Twisted Cubics on Cayley's Surface

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Cayley's ruled surface

Cayley's (ruled cubic) surface is, to within collineations of $\mathbb{P}_3(\mathbb{R})$, the surface F with equation

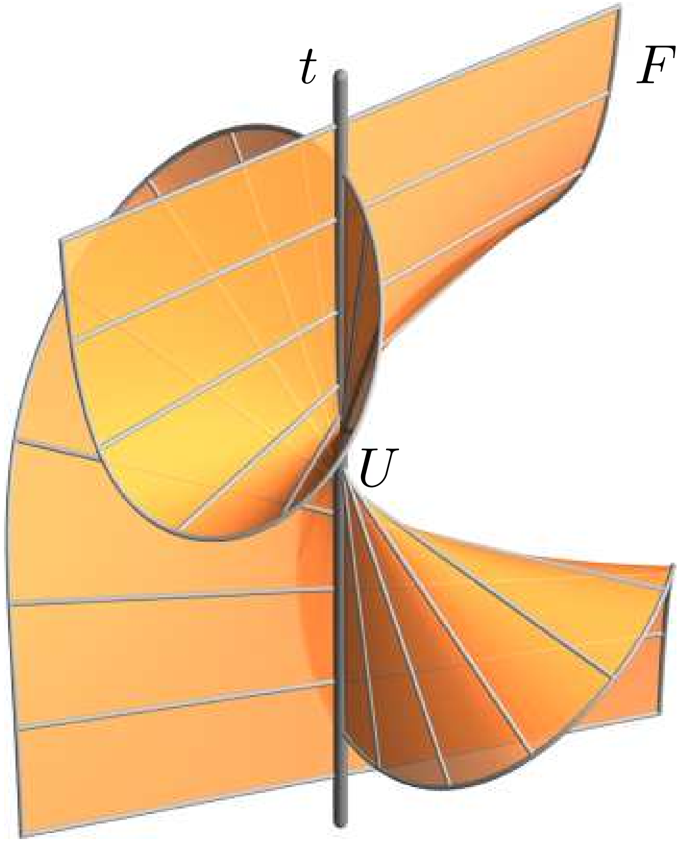
$$3x_0x_1x_2 - x_1^3 - 3x_3x_0^2 = 0.$$

The line $t : x_0 = x_1 = 0$ is a *torsal generator* of second order and a *directrix* for all other generators of F.

$U = \mathbb{R}(1,0,0,0)^{\mathrm{T}}$ is the *cuspidal point* on t.

The plane $\omega : x_0 = 0$ is the *tangent plane* at U. We consider it as *plane at infinity* (but not in all figures).

Affine chart for figure: $x_3 \neq 0$.



Each triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ with $\beta \neq 0$ gives rise to a function

$$\Phi_{\alpha,\beta,\gamma} : \mathbb{R}^{2\times 1} \to \mathbb{R}^{4\times 1} : (u_0, u_1)^{\mathrm{T}} \mapsto$$
$$\left(u_0^3, u_0^2(u_1 - \gamma u_0), \frac{u_0(u_1^2 + \alpha u_0^2)}{\beta}, \frac{(u_1 - \gamma u_0)}{3\beta} \left(3(u_1^2 + \alpha u_0^2) - \beta(u_1 - \gamma u_0)^2\right) \right)^{\mathrm{T}}.$$

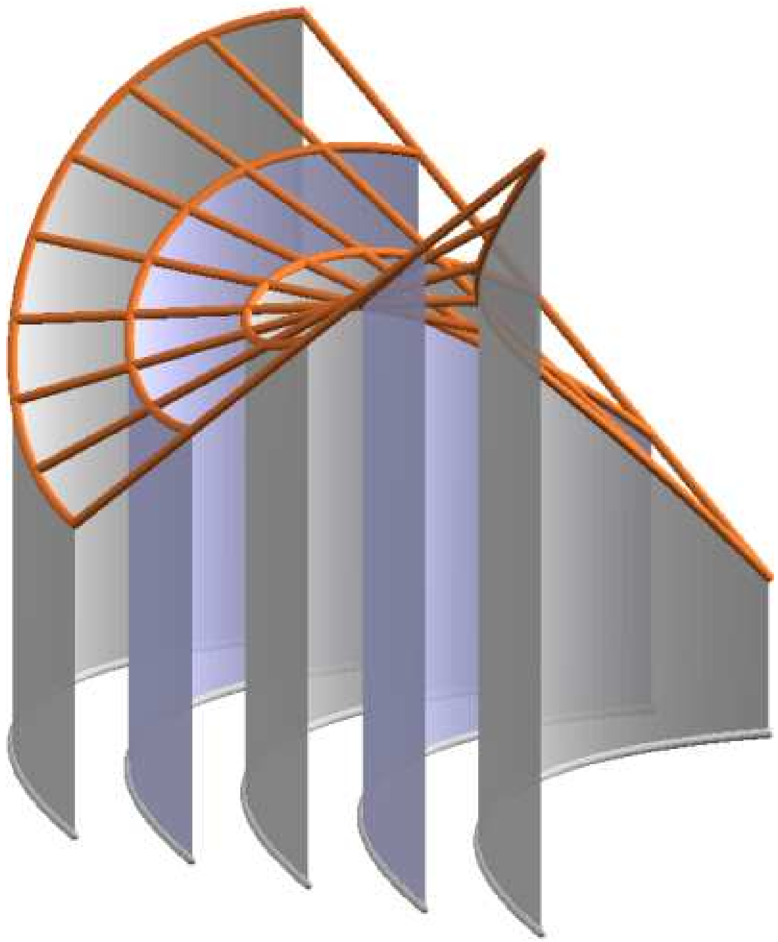
In projective terms we obtain a curve $c_{\alpha,\beta,\gamma}$ on the Cayley surface, lying on the *parabolic cylinder* with equation $\alpha x_0^2 - \beta x_0 x_2 + (x_1 + \gamma x_0)^2 = 0$.

$\beta \neq 3$: *cubic parabola* $c_{\alpha,\beta,\gamma} \subset F$.

$\beta = 2,\ \gamma = 0$: *asymptotic curves* $c_{\alpha,2,\gamma}$ of F.

$\beta = 3$: $\Phi_{\alpha,3,\gamma}((1, u_1)^{\mathrm{T}})$ . . . affine part of a *parabola* $c_{\alpha,3,\gamma} \subset F$.

A one-parameter family of cubic parabolas



Cubic parabolas $c_{\alpha,\beta,0}$, where α ranges in $\{-\frac{3}{2}, -\frac{3}{4}, 0, \frac{3}{4}, \frac{3}{2}\}$, and $\beta = \frac{3}{2}$.

Claims on higher order contact

H. Neudorfer (1925): *The asymptotic curves of F (other than generators) have fourth order contact at U.*

H. Brauner (1964): *Given cubic parabolas $c_{\alpha,\beta,\gamma}$ and $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ the following assertions hold:*

- $\beta = \overline{\beta}$ $\Rightarrow$ *second order contact at U;*
- $\beta = \overline{\beta}$ *and* $\gamma = \overline{\gamma}$ $\Rightarrow$ *third order contact at U;*
- *contact of order four at U* $\Rightarrow$ *curves are identical.*

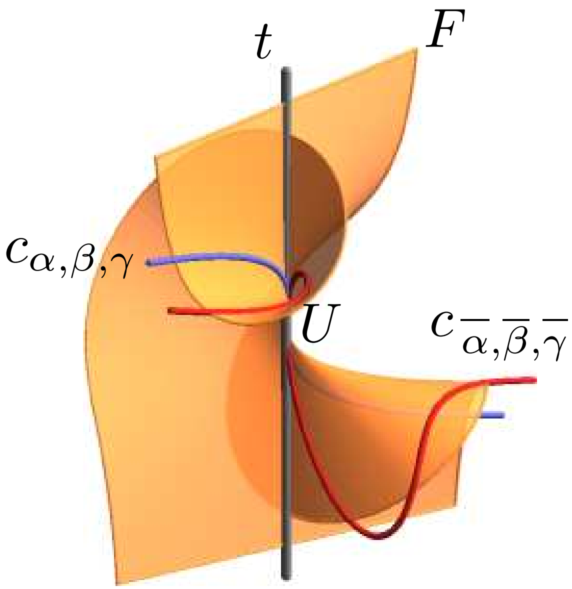
Higher order contact

Theorem. *Distinct cubic parabolas $c_{\alpha,\beta,\gamma}$ and $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ on Cayley's ruled surface have*

- *second order contact at U* $\Leftrightarrow$ $\beta = \overline{\beta}$ *or* $\beta = 3 - \overline{\beta}$*;*
- *third order contact at U* $\Leftrightarrow$ $\beta = \overline{\beta}$ *and* $\gamma = \overline{\gamma}$*, or* $\beta = \overline{\beta} = \frac{3}{2}$*;*
- *fourth order contact at U* $\Leftrightarrow$ $\beta = \overline{\beta} = \frac{3}{2}$ *and* $\gamma = \overline{\gamma}$*.*

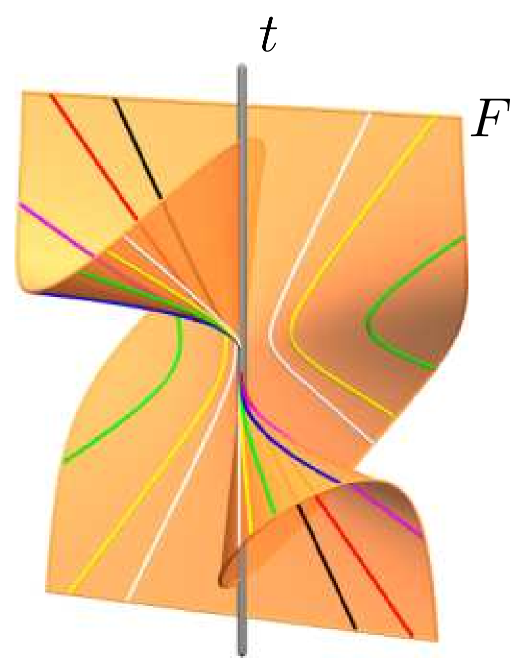
Proof. A long, even though straightforward calculation (Maple).

Affine chart for figures: $x_3 \neq 0$.



Second order contact at U:

$$(\alpha, \beta, \gamma) = (0, \tfrac{1}{10}, 0),$$
$$(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = (1, 3 - \tfrac{1}{10}, \tfrac{1}{10}).$$



Fourth order contact at U.

$$\alpha = -3, -2, \ldots, 3,$$
$$\beta = \tfrac{3}{2},\ \gamma = 0.$$

What means $\beta = \frac{3}{2}$?

ω . . . *isotropic plane* with absolute flag (U, t).

$\mathbb{P}_3(\mathbb{R})$ . . . *two-fold isotropic space* with absolute flag (U, t, ω).

Theorem. *Among all cubic parabolas $c_{\alpha,\beta,\gamma}$ on the Cayley surface F, the cubic parabolas with $\beta = \frac{3}{2}$ are precisely those with maximal twofold isotropic conical curvature.*

Proof. The *tangent surface* of a cubic parabola $c_{\alpha,\beta,\gamma}$ meets the plane ω in t and in an *isotropic circle* with *isotropic curvature*

$$\frac{1}{2}\beta(3-\beta) \leq \frac{9}{8}.$$

Hence $c_{\alpha,\beta,\gamma}$ has the twofold isotropic conical curvature $\frac{1}{2}\beta(3-\beta) \leq \frac{9}{8}$.

The dual Cayley surface

A plane with coordinates $\mathbb{R}(y_0, y_1, y_2, y_3)$ is tangent to the Cayley surface if, and only if,

$$3y_0y_3^2 - 3y_1y_2y_3 + y_2^3 = 0.$$

So, all tangent planes comprise a Cayley surface in the dual space.

All osculating planes of a twisted cubic form a twisted cubic in the dual space.

Dual contact of order k $\Leftrightarrow$ contact of order k of the dual curves.

Higher order dual contact

Theorem. *Distinct cubic parabolas* $c_{\alpha,\beta,\gamma}$ *and* $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ *on Cayley's ruled surface have*

- *second order dual contact at* ω $\Leftrightarrow$ $\beta = \overline{\beta}$;
- *third order dual contact at* ω $\Leftrightarrow$ $\beta = \overline{\beta}$ *and* $\gamma = \overline{\gamma}$, *or* $\beta = \overline{\beta} = \frac{5}{2}$;
- *fourth order dual contact at* ω $\Leftrightarrow$ $\beta = \overline{\beta} = \frac{7}{3}$ *and* $\gamma = \overline{\gamma}$.

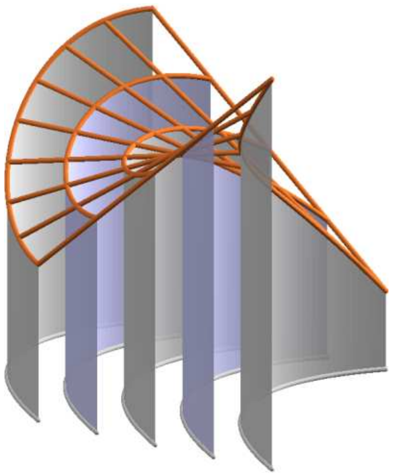
In particular, two cubic parabolas of this kind, with fourth order contact at U and fourth order dual contact at ω, are identical.

Dual contact vs. contact

Theorem. *Let $\beta \neq 0, 3$ be a fixed real number. The image of the affine part of the Cayley surface F under the mapping*

$$P \in c_{\alpha,\beta,0} \setminus \{U\} \xmapsto{\Sigma} \textit{osculating plane of } c_{\alpha,\beta,0} \textit{ at } P$$

consists of tangent planes of a Cayley surface F' for $\beta \neq 0, 3, \frac{8}{3}$, and of tangent planes of a hyperbolic paraboloid for $\beta = \frac{8}{3}$.



Each $c_{\alpha,\beta,0} \xrightarrow{\Sigma} D_\beta(c_{\alpha',\beta',0})$ with D_β a duality,

$$\alpha' := \alpha(\beta - 3), \quad \text{and} \quad \beta' := \frac{3\beta - 8}{\beta - 3},$$

whence

$$\beta = \frac{7}{3} \mapsto \beta' = \frac{3}{2}.$$

A final remark

There remains the problem to find a geometric interpretation of the value

$$\beta = \overline{\beta} = \frac{5}{2}$$

which guarantees third order dual contact at U.