ANALYSIS AND DESIGN OF CURVED SUPPORT STRUCTURES

CHENGCHENG TANG, MARTIN KILIAN, PENGBO BO, JOHANNES WALLNER, AND HELMUT POTTMANN

ABSTRACT. Curved beams along freeform skins pose many challenges, not least on the level of basic geometry. A prototypical instance of this is presented by the glass facades of the Eiffel tower pavilions, and the interrelation between the differential-geometric properties of the glass surface on the one hand, and the layout of beams on the other hand. This paper discusses how curved beams are represented by developable surfaces, and studies geometric facts relevant to beam placement along guiding surfaces. Surprisingly, many of the curves which are interesting from the viewpoint of pure geometry (geodesics, principal curves, etc.) occur in this context too. We discuss recent advances in the modeling of developable surfaces, and show how they permit the interactive design of arrangements of curved beams, in particular the design of so-called geometric support structures.

KEY WORDS. developable surfaces, support structures, interactive design, Darboux frame.

1. DEVELOPABLE SURFACES IN FREEFORM ARCHITECTURE

Our objects of study are composed of *developable* surfaces, which may be informally introduced as surfaces that can be flattened without stretching or tearing. We first discuss different kinds of developables occurring in freeform architecture.

Freeform skins composed of developables. Inextensible materials like paper and sheet metal naturally assume developable shapes, so it is not surprising that freeform skins consisting of developables have been built – see (Pottmann et al.,

FIGURE 1. Freeform skins composed of developables. *Left:* The Disney Concert Hall consists of large near-developable pieces. *Center:* The Fondation Louis Vuitton, Paris, is composed of strip sequences, each strip being made from cylindrical glass panels and approximating a continuous developable. *Right:* The interior of the Burj Khalifa, Dubai, exhibits a paneling by "geodesic" developable elements, using the terminology of (Pottmann et al., 2008) and (Wallner et al., 2010). All three designs are by F. Gehry.

FIGURE 2. Curved-Crease Sculptures. *Left:* Arum installation, 2012 Venice biennale, by Zaha Hadid architects and Robofold. Its form is defined by sheet metal folded along curved creases. *Right:* This virtual model of annuli folded along concentric rings by (Tang et al., 2016) is motivated by actual paper objects, cf. (Demaine and Demaine, 2012).

2015, §3) for a short overview of this topic, and Figure 1 for examples. Material properties however are not the only reason why developables occur: (Liu et al., 2006) successfully exploited the viewpoint that a sequence of planar quadrilateral panels approximates a developable. So does a sequence of cylindrical glass panels. Such sequences occur e.g. in the 2007 Strasbourg railway station, the Eiffel tower pavilions, see (Baldassini et al., 2013), or the 2015 Fondation Louis Vuitton, see Figure 1, center.

Some piecewise-smooth surfaces can be flattened without even cutting them along creases. One distinguishes two cases: (i) surfaces which locally around every point can be flattened, but a global flattening requires a certain number of cuts (Figure 2 left); (ii) surfaces capable of flattening without a single cut (Figure 2 right). The behaviour of such curved-crease sculptures, especially regarding degrees of freedom in modeling, is entirely different from the skins of Figure 1.

Non-skin arrangements of developables. Developables may have other functions, in particular when they are positioned transverse to a freeform skin. Figure 5 shows the Eiffel tower pavilions, where the sides of curved beams supporting the glass facade contain developables *orthogonal* to that facade, see (Schiftner et al., 2012). Figure 3 shows so-called geometric support structures – using the terminology of (Pottmann et al., 2015, §6.1) – which can either be smooth like the curved beams of Figure 3, left, or discrete like the shading elements in Figure 3, right.

2. DIFFERENTIAL GEOMETRY OF STRIPS

We are interested in the degrees of freedom available to a designer who wishes to lay out developables positioned either tangential to a given surface Φ, or transverse to it (see Figures 1 and 3, respectively). This discussion requires studying the well
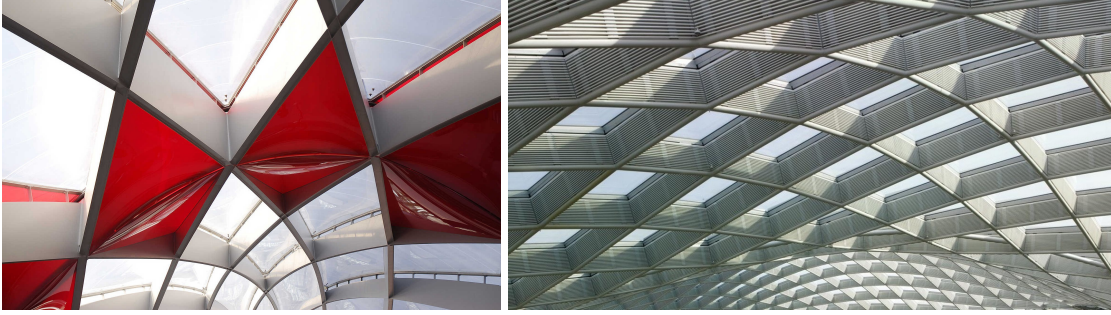
FIGURE 3. Developables which lie transverse to freeform skins. *Left:* Mediacité retail centre, Liege (Ron Arad Architects, structural engineering: Buro Happold). Here both families of intersecting curved beams are modeled as developables. *Right:* Kogod courtyard, Smithsonian Institution (Foster and partners). Quadrilateral shading elements approximate developables, and their lines of intersection approximate rulings.

known movement of the so-called *Darboux frame* which is adapted to a curve $\mathbf{c}$ lying in Φ. We give an introduction to this frame, for more details see textbooks like (O'Neill, 2006) or (Strubecker, 1969).

We assume that s is an arc length parameter and $\mathbf{c}(s)$ is a point of the curve under consideration. Consider the unit tangent vector $\mathbf{t}(s)$, the vector $\mathbf{n}(s)$ orthogonal to the reference surface Φ, and the sideways vector $\mathbf{u}(s) = \mathbf{n}(s) \times \mathbf{t}(s)$. These vectors are used to describe a developable Ψ which follows the curve $\mathbf{c}$. If Ψ is orthogonal to Φ, it is the envelope of the plane spanned by $\mathbf{t}$ and $\mathbf{n}$. If Ψ is tangential to Φ, it is enveloped by the plane $[\mathbf{t}, \mathbf{u}]$. Figure 4 illustrates this situation.

The goal of the computations which follow below is to find out the *rulings* of the developable Ψ. Their position is relevant to manufacturing by bending from a flat state. Figure 4 illustrates the Darboux frame $\{\mathbf{t}, \mathbf{u}, \mathbf{n}\}$ for a particular choice of curve and developable. The rulings are indicated by thin lines – it is important to note that a ruling does not have to be parallel to $\mathbf{n}$.

The motion of the Darboux frame along a curve in a reference surface. The rotational movement of the Darboux frame is governed by a *vector of angular velocity*, called $\mathbf{d}$. Any $\mathbf{x}$ moving with the Darboux frame has a rate of change expressed in terms of the angular velocity as $\mathbf{x}' = \frac{d}{ds}\mathbf{x}(s) = \mathbf{d} \times \mathbf{x}$. Thus,

$$\mathbf{d} = \tau_g\mathbf{t} - \kappa_n\mathbf{u} + \kappa_g\mathbf{n} \implies \begin{cases} \mathbf{t}' = \mathbf{d} \times \mathbf{t} = \qquad\qquad \kappa_g\mathbf{u} + \kappa_n\mathbf{n} \\ \mathbf{u}' = \mathbf{d} \times \mathbf{u} = -\kappa_g\mathbf{t} \qquad\quad + \tau_g\mathbf{n} \\ \mathbf{n}' = \mathbf{d} \times \mathbf{n} = -\kappa_n\mathbf{t} - \tau_g\mathbf{u} \end{cases}$$

It is well known that the coefficients of $\mathbf{d}$ are the curve's normal curvature κ_n, the curve's geodesic curvature κ_g, and its geodesic torsion τ_g. An insect crawling on the surface Φ can freely choose the geodesic curvature κ_g by turning to the left
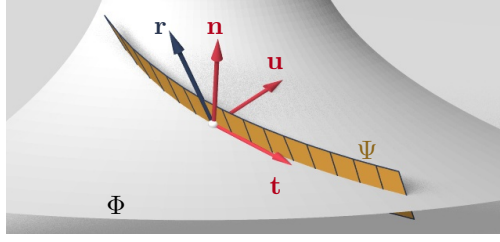
FIGURE 4. The Darboux frame associated with a curve in a surface Φ consists of the curve's unit tangent vector $\mathbf{t}$, the surface's normal vector $\mathbf{n}$ and the vector $\mathbf{u} = \mathbf{t} \times \mathbf{n}$. We also consider the developable surfaces Ψ which follows that curve and which is orthogonal to the reference surface Φ. Its rulings are indicated by the vector $\mathbf{r}$.

or to the right, in the manner of the driver of a car can use the steering wheel. Walking straight ($\kappa_g = 0$) produces a geodesic curve on the surface.

On the other hand, the normal curvature equals $\kappa_n = \mathrm{II}(\mathbf{t})/\mathrm{I}(\mathbf{t})$, where I, II are the first and second fundamental forms of the surface, respectively. Therefore κ_n is determined by the direction of the tangent vector $\mathbf{t}$ alone and can only be influenced by taking a completely different route. In negatively curved regions there are two *asymptotic* directions at each point where $\mathrm{II}(\mathbf{t}) = 0$, which are found by intersecting an infinitesimal piece of surface with its own tangent plane. In that case the above-mentioned insect can decide to follow the asymptotic line field to achieve $\kappa_n = 0$ if necessary. In positively curved regions this is not possible.

Similarly, also the geodesic torsion is already determined by the direction $\mathbf{t}$: It is known that $\tau_g = \frac{1}{2}(\kappa_2 - \kappa_1)\sin 2\phi$, where κ_1, κ_2 are the principal curvatures and ϕ is the angle between $\mathbf{t}$ and the vector indicating the principal direction. We see that τ_g vanishes for the two principal directions, and is highest exactly in between (i.e., if $\mathbf{t}$ bisects the principal directions).

Developable strips along a curve. Considerations regarding flattening. A developable surface Ψ which is orthogonal to the reference surface Φ and contains the guiding curve $\mathbf{c}$, is enveloped by the planes with normal vector $\mathbf{u}$ – see Figure 4. Thus the direction $\mathbf{r}$ of rulings is computed as $\mathbf{r} = \mathbf{u} \times \mathbf{u}'$. In more detail,

$$\mathbf{r} = \mathbf{u} \times \mathbf{u}' = \mathbf{u} \times (\mathbf{d} \times \mathbf{u}) = \mathbf{d} - (\mathbf{d} \cdot \mathbf{u})\,\mathbf{u}$$

$$= (\tau_g \mathbf{t} - \kappa_n \mathbf{u} + \kappa_g \mathbf{n}) - \big((\tau_g \mathbf{t} - \kappa_n \mathbf{u} + \kappa_g \mathbf{n}) \cdot \mathbf{u}\big)\,\mathbf{u} = \kappa_g \mathbf{n} + \tau_g \mathbf{t}.$$

A similar computation applies if Ψ encloses the constant angle α with the reference surface (we take planes with normal vector $\mathbf{n}^\alpha = \cos\alpha\,\mathbf{n} + \sin\alpha\,\mathbf{u}$ instead of $\mathbf{u}$, and get rulings $\mathbf{r}^\alpha = \tau_g \mathbf{t} + (\kappa_g \sin\alpha + \kappa_n \cos\alpha)(\sin\alpha\,\mathbf{n} - \cos\alpha\,\mathbf{u})$). For developables tangent to the guiding surface (i.e., $\alpha = 0$), we get $\mathbf{r} = \tau_g \mathbf{t} - \kappa_n \mathbf{u}$.

We would also like to say some words about *development*, i.e., flattening of a developable strip Ψ. It is known that geodesic curvatures of curves are invariant in this process. Thus the guiding curve $\mathbf{c}$ is flattened to a straight line if and only if its geodesic curvature w.r.t. Ψ (not w.r.t. Φ) vanishes. In the notation

employed above, this curvature of the development equals $-\kappa_g \cos \alpha + \kappa_n \sin \alpha$; for developables orthogonal to Φ, it equals κ_n. We summarize:

Proposition. *Consider a developable Ψ through a guiding curve $\mathbf{c}$ which itself lies in a reference surface Φ. The rulings of Ψ are called good resp. bad, if they are orthogonal resp. tangential to $\mathbf{c}$. Then we have the following properties:*

If the curve $\mathbf{c}$ in Φ is . . .	*then a developable Ψ through $\mathbf{c}$, tangential to Φ, has . . .*	*and a developable Ψ through $\mathbf{c}$, orthogonal to Φ, has . . .*
geodesic $(\kappa_g = 0)$	*straight development*	*bad rulings*
asymptotic $(\kappa_n = 0)$	*bad rulings*	*straight development*
principal $(\tau_g = 0)$	*good rulings*	*good rulings*

Actually, the conclusion about $\tau_g = 0$ applies for all angles between the reference surface Φ and the developable Ψ, not only in the special cases $\alpha = 0$ and $\alpha = 90°$.

3. Behaviour of developables aligned with reference surfaces

Manufacturing considerations. The mathematical considerations of the previous section have practical implications regarding manufacturing and design, especially design freedom. We discuss these issues in the following paragraphs. Obviously a developable strip is more easily manufactured if it can be flattened to a *straight* planar piece. This is because it will fit into a smaller rectangular sheet in its flattened state. We mention that examples below (Figures 8 and 9) are based on developables which will unfold not to straight planar strips, but to circular ones. The individual strip even develop to circular strips of the same radius. This property is of interest for manufacturing because it means that the unfolded state of strips has simple geometry.

The reason why *rulings* are called good or bad is that they can be seen as the infinitesimal axes of bending, when producing a developable surface from its flat state. If the rulings are tangential to the reference surface, we would have to bend longish sheets along the sheet instead of across. Obviously bending a strip is easier if the ifinitesimal axis of bending runs across that strip.

Examples which have been built. For a curved beam with rectangular cross-section which stays tangential/orthogonal to a reference surface Φ, optimal rulings are achieved if the beam follows the principal curves of Φ. This is exactly the case for the Eiffel Tower pavilions, see Figure 5.

Wooden panels which comprise a skin like the one shown by Figure 1 for the Burj Khalifa have straight development (simply because they were originally straight panels, before they were bent in order to fit the reference surface). They therefore follow geodesics of the reference surface. In the Burj Khalifa case the rulings on these panels are never bad, since the skin has no asymptotic directions, being of positive curvature. We would also like to point to previous work on geodesic timber constructions, see (Pirazzi and Weinand, 2006) and follow-up work.
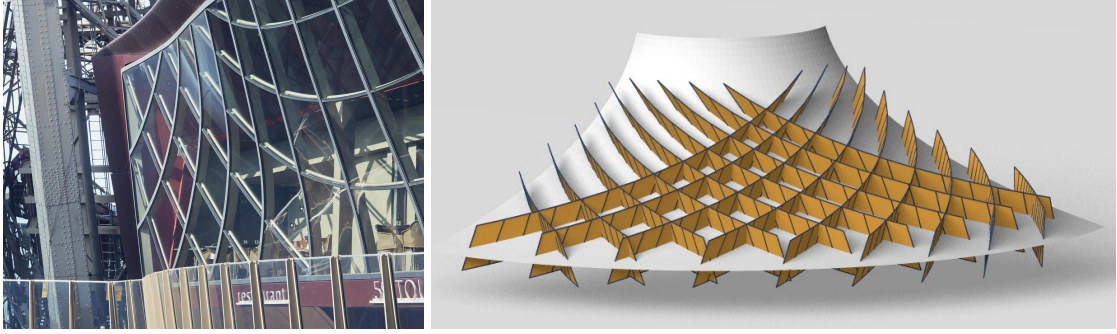
FIGURE 5. Strips with different kinds of optimality properties. *Left:* Eiffel tower pavilions (Moatti et Rivière architects, engineering by RFR). The top-down beams have a rectangular cross-section and are thus modeled as a union of four developable strips – two orthogonal to the glass surface Φ, two tangential to them. The guiding curves are principal for Φ, implying optimal rulings. Image courtesy RFR. *Right:* A minimal surface with two families of developable strips guided by curves with $\kappa_n = 0$, implying straight development. Rulings are not optimal, but far from bad. Further, transverse strips intersect not along rulings.

Mutual exclusivity of "good" properties. The beneficial properties of strips which are mentioned in the proposition unfortunately are mutually exclusive. For developables *orthogonal* to a reference surface Φ, optimal rulings are impossible if we are to have a straight development (principal curves are never asymptotic except in the special case of Φ being developable). Conversely, a straight development might imply bad rulings, if asymptotic curves happen to be geodesic (this could happen if Φ is ruled but not developable).

Developables tangential to Φ with optimal rulings rarely have straight development (only if Φ is one of Monge's *surfaces moulures*, principal curves are geodesics). Straight developments might lead to bad rulings if accidentally we choose a geodesic which is asymptotic (that can happen if Φ is ruled).

The reader is advised that the previous paragraphs heavily draw from knowledge of the manifold interesting properties of curves in surfaces which are discussed in older textbooks like (Blaschke, 1921).

The loss of design freedom. If one insists on optimal rulings (orthogonal to guiding curves) then the only possibility is that the guiding curves are principal, which are uniquely determined by the reference surface Φ. If Φ is already known, there is no design freedom left. This dilemma had to be solved for the Eiffel tower pavilions, see (Schiftner et al., 2012).

A similar dilemma occurs if we want to construct a family of developable strips orthogonal to the reference surface which have straight development. We are stuck with using the asymptotic curves which are uniquely determined by Φ.

The fact that both the "optimal rulings" and the "straight development" requirements determine the strip layout has another consequence besides the inconvenience of loss of design freedom: this layout may be unusable. While the principal network is always right-angled, the network of asymptotic curves has no such property. Only for very special surfaces like the one of Figure 5, right, it looks nice in the sense that the angle of intersection of different asymptotic curves is close to 90 degrees (the surface shown is a minimal surface, where the asymptotic curves are exactly orthogonal).

4. Geometric modeling with developables

Developability as a constraint on spline surfaces. Geometric modeling of developable surfaces has been a topic of interest for a long time. We refrain from giving a history of the extensive previous work in this area. The commonly used degree $m \times n$ Bézier surfaces and B-spline surfaces (see Figure 6) make it easy to produce ruled surfaces – simply let $n = 1$, in both the polynomial and the rational cases. The conditions on the control points of these surfaces which ensure developability are not difficult, see (Lang and Röschel, 1992), but the nonlinear nature of these constraints has prevented truly interactive modeling until recently. Similarly, approaches to modeling of developables via discretization and differential-geometric analysis were too slow for interactive modelling.

The constraints expressing developability enjoy mathematical properties that correspond directly to geometric design: The system has a high-dimensional solution manifold, implying design freedom. (Tang et al., 2016) showed how to solve these constraints quickly enough for interactive modeling. Following earlier work (Tang et al., 2014; Jiang et al., 2015), they modify constraints so that they are at
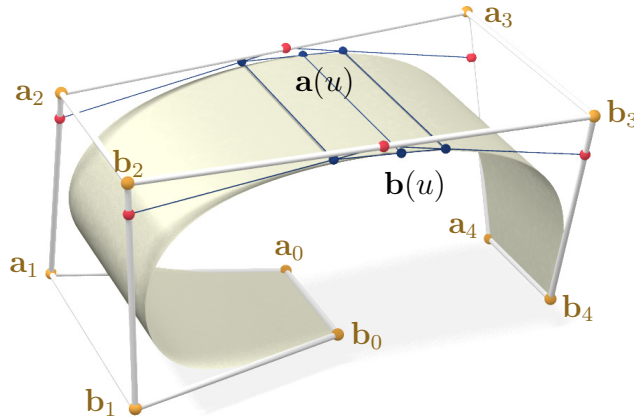

FIGURE 6. Developable strips as ruled spline surfaces which connect two B-spline curves. Here a B-spline curve $\mathbf{a}(u)$ is defined by its control points $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_N$. This image shows the evaluation of the curve for a certain parameter value $\mathbf{a}(u)$, and similar for a spline curve $\mathbf{b}(u)$, cf. (Tang et al., 2016).
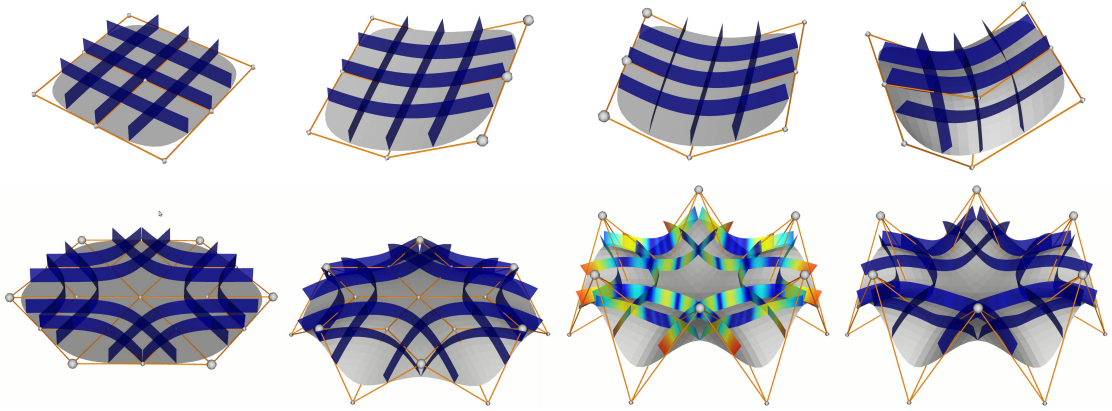
FIGURE 7. *Interactive modeling of curved support structures. Top row:* A configuration of strips follows guiding curves in a reference surface Φ. Starting from a very simple configuration, modeling is done by modifying the parametric representation of Φ and the guiding curve network connected to Φ. The strips follow their respective guiding curves, with their actual position in space being defined by the constraint solver. *Bottom row:* Such deformations destroy developability, as indicated by the color coding (blue to green indicates sufficient developability for manufacturing purposes). After each deformation applied by the user the constraint solver re-establishes developability within seconds.

most quadratic and still sparse; and they employ fairness energies as a regularizer for a Newton-type method in order to guide the user towards "sensible" parts of the solution manifold.

Setup of variables. We describe our computational setup which follows (Tang et al., 2016). A strip is modeled as a degree 3×1 cubic B-spline surface $\mathbf{b}(u, v)$ of C^2 smoothness, whose shape is determined by two rows $\mathbf{a}_0, \ldots, \mathbf{a}_N$ and $\mathbf{b}_0, \ldots \mathbf{b}_N$ of control points; each row being the control polygon of the upper and lower boundary $\mathbf{a}(u)$, $\mathbf{b}(u)$, see Figure 6. We always assume that the boundary curve $\mathbf{a}$ is following a guiding curve $\mathbf{c}$ which lies in the reference surface Φ. It does not matter if the actual strip which is to be used in applications has boundaries different from $\mathbf{a}, \mathbf{b}$, since developable strips may be freely extended and cropped to either side.

Setup of developability constraints. Developability is expressed by existence of a unit vector $\mathbf{n}_u$, for all parameter values u, which is orthogonal to $\mathbf{b} - \mathbf{a}$ and to the derivatives $\mathbf{a}'$, $\mathbf{b}'$. These conditions read $\mathbf{n} \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{n} \cdot \mathbf{a}' = \mathbf{n} \cdot \mathbf{b}' = 0$ and are required to hold only for a finite number of values $u_1, u_2, \ldots$ and corresponding normal vectors $\mathbf{n}_1, \mathbf{n}_2, \ldots$, because the equivalent condition $\det(\mathbf{a}', \mathbf{b}', \mathbf{b} - \mathbf{a}) = 0$ is piecewise-polynomial of degree not exceeding 6, cf. the analogous discussion by (Tang et al., 2016).
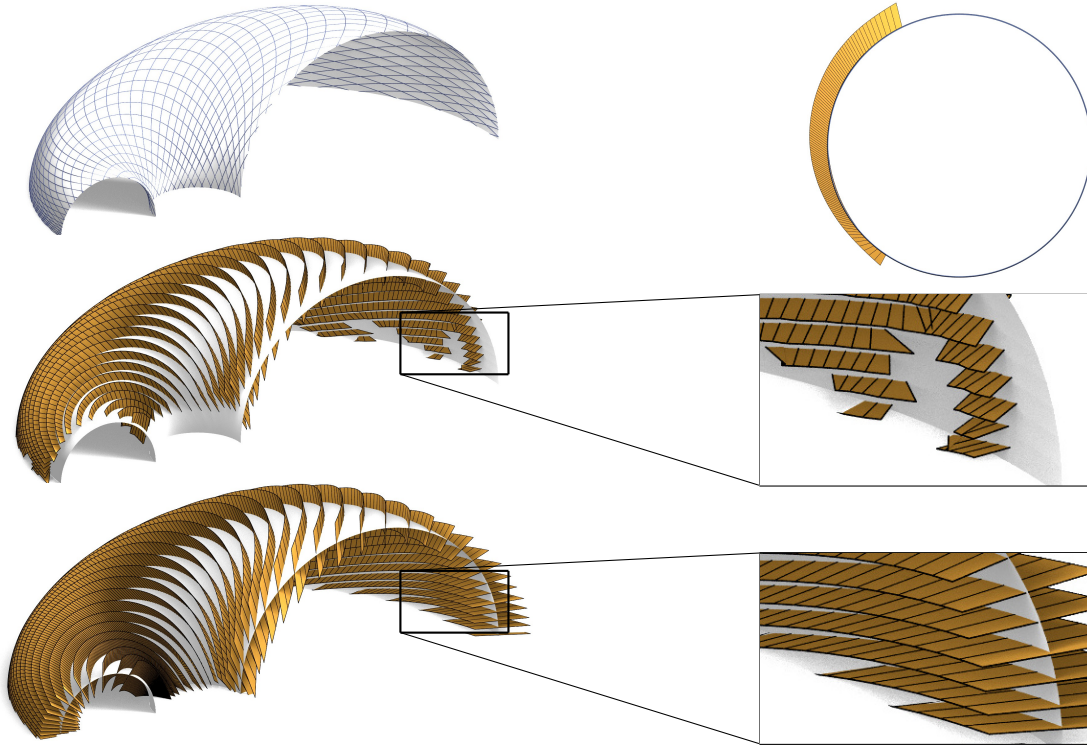
FIGURE 8. *Strips which follow guiding curves.* The top left image shows curves on a reference surface Φ with a constant nonzero value of κ_n. Developables guided by these curves (middle row) have circular development. Unfortunately the rulings of these developables are in several places rather bad (the strips are interrupted there). Our constraint solver finds a sequence of strips which, as far as they can, stay orthogonal to Φ and close to the guiding curves (bottom row). The setup of surfaces in this procedure automatically ensures good behaviour of rulings, but entails changes in the geometry. Nevertheless the development of a sample strip is still rather circular (top, right).

An arrangement of developable strips is defined by additional constraints like common intersection of strips (this corresponds to linear equations involving control points), and smooth transition of strips (more linear equations involving control points). For example, the six developable strips in Figure 7 (top row) which appear to intersect in 9 rulings are actually 24 individual strips with common boundary rulings which join smoothly.

Constraint solving. (Tang et al., 2016) show how to solve the system of constraints quickly, by linearizing the constraints and solving the resulting linear system (which at the same time is under-determined and has redundant equations) via regularization. The regularizer is a fairness energy, thus pushing the solver towards "sensible" solutions of the system. We extended their interactive modeling system for developable skins to the case of non-skin strip arrangements.
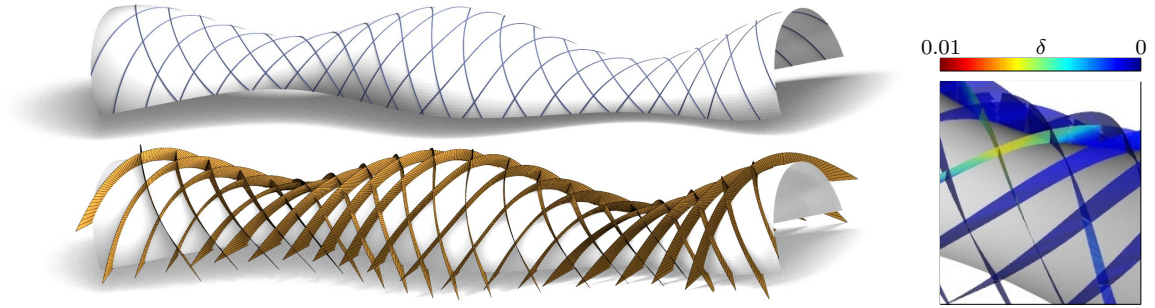
FIGURE 9. *Strips which follow guiding curves.* The top left image shows curves on a reference surface Φ which enjoy a constant nonzero value of κ_n, similar to Figure 8. Constraint solving produces developable strips which, as far as they can, follow these guiding curves in addition to being orthogonal to Φ. The detail at right illustrates the degree of developability of the strips which are achieved in this way. It is measured via a quad mesh produced by regular sampling of strips. For each face of this quad mesh, we compute the value $\delta = \frac{distance\ of\ diagonals}{average\ of\ 2\ short\ edgelengths}$. This example is sufficiently developable for manufacturing.

Positioning constraints. Besides developability, further constraints can be imposed on an arrangement of strips. It is important that these constraints are linear or quadratic – otherwise the method of (Tang et al., 2016) becomes slow.

An example of such a constraint is that a strip Ψ encloses a certain angle θ with the reference surface Φ. If the normal vectors $\mathbf{n}_i$ and $\mathbf{n}_i^*$ of Ψ resp. Φ in selected points are available, then we require $\mathbf{n}_i \cdot \mathbf{n}_i^* = \cos\theta$. This condition is neither linear nor quadratic if the dependence of $\mathbf{n}_i^*$ on he control points is nonlinear, but we can use a standard trick to make it linear: We simply consider $\mathbf{n}_i^*$ fixed during each pass of the iterative solver.

Another constraint is that a strip boundary $\mathbf{a}$ follows a guiding curve $\mathbf{c}$. We consider a sample $\mathbf{a}(u_i)$ of boundary points. The condition of closeness to $\mathbf{c}$ is highly nonlinear. Also here a well known trick can be applied. By computing the closest point $\mathbf{c}_i^*$ on the guiding curve and the tangent vector $\mathbf{t}_i^*$ there, we require $(\mathbf{a}(u_i) - \mathbf{c}_i^*) \times \mathbf{t}_i^* = 0$. This equation expresses the requirement that the point $\mathbf{a}_i^*$ lies on the tangent of the guiding curve. It becomes linear, if $\mathbf{c}_i^*, \mathbf{t}_i^*$ are recomputed before each pass of the iterative solver and are kept constant.

5. RESULTS AND DISCUSSION

Applications. We discuss two main applications of the constraint solving procedure: One is the establishment of developables which follow a pre-selected curve network on a reference surface. We show two examples, namely Figures 8 plus 10, and Figure 9. Using the formulae of Section 2, it is not difficult to compute rulings of developables which however are not everywhere nicely transverse to Φ. Using this data as input for the constraint solver described in Section 4 yields a
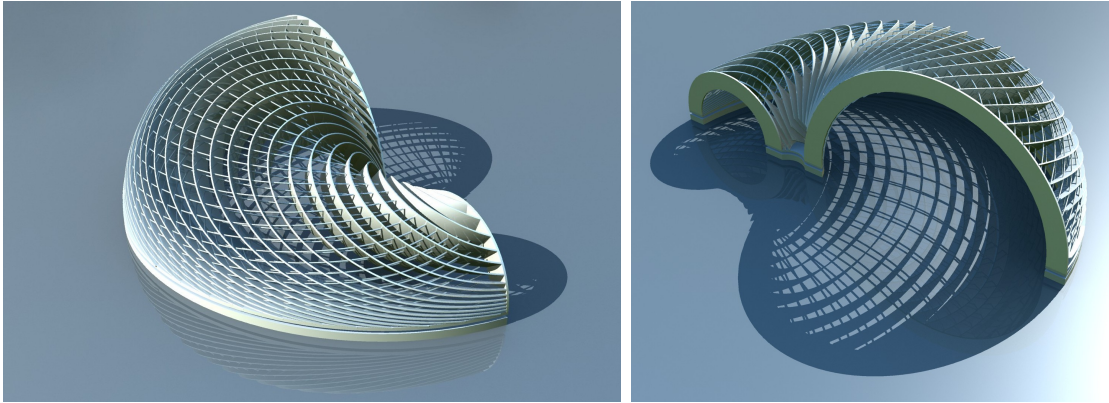
FIGURE 10. The strip sequence of Figure 8 is the basis of this system of curved beams. The individual strips, being developable with "good" rulings, can be manufactured from flat pieces by bending. As an additional geometric property, each beam unfolds into a circular strip of the same radius. The members transverse to the beams follow the system of curves shown in Figure 8 (even if there is no particular reason to do so).

sequence of developable strips which approximates the original setup and (from construction) has nice rulings throughout.

The other main application is interactive modeling, made possible by *quickly* solving developability constraints. Figures 7 illustrates how it works.

Limitations. The limitations which have been explained in Section 2 and Section 3 apply generally, especially the paragraph on mutual exclusivity of desirable properties. Since our procedure of computing developables always produces surfaces with "good" rulings, it is not possible to faithfully approximate developables which might have certain geometric properties, but bad rulings. The result of the computations either is not fully developable, or does not entirely have the desired properties. Numerical solvers usually achieve a compromise between competing constraints. It is therefore advisable to check after computation if some properties have been lost. In fact our implementation in its current state provides real-time feedback to the user, e.g. by color coding the surfaces according to developability, see Figure 7. The user is able to decide on the importance of individual constraints by tuning weights which govern the constraint solving.

Examples of these limitations are shown by Figure 9, where we almost lose developability, and by Figure 8 where a circular development is not achieved exactly but only approximately. Since in the real world mathematical equalities are true only up to tolerances, such imperfections often are no obstacle.

6. CONCLUSION

We have presented an overview of the use of developables in freeform architecture – both for freeform skins and other kinds of arrangements of developable

strips. After a differential-geometric discussion we showed how the computational framework of (Tang et al., 2016) can be extended and subsequently applied to strip arrangements – both for modeling and for computing arrangements defined by guiding curves. It is the purpose of this paper to further the understanding of the complex geometry of developable surfaces and to show the currently available possibilities of geometric modeling.

Acknowledgments

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