Möbius differential geometry

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Basics
Möbius geometry is the geometry of the group of Möbius transformations, that is, hypersphere preserving (point) transformations, acting on the n-sphere \( S^n \) as a base manifold. The elements of Möbius geometry are points (elements of the first kind) and hyperspheres (elements of the second kind).

Models
Models serve a uniform description of the elements of (Möbius) geometry (points, hyperspheres) and derived objects (for example, k-spheres) as well as a description of the Möbius transformations as linear, fractional linear, or spin transformations.

The classical (projective) model: the conformal n-sphere \( S^n \) as a linear, fractional linear, or spin transformation.

As quaternionic Hermitian forms, in the quaternionic approach \( S^n \) is the space of hyperspheres as the “outer space” \( S^n_{n+1}/\mathbb{Z} \subset \mathbb{R}^{n+1} \); the Lorentz sphere \( S^n_{n+1} = \{ v \in \mathbb{R}^{n+2} \mid |v|^2 = 1 \} \) can be interpreted as the space of oriented hyperspheres. Möbius transformations become Lorentz transformations, resp. projective transformations that preserve \( S^n \subset \mathbb{R}^{n+1} \).

The quaternionic approach: the conformal 4-space as the quaternionic projective line, \( S^3 \cong \mathbb{H}^1 \), and the space of quaternionic Hermitian forms \( \mathcal{H}(\mathbb{H}) \cong \mathbb{R}^3 \) with \( \mathcal{H}^3 \cong \mathbb{R}^3 \) (w.r.t. some basis) so that 3-spheres are quaternionic Hermitian forms. Möbius involutions \( S \in \mathcal{H}(\mathbb{H}) \), \( S^2 = -id \), are 2-spheres. Orientation preserving Möbius transformations are fractional linear transformations, or special linear transformations (on homogeneous coordinates \( v \in \mathbb{H}^2 \)).

A Clifford algebra model: the coordinate Minkowski space \( \mathcal{H}^2 \) of the projective model is embedded into its Clifford algebra \( \mathcal{A}_{n+2} \). Möbius transformations are \( (m,\bar{m}) \)-spin transformations.

The Vahlen matrix approach: the Clifford algebra \( \mathcal{A}_{n+2} \) is described in terms of 2x2-matrices with entries from the Clifford algebra \( \mathcal{A}_{n} \) of Euclidean n-space. Möbius transformations are fractional linear transformations, given by Vahlen matrices.

Points
We consider \( \mathcal{R}^{n+2} = \mathbb{R} \times \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^{n} \times \mathbb{R} \) with the Minkowski product \((\eta_0, y) \cdot (\eta_1, y_1) = -\eta_0^2 + |y|^2 \). The following are descriptions of points in different models.

As points of the absolute quadric in the projective model:
\[
\mathcal{R}^{n+2} \cong S^n \ni y \quad \mapsto \quad \mathcal{R}(1, y) \quad \in S^n \subset \mathbb{R}^{n+1}.
\]

As quaternionic Hermitian forms, in the quaternionic approach (\( \mathcal{R}^2 \cong \mathbb{H} \) can be identified with the affine slice \( v_2 = 1 \)),
\[
S^1 \cong \mathbb{H}^1 \ni (v_1, v_2) \quad \mapsto \quad \mathcal{R}(v_1^2 - v_2^2) \quad \subset \mathcal{H}(\mathbb{H}).
\]

As 2x2-Clifford algebra matrices in the Vahlen matrix approach:
\[
\mathcal{R}^{n} \ni x \quad \mapsto \quad \mathcal{A}(x, -x) \quad \in \mathcal{A}_{n+2} \cong M(2 \times 2, \mathcal{A}_{n}).
\]

Hyperspheres
A hypersphere with center \( m \in S^n \subset \mathbb{R}^{n+1} \) and radius \( r \in (0, \pi) \):
\[
S = \frac{1}{\sin r} (\cos \theta, m) \subset S^{n+1}_{n+1};
\]

A hypersphere with center \( m \in \mathbb{R}^n \) and radius \( r \neq 0 \):
\[
S = \frac{1}{r} (1 + r^2 \cdot x, m, -\frac{1}{2} r^2 \cdot x^2) \subset S^{n+1}_1,
\]

and a hyperplane with normal \( n \in S^{n-1} \subset \mathbb{R}^n \) and (directed) distance \( d \in \mathbb{R} \) from the origin:
\[
T = (d, n, -d) \subset S^{n+1}_1.
\]

As Vahlen matrices:
\[
S = \frac{1}{r} \left( \begin{array}{cc} 1 - m^2 \cdot r^2 & -m \cdot r^2 \\ \frac{1}{2} \cdot r^2 \cdot m^2 & 1 \end{array} \right) \quad S^{n+1}_1;
\]

and as quaternionic Hermitian forms:
\[
s = \frac{1}{r} \left( \begin{array}{cc} 1 - m^2 \cdot r^2 & -m \cdot r^2 \\ \frac{1}{2} \cdot r^2 \cdot m^2 & 1 \end{array} \right) \quad \in \mathcal{H}(\mathbb{H}),
\]

2-spheres (or planes) of complex involutions as Möbius involutions:
\[
S = \frac{1}{r} \left( \begin{array}{cc} 1 - m^2 \cdot r^2 & -m \cdot r^2 \\ \frac{1}{2} \cdot r^2 \cdot m^2 & 1 \end{array} \right) \quad \in \mathcal{S}(\mathbb{H}).
\]

Note that \( S \) and \( T \) are symmetric w.r.t. \( \mathbb{R}^n \cong \mathbb{R}^n \) for the following are described in the classical (projective) picture, this is the group of projective transformations that map \( S^n \subset \mathbb{R}^{n+1} \) to itself.

The Möbius group
The Möbius group \( \text{Mob}(S^n) \) is the conformal group \( \text{Conf}(S^n) \) of \( S^n \); in the classical (projective) picture, this is the group of projective transformations that map \( S^n \subset \mathbb{R}^{n+1} \) to itself.

Inversions
The inversion at a hypersphere \( S \subset S^n \) is the polar reflection at \( S \in \mathbb{R}^{n+1} \); in homogeneous coordinates, \( p \in \mathbb{R}^n \) and \( S \in S^{n+1}_1 \).

In terms of Vahlen matrices:
\[
\mathcal{R}^n \ni x \quad \mapsto \quad x - (x^2 - r^2) \quad \in \mathcal{A}_{n+2} \cong M(2 \times 2, \mathcal{A}_n).
\]

\( \mathcal{S}(2, \mathbb{H}) \) does not provide (orientation reversing) inversions.

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Stereographic projection

Let $K_0 = (1, 0, -1) \in \mathbb{Q}$ be the “south pole” in the round n-sphere $S^n \cong \mathbb{Q}$ given by $K_0 = (1, 0, 0)$:

$$Q_0 \equiv \left[\begin{array}{ccc} 1 + |x|^2 & x \cdot 1 - |x|^2 \\ x \cdot 1 + |x|^2 & 1 - |x|^2 \end{array}\right] \in \mathbb{Q}.$$  

$q_1 \in \mathbb{K} \supset (1,y_1, 2y_2) \in Q_0$

then yields the classical stereographic projection. More generally, $S^n \supset \mathbb{R}^n \to \mathbb{K} \cong \mathbb{Q}$ can be considered as a stereographic projection from (point of the conformal n-sphere onto a quadratic form of constant curvature.

With $\nu_x, \nu_y \in (H^2)^\ast$ a notion of stereographic projection is given by

$$\nu_x \nu_y \in (H^2)^\ast$$

The cross ratio

Four points $p_i \in S^n$ always lie on a 2-sphere that can be considered as a Riemann sphere, so that their complex cross ratio $[p_1; p_2; p_3; p_4]$ can be defined up to complex conjugation (orientation of $S$). In the following $[p_1; p_2; p_3; p_4] \in C$ is obtained by taking $[p_1; p_2; p_3; p_4] := \Re \{\nu (\nu_1 \nu_2 \nu_3 \nu_4)\}$ where appropriate.

Expressing the cross ratio in terms of the distances $|x_i - x_j|^2 = -2v_i \nu_j$, where $v_k = (1 - |x|^2) \nu_k$ of the four points in $\mathbb{R}^n$, one arrives at

$q = (v_i, 1 - |x|^2)(v_j, 1 - |x|^2) = (v_i, 1 - |x|^2)(v_j, 1 - |x|^2)$

Using the Clifford algebra setup, the ratio is obtained from

$q = \frac{(v_i, 1 - |x|^2)(v_j, 1 - |x|^2)}{(v_i, 1 - |x|^2)(v_j, 1 - |x|^2)}$.

The mean curvature $H$ of a hypersphere $S \subset S^n_{1+2}$ is given by

$$H = (S, K),$$

in particular, $S$ is a hypersphere if $K$ is a sphere of the space complex $K$, $S \subset K$.

A $k$-sphere $S = S_1 \ldots S_n$, $S_{n-k}$ is a $k$-plane in $Q_n$ iff all $S_j \subset K \in \mathbb{K}$ span $\{v_i = 1, \ldots, k+2\}$

Two 2-spheres $S \subset \mathbb{K}$ is a 2-plane in $Q_n$ given by $K \subset \mathbb{K}$ iff $S$ is skew wr.t. $K$; more generally, its mean curvature is given by

$$|H|^2 = |K_j|^2,$$

where $K_j = \{K(S, +S_j)\}$. A Möbius transformation that fixes the sphere complex $K$ (the hyperplanes of $Q_n$) is an isometry of $Q_n$ if $\kappa \neq 0$ or a similarity of $Q_n$, respectively;

$\text{Isom}(Q_n) = \{u \in O(n+2) | \kappa(K) = K\}$

is the group of isometries of $Q_n$, i.e. $\kappa \neq 0$, it is the group of isometries that extend smoothly through the infinity sphere $\mathbb{K}$.

Sphere congruences and envelopes

A sphere congruence is a smooth map $S : M^m \to S^{n+1}_{1+2}$, and a smooth map $f : M^m \to S^n$ is said to envelope $S$ if $f(p) \in S(p)$ and $df(T_pM^m) \subset T_pS(p)$ for all $p \in M^m$.

For hypersurfaces, $m = n - 1$, this reads

$$0 = (f, S) = \frac{1}{2} S(Sf - f)$$

and $0 \equiv df(S) = \frac{1}{2} S(Sdf - df)$, when considering $f : S : M^{n+1} - \to S^{n+1}_{1+2} \subset \mathbb{A}_2$ an immersed congruence $S : M^{n+1} - \to S^{n+1}_{1+2}$ has two envelopes iff $(dS, dS)$ is positive definite. For $f : M^3 \to H^3$ and $S : M^3 \to G(H^2)$ the enclosing condition reads

$$0 = (S, f) \quad \text{and} \quad 0 \in S(df, df) + S(df, df).$$

A 2-sphere congruence $S : M^2 \to \mathbb{E}(H^2)$ is enveloped by $f$ if $S \cdot f \parallel f$ and $dS \cdot f \parallel f$
or, equivalently, if $f$ envelopes every hypersphere congruence (section) in the congruence of elliptic sphere pencils given by $S$. Similarly, a m-sphere congruence $S : M^m \to \Lambda_m \mathbb{R}^{m+1}_{1+2}$ is enveloped by $f : M^m \to \Lambda_m \mathbb{R}^{m+1}_{1+2}$ if $f$ envelopes any section of $S$ (hypersphere congruence in $S$). With the contact elements $\wp = \wp(p) \cdot dp(f(e_1), \ldots, dp(f(e_m)))$, $(e_1, \ldots, e_m)$ orthonormal, of an immersion $R : M^m \to S^n$, the enveloping condition reads

$$\nu \in \nu(S) \quad \text{iff} \quad R \nu \in \mathbb{A}_2.$$

The central sphere congruence $Z : M^m \to \Lambda_m \mathbb{R}^{m+1}_{1+2}$ is given by

$$Z = (S, f) \cdot f$$

The central sphere congruence $Z : M^m \to \Lambda_m \mathbb{R}^{m+1}_{1+2}$ is an immersion $R : M^m \to S^n$ given by

$$v \in v(S) \quad \text{iff} \quad R v \in \mathbb{A}_2.$$