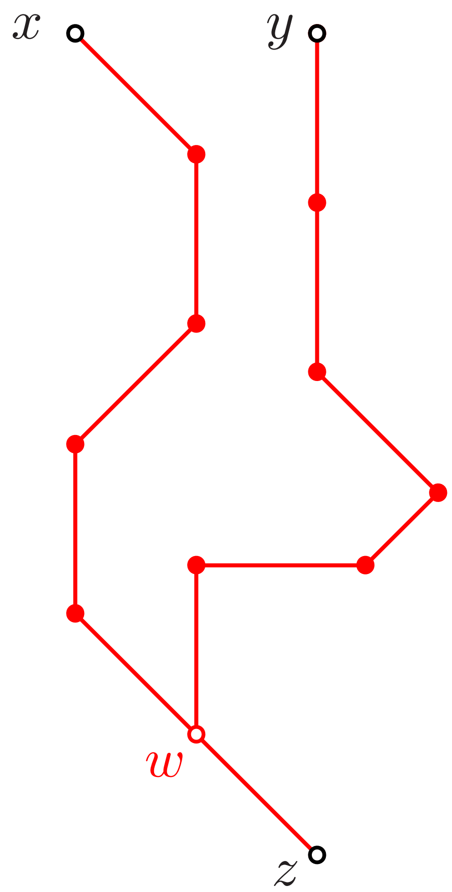

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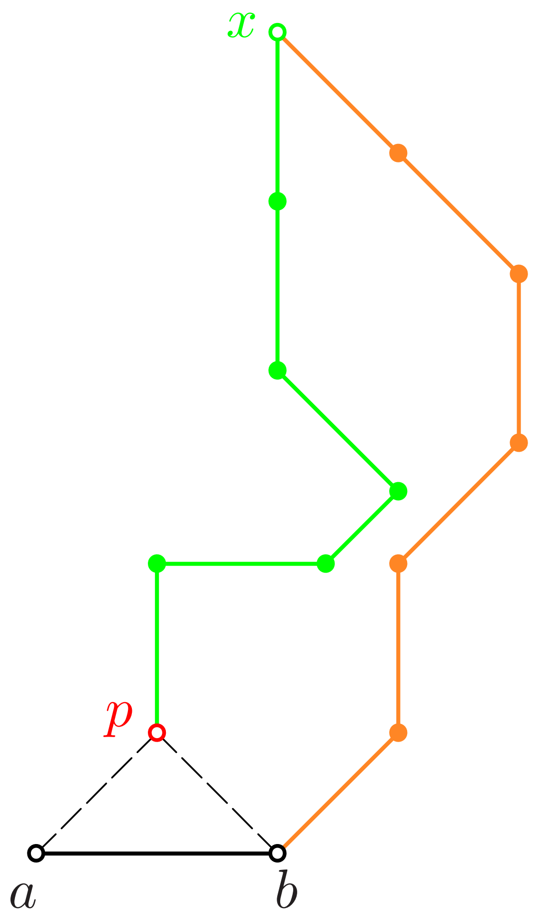

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A Lemma about Adjacency

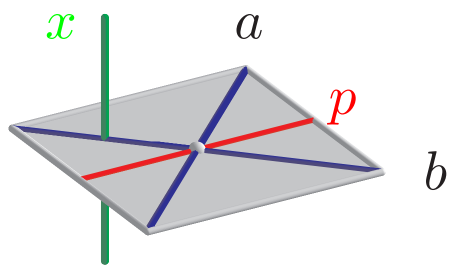
Lemma. *Let* $\Gamma = (\mathcal{P}, \mathcal{E})$ *be a graph which satisfies conditions* (A1)–(A4). *Suppose that* $a, b \in \mathcal{P}$ *are distinct points with the following property:*

$$\exists\, p \in \mathcal{P} \setminus \{a, b\}\ \forall x \in \mathcal{P}\ :$$

$$d(x, p) = \operatorname{diam}\Gamma \quad \Rightarrow \quad \big(d(x, a) = \operatorname{diam}\Gamma \ \vee\ d(x, b) = \operatorname{diam}\Gamma\big). \tag{1}$$

Then a *and* b *are adjacent.*

Illustration from a projective point of view for $\mathcal{G}_{4,2}$, *i. e.*, the Grassmannian of lines in a three-dimensional space:



Condition (A5) just guarantees that (1) holds for any two adjacent points $a, b \in \mathcal{P}$.

Final remarks

Characterisations of geometric transformations under mild hypotheses.

W. Benz, *Geometrische Transformationen*, 1992.

Z.-X. Wan: Geometry of Matrices, 1996.

Preservation theorems can be seen as as consequences of first-order definability, V. Pambuccian, 2000.

Generalisation from division rings to rings.

L. P. Huang: Geometry of Matrices over Ring, 2006.

M. Pankov, Grassmannians of Classical Buildings, to appear.

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