Matrix Spaces and Grassmannians

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Part 1

Rectangular Matrices

The first part deals with some basic notions and results from the Geometry of Rectangular Matrices. Square matrices are not excluded, and their particular properties will be exhibited in due course.

Our exposition follows the book of Z.-X. Wan [22].

Basic Notions

- Let F be a field (not necessarily commutative) or, said differently, a division ring.
- We denote by F^n the left vector space of row vectors $x = (x_1, x_2, \ldots, x_n)$ with entries from F.
- Let $F^{m\times n}$, $m, n \geq 1$, be the set of all $m \times n$ matrices over a division ring F.

There is yet no structure on the set $F^{m\times n}$.

A Single Matrix

- Each matrix $A \in F^{m \times n}$ determines a linear mapping

$$f_A : F^m \to F^n : x \mapsto xA.$$

- All linear mappings $F^m \to F^n$ arise in this way.
- The left row space of A is the subspace of F^n which is generated by the rows of A. It equals the image of the linear mapping f_A.
- The dimension of the left row space of A is called the left row rank of A.

The Dual Approach

Each column vector (single column matrix) $a^* \in F^{m\times 1} =: F^{m*}$ determines a linear form $F^m \to F : x \mapsto x \cdot a^*$. The elements of F^{m*} can be identified with the dual vector space of F^m, which is a right vector space over F.

This yields our second interpretation: Any matrix $A \in F^{m\times n}$ determines a linear mapping between dual vector spaces, viz.

$$f_A^{\mathrm{T}} : F^{n*} \longrightarrow F^{m*} : y^* \mapsto Ay^*$$

which is known as the transpose (or dual) of the mapping $f_A : x \mapsto xA$.
We obtain, mutatis mutandis, the notions right column space and right column rank of A.

Remarks

For any matrix one may introduce four notions of rank (left / right, row / column).

- The left row rank equals the right column rank of A. Either of these numbers will simply be called the rank of A, in symbols $\operatorname{rk} A$.

- The right row rank equals the left column rank of A.
  We shall not make use of these ranks.

- The left row rank and the right row rank of A may be different.

Example The matrix

$$\begin{pmatrix} 1 & j \\ i & k \end{pmatrix}$$

over the real quaternions $\mathbb{H}$ has left row rank 1 and right row rank 2, because

$$i(1, j) = (i, k), \quad \text{whereas} \quad (1, j)i = (i, -k) \neq (i, k).$$

Vector Space on $F^{m\times n}$

The sum of two matrices $A, B \in F^{m\times n}$ corresponds in a natural way to the sum of the associated mappings $f_A + f_B$ (and dually).

Even though a matrix A can be multiplied by a scalar $\lambda \in F$ from the left hand side (λA) or the right hand $(A\lambda)$, these products are in general not useful in terms of our interpretations of matrices as linear mappings:

"The λ is never where it should be!"

Only when λ is in the centre of F, in symbols $\lambda \in Z(F)$, then $\lambda A = A\lambda$ may be viewed as the product of λ and either of the two linear mappings given by A:

$$(\lambda f_A) : x \mapsto \lambda(xA) = x(\lambda A), \qquad (f_A^{\mathrm{T}}\lambda) : y^* \mapsto (Ay^*)\lambda = (\lambda A)y^*.$$

Hence $F^{m\times n}$ is a (left or right) vector space over $Z(F)$. This will be of some importance in what follows.

Rank One Matrices

Given a column vector $a^* = (a_1^*, a_2^*, \ldots, a_m^*)^{\mathrm{T}}$ (i. e. a linear form on F^m) and a vector $c = (c_1, c_2, \ldots, c_n)$ we obtain the linear mapping

$$F^m \longrightarrow F^n : x \mapsto x \cdot a^* \cdot c.$$

Its matrix is therefore

$$a^* \cdot c = \begin{pmatrix} a_1^* c_1 & a_1^* c_2 & \ldots & a_1^* c_m \\ a_2^* c_1 & a_2^* c_2 & \ldots & a_2^* c_m \\ \multicolumn{4}{c}{\dotfill} \\ a_n^* c_1 & a_n^* c_2 & \ldots & a_n^* c_m \end{pmatrix}.$$

This matrix has rank one provided that $a^* \neq 0$ and $c \neq 0$. All matrices with rank ≤ 1 arise in this way.

Graph on $F^{m\times n}$

Let $F^{m\times n}$, $m,n \geq 2$, be the set of all $m \times n$ matrices over a field F. Hence $F^{m\times n}$ contains matrices of rank ≥ 2.

- Two matrices A and B are called *adjacent* if $A - B$ is of rank one.
- We consider $F^{m\times n}$ as the set of vertices of an undirected graph the edges of which are precisely the (unordered) pairs of adjacent matrices.
- Two matrices A and B are at the graph-theoretical distance $k \geq 0$ if, and only if,

$$\mathrm{rk}(A - B) = k.$$

Almost a “Middle Product”

Given $a^* \in F^{m*} \setminus \{0\}$, $c \in F^n \setminus \{0\}$, and $\lambda \in F$ one may “multiply the rank one matrix $A := a^* c$ by $\lambda \in F$ from the middle” as follows:

$$(a^* \lambda) c = a^* (\lambda c) =: a^* \lambda c$$

This “product” in general depends on the vectors which are chosen to factorise A. Indeed, we have

$$A = (a^* \alpha)(\alpha^{-1} c) \quad \text{for all} \quad \alpha \in F \setminus \{0\},$$

and

$$(a^* \alpha) \lambda (\alpha^{-1} c) = a^* (\alpha \lambda \alpha^{-1}) c.$$

Nevertheless, the set of matrices

$$\{a^* \lambda c \mid \lambda \in F\}$$

depends only on the rank one matrix A and the ground field F.

Lines

Given $a^* \in F^{m*} \setminus \{0\}$, $c \in F^n \setminus \{0\}$ and any matrix $R \in F^{m\times n}$ the set

$$\{a^*\lambda c + R \mid \lambda \in F\}$$

is called a *LINE* of $F^{m\times n}$.

Let $\mathcal{L}$ be the set of all such lines. Then $\left(F^{m\times n}, \mathcal{L}\right)$ is a partial linear space, called the *space of $m \times n$ matrices over F.*

In this context the elements of $F^{m\times n}$ will also be called *POINTS*.

Two matrices A and B are adjacent if, and only if, they are distinct and COLLINEAR.

In this case the unique LINE joining A and B equals $\{A, B\}^{\sim\sim}$, where

$$\mathcal{M}^{\sim} := \{X \mid \forall\, Y \in \mathcal{M} : X \text{ is adjacent or equal to } Y\}.$$

Example

We consider the real quaternions $\mathbb{H}$. The LINE joining the 2×2 zero matrix and the matrix

$$\begin{pmatrix} 1 \\ i \end{pmatrix} \begin{pmatrix} 1 & i \end{pmatrix} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} =: A$$

equals the set of all matrices

$$\begin{pmatrix} 1 \cdot \lambda \cdot 1 & 1 \cdot \lambda \cdot i \\ i \cdot \lambda \cdot 1 & i \cdot \lambda \cdot i \end{pmatrix} = \begin{pmatrix} \lambda & \lambda i \\ i\lambda & i\lambda i \end{pmatrix},$$

where λ ranges in $\mathbb{H}$. The matrices (POINTS) of this LINE are in general neither left proportional nor right proportional to A.

Example

We consider the space of 2×2 matrices over the Galois field $\mathrm{GF}(2)$. All its rank one matrices can be read off from the following table:

$$\begin{array}{c|ccc}
 & \begin{pmatrix} 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \end{pmatrix} \\
\hline
\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\end{array}$$

Thus there are nine LINES through the zero matrix, each comprising two POINTS. The space of 2×2 over $\mathrm{GF}(2)$ matrices is a *partial affine space*, viz. the affine space on $\mathrm{GF}(2)^{2 \times 2}$ with six parallel classes of lines removed.

Summary

- The space $(F^{m\times n}, \mathcal{L})$ is a connected partial linear space.

- If F is a proper skew field then $F^{m\times n}$ can be considered as a vector space (affine space) over F from the left and right hand side, and (more naturally) as a vector space over the centre $Z(F)$. The LINES of $\mathcal{L}$ are in general not lines of any of these affine spaces.

- If F is a commutative field then $F^{m\times n}$ can be considered as a (left or right) vector space (affine space) over $F = Z(F)$. The LINES of $\mathcal{L}$ comprise some of the parallel classes of lines of this affine space.

Automorphisms

An *automorphism* of the space $(F^{m\times n}, \mathcal{L})$ is a bijection

$$\varphi : F^{m\times n} \to F^{m\times n} : X \mapsto X^{\varphi}$$

preserving adjacency in both directions. Consequently, LINES are mapped onto LINES under φ and φ^{-1}.

Examples

- Translations: $X \mapsto X + R$, where $R \in F^{m\times n}$.
- Equivalence transformations: $X \mapsto PXQ$, where $P \in \mathrm{GL}_m(F)$ and $Q \in \mathrm{GL}_n(F)$.
- Field automorphisms: $X \mapsto X^{\sigma}$, where σ is an automorphism of F acting on the entries of X.
- σ-Transpositions: $X \mapsto (X^{\sigma})^{\mathrm{T}}$, where σ is an antiautomorphism of F acting on the entries of X. (Only for $n = m$ provided that such a σ exists.)

Remarks on Automorphisms

- If $m = n$ and F is a commutative field then the transposition $X \mapsto X^{\mathrm{T}}$ is an automorphism.
- If $m = n$ and F is a proper skew field then $X \mapsto X^{\mathrm{T}}$ need not be automorphism. E. g., over the real quaternions $\mathbb{H}$ we already saw that

$$\operatorname{rk}\begin{pmatrix} 1 & j \\ i & k \end{pmatrix} = 1, \quad \text{whereas} \quad \operatorname{rk}\begin{pmatrix} 1 & j \\ i & k \end{pmatrix}^{\mathrm{T}} = \operatorname{rk}\begin{pmatrix} 1 & i \\ j & k \end{pmatrix} = 2.$$

- If $m = n$, F is a proper skew field, and σ is an antiautomorphism then $X \mapsto X^{\sigma}$ need not be an automorphism. E. g., letting $\sigma = \bar{\ }$ to be the conjugation of $\mathbb{H}$ gives

$$\operatorname{rk}\begin{pmatrix} 1 & j \\ i & k \end{pmatrix} = 1, \quad \text{whereas} \quad \operatorname{rk}\overline{\begin{pmatrix} 1 & j \\ i & k \end{pmatrix}} = \operatorname{rk}\begin{pmatrix} 1 & -j \\ -i & -k \end{pmatrix} = 2.$$

- There are proper skew fields without any antiautomorphism [4].

Fundamental Theorem

Theorem (L. K. Hua 1951 et al.) *Every bijective mapping*

$$\varphi : F^{m\times n} \to F^{m\times n} : X \mapsto X^{\varphi}$$

preserving adjacency in both directions is of the form

$$X \mapsto P X^{\sigma} Q + R,$$

where $P \in \mathrm{GL}_m(F)$, $Q \in \mathrm{GL}_n(F)$, $R \in F^{m\times n}$, and σ is an automorphism of F.

If $m = n$, then we have the additional possibility that

$$X \mapsto P(X^{\sigma})^{\mathrm{T}} Q + R$$

where P, Q, R are as above, σ is an antiautomorphism of F, and T denotes transposition.

The assumptions in Hua's fundamental theorem can be weakened.

W.-l. Huang and Z.-X. Wan [18], P. Šemrl [20].

Avoiding Matrices

From a theoretical viewpoint one may define the space of $m \times n$ matrices over F in a coordinate free way.

with coordinates / matrices	without coordinates / matrices
F^m	$V \ldots m$-dimensional left vector space over F
F^n	$W \ldots n$-dimensional left vector space over F
$F^{m\times n}$	$\mathrm{Hom}_F(V,W) \cong V^* \otimes_F W \ldots$ tensor product
$a^* \cdot c$	$a^* \otimes c \ldots$ pure tensor
rank of a matrix	rank of a linear mapping

Part 2

Grassmannians

We establish an embedding of any space of rectangular matrices in an appropriate Grassmann space. For square matrices this embedding will reveal neat connections with the projective lines over matrix rings.

Projective Space on F^{s+1}

Let $\mathrm{PG}(s, F)$ be the projective space over the left vector space F^{s+1}, where F is a field.

- In what follows we do not distinguish between subspaces of F^{s+1} and subspaces of $\mathrm{PG}(s, F)$.

- The *dimension* $\dim W$ of a subspace W is always understood as the “projective dimension”, which is one less than the vector space dimension.

- Subspaces of dimension $0, 1, 2, 3$, and $s-1$ are called *points*, *lines*, *planes*, *solids*, and *hyperplanes*, respectively.

- We use the shorthand *d-subspace* for a d-dimensional subspace.

Grassmann Graph on $\mathcal{G}_{s,d}$

Let $\mathcal{G}_{s,d}(F)$ be the Grassmannian of all d-subspaces of $\mathrm{PG}(s,F)$. We assume $1 \leq d \leq s-2$ in order to avoid trivial cases.

- Two d-subspaces W_1 and W_2 are called *adjacent* if $\dim W_1 \cap W_2 = d-1$.
- We consider $\mathcal{G}_{s,d}(F)$ as the set of vertices of an undirected graph the edges of which are the (unordered) pairs of adjacent d-subspaces.
- Two d-subspaces W_1 and W_2 are at graph theoretical distance $k \geq 0$ if, and only if,
$$\dim W_1 \cap W_2 = d-k.$$
- For any subset $\mathcal{M} \subset \mathcal{G}_{s,d}(F)$ we define
$$\mathcal{M}^{\sim} := \{X \mid \forall Y \in \mathcal{M} : X \text{ is adjacent or equal to } Y\}.$$

Grassmann Space on $\mathcal{G}_{s,d}$

Given a $(d-1)$-subspace U and a $(d+1)$-subspace V of $\mathrm{PG}(s, F)$ with $U \subset V$ the set

$$\{W \in \mathcal{G}_{s,d}(F) \mid U \subset W \subset V\}$$

is called a *pencil*.

The set $\mathcal{G}_{s,d}(F)$, considered as a set of *POINTS*, together with the set $\mathcal{P}$ of all its pencils, considered as its set of *LINES*, is called the *Grassmann space* of d-subspaces of $\mathrm{PG}(s, F)$.

The Grassmann space $(\mathcal{G}_{s,d}(F), \mathcal{P})$ is a connected partial linear space.

Two d-subspaces W_1 and W_2 are adjacent if, and only if, they are distinct and COLLINEAR. In this case the unique LINE joining W_1 and W_2 equals $\{W_1, W_2\}^{\sim\sim}$.

Fundamental Theorem

(W. L. Chow 1949) *Every bijective mapping*

$$\varphi : \mathcal{G}_{s,d}(F) \to \mathcal{G}_{s,d}(F) : X \mapsto X^{\varphi}$$

preserving adjacency in both directions is of the form

$$X \mapsto \{x^{\sigma}P \mid x \in X \subset F^{s+1}\},$$

where $P \in \mathrm{GL}_m(F)$ and σ is an automorphism of F.

If $s = 2d + 1$, then we have the additional possibility that

$$X \mapsto \{y \in F^{s+1} \mid yP(x^{\sigma})^{\mathrm{T}} = 0 \text{ for all } x \in X \subset F^{s+1}\},$$

where P is as above, σ is an antiautomorphism of F, and T denotes transposition.

The assumptions in Chow's fundamental theorem can be weakened.

W.-l. Huang [11].

An Embedding

We adopt the assumptions from Part 1. The $m \times m$ identity matrix will be denoted by I_m. Horizontal augmentation of (suitable) matrices A, B is written as $A|B$.

$F^{m\times n}$ can be embedded in the Grassmannian $\mathcal{G}_{m+n-1,m-1}(F)$ as follows:

$$\begin{array}{ccccc} F^{m\times n} & \longrightarrow & F^{m\times(m+n)} & \longrightarrow & \mathcal{G}_{m+n-1,m-1}(F) \\ X & \longmapsto & X|I_m & \longmapsto & \text{left rowspace of } X|I_m \end{array}$$

- Matrices $X, Y \in F^{m\times n}$ are adjacent if, and only if, their images in $\mathcal{G}_{m+n-1,m-1}(F)$ are adjacent.
- LINES of matrices are mapped to LINES (pencils) of the Grassmann space with one element removed.

Projective Matrix Spaces

Each element of the Grassmannian $\mathcal{G}_{m+n-1,m-1}(F)$ can be viewed as the left row space of a matrix $X|Y$ with rank m, where $X \in F^{m\times n}$ and $Y \in F^{m\times m}$.

- $X|Y$ and $X'|Y'$ have the same left row space, if and only if, there is a $T \in \mathrm{GL}_m(F)$ with $X' = TX$ and $Y' = TY$.

- One may consider a pair $(X, Y) \in F^{m\times n} \times F^{m\times m}$ as left homogeneous coordinates of an element of $\mathcal{G}_{m+n-1,m-1}(F)$ provided that $\mathrm{rk}(X|Y) = m$.

  This means that $X|Y$ possesses an invertible $m \times m$ submatrix. (This submatrix need not be Y).

The Grassmann space on $\mathcal{G}_{m+n-1,m-1}(F)$ is often called the *projective space* of $m \times n$ matrices over F, even though it is not a projective space in the usual sense.

Points at Infinity

- A subspace with coordinates (X, Y) is in the image of the embedding

$$F^{m\times n} \to \mathcal{G}_{m+n-1,m-1}(F)$$

  if, and only if, Y is invertible. In this case its only preimage is the matrix $Y^{-1}X \in F^{m\times n}$.

- All subspaces with coordinates (X, Y), where $Y \notin \mathrm{GL}_m(F)$, are called *POINTS at infinity* of the Grassmann space.

  Clearly, this notion depends on the chosen embedding.

- There is a distinguished $(n-1)$-subspace of $\mathrm{PG}(m+n-1, F)$ given by the left row space of the $n \times (m+n)$ matrix $I_n|0$.

- An element of $\mathcal{G}_{m+n-1,m-1}(F)$ is at infinity, precisely when it has at least one common point with this $(n-1)$-subspace.

See also R. Metz [19].

Example

The space of 2×2 matrices over $\mathrm{GF}(2)$ comprises 16 elements. It can be embedded in the Grassmann space of lines in $\mathrm{PG}(3,2)$. Note that $\#\mathcal{G}_{3,1}(\mathrm{GF}(2)) = 35$.

There is a unique distinguished line, viz. the row space of $I_2|0$. There are

$$3 \cdot 6 + 1 = 19$$

lines which have at least one common point with this line. These are the POINTS at infinity of the Grassmann space.

The $35 - 19 = 16$ lines which are skew to the line with coordinates $(I_2, 0)$ are in one-one correspondence with the 16 matrices of $\mathrm{GF}(2)^{2\times 2}$.

Example

The space of 2×3 matrices over $\mathrm{GF}(2)$ comprises 64 elements. It can be embedded in the Grassmannian of lines in $\mathrm{PG}(4,2)$. Note that $\#\mathcal{G}_{4,1}(\mathrm{GF}(2)) = 155$.

There is a unique distinguished plane, viz. the row space of $I_3|0$. There are

$$7 \cdot 12 + 7 = 91$$

lines which have at least one common point with this plane. They are the POINTS at infinity of the Grassmann space.

The $155 - 91 = 64$ lines which are skew to the plane with coordinates $(I_3, 0)$ are in one-one correspondence with the 64 matrices of $\mathrm{GF}(2)^{2\times 3}$.

Square Matrices

We consider square matrices ($m = n \geq 2$) and the full matrix algebra $R := (F^{n\times n}, +, \cdot)$ over $Z(F)$.

In terms of our left-homogeneous coordinates $(X, Y) \in R^2$ the POINT set of the Grassmannian $\mathcal{G}_{2n-1,n-1}(F)$ is the same as the POINT set of the projective line $\mathbb{P}(R)$ over the full matrix algebra R (up to irrelevant differences). Cf. [2].

There is one difference though:

- The basic notion in the Grassmann space is adjacency: $\dim W_1 \cap W_2 = n - 2$.
- The basic notion in ring geometry is being distant: $\dim W_1 \cap W_2 = -1$.

Each of these relations can be expressed in terms of the other. A. Blunck, H. H. [1], W.-l. Huang, H. H. [15].

Hence the two structural approaches are essentially the same.

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The book [22] is equipped with an extensive bibliography covering the relevant literature up to the year 1996. See [20] for a more recent survey.