Conchoid surfaces of rational ruled surfaces

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Abstract

The conchoid surface G of a given surface F with respect to a point O is roughly speaking the surface obtained by increasing the radius function of F with respect to O by a constant d. This paper studies real rational ruled surfaces in this context and proves that their conchoid surfaces possess real rational parameterizations, independently on the position of O. Thus any rational ruled surface F admits a rational radius function $r(u, v)$ with respect to any point in space. Besides the general skew ruled surfaces and examples of low algebraic degree we study ruled surfaces generated by rational motions.

Keywords: rational ruled surface, rational conchoid surface, polar representation, rational radius function, pencil of conics.

1. Introduction

The conchoid is a classical geometric construction and dates back already to the ancient Greeks. Given a planar curve C, a fixed point O and a constant distance d, the conchoid D of C with respect to O at distance d is the set of points Q in the line OP at distance d of a point P varying at the curve C,

$$D = \{Q \in OP \text{ with } P \in C, \text{ and } \overline{QP} = d\}^*, \tag{1}$$

where the asterisk denotes the Zariski closure. For a formal definition of the conchoid in terms of diagrams of incidence we refer to [11, 12]. The definition of the conchoid surface to a given surface F in space with respect to a given point O and distance d follows analogous lines. Note that the definition of the conchoid (either for curves or for surfaces) by means of diagrams of incidence, in combination with the Closure Theorem (see [2] p. 122), imply that elimination theory techniques, such as Gröbner bases, provide the equations of the conchoid.

1.1. Polar representation of conchoids

We briefly turn to the curve case to discuss some important properties. For an analytic representation it is appropriate to choose $O = (0, 0)$. Using a representation of a curve C in terms of polar coordinates $\mathbf{c}(t) = r(t)(\cos t, \sin t)$, its conchoid curve D with respect to O and distance d is obtained by $\mathbf{d}(t) = (r(t) \pm d)(\cos t, \sin t)$. More generally we can consider any parameterization $\mathbf{k}(t)$ of the unit circle S^1. The curve C and its conchoid curves D are represented by

$$\mathbf{c}(t) = r(t)\mathbf{k}(t) \text{ and } \mathbf{d}(t) = (r(t) \pm d)\mathbf{k}(t), \text{ with } \|\mathbf{k}\| = 1. \tag{2}$$

Surfaces and their conchoids are analogously represented. With respect to a given point $O = (0,0,0) \in \mathbb{R}^3$ and a parameterization $\mathbf{k}(u,v)$ of the unit sphere S^2, a surface F is parameterized by a polar representation $\mathbf{f}(u,v) = r(u,v)\mathbf{k}(u,v)$. The conchoid surface G of F at distance d is consequently represented by

$$\mathbf{g}(u,v) = (r(u,v) \pm d)\mathbf{k}(u,v). \tag{3}$$

We consider trigonometric parameterizations $\mathbf{k} = (\cos u \cos v, \sin u \cos v, \sin v)$ of S^2 or rational ones, for instance $\mathbf{k} = (2ac, 2bc, a^2+b^2-c^2)/(a^2+b^2+c^2)$. This rational parameterization originates from a stereographic projection with center $(0,0,-1)$ which maps points $(a/c, b/c, 0)$ of the plane $z = 0$ to points $\mathbf{k}$ in S^2. Here, a, b and c can be considered as polynomials, for instance $a = u$, $b = v$ and $c = 1$. More generally the unit sphere S^2 admits rational parameterizations $\mathbf{k} = (A/D, B/D, C/D)$, with

$$A = 2(ac+bd),\ B = 2(bc-ad),\ C = a^2+b^2-c^2-d^2,\ D = a^2+b^2+c^2+d^2,$$

where a, b, c and d are assumed to be polynomials in u and v.

Conchoids of curves can be found in several monographs on curves and surfaces, see for instance [4]. The classical conchoid of Nicomedes, that is the conchoid of a line, appeared already 200 B.C. The conchoid of a circle with respect to one of its points is also a well known curve, called Limacon of Pascal. A recent careful investigation of algebraic properties of conchoid curves and conditions for their rationality, as well as direct parameterization algorithms, can be found in [11, 12].

The conchoid of a curve can be considered as special case of the cissoid of two curves. Given two curves A and B and a fixed point O, and let P and Q be two points in A and B, respectively, which are collinear with O. The cissoid of A and B with respect to O is the set of points X in the lines OP for $P \in A$ which satisfy the relation $\overline{OX} = \overline{OP} - \overline{OQ}$. By specializing B as circle of radius d centered at O, we arrive at the definition of the conchoid of A with respect to O. Some authors use the sum instead of the difference in the cissoid's definition. By reflecting B at O these definitions are equivalent. A recent publication dealing with this topic is [1].

Let a surface F in $\mathbb{R}^3$ be represented in polar coordinates by $\mathbf{f}(u,v) = r(u,v)\mathbf{k}(u,v)$, with $\|\mathbf{k}\| = 1$. We note that the conchoid surfaces are always computed with respect to the origin $O = (0,0,0)$ as focus point. The construction itself is invariant with respect to rotations and scaling with center O. For general rational surfaces the rationality of the conchoids is typically dependent on the position of the focus point O with respect to the surface. For ruled surfaces we prove that their conchoids are rational independent on the position of the focus point.

Typically we use Cartesian coordinates $\mathbf{x} = (x,y,z)$ to represent points in $\mathbb{R}^3$. The dot product of $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\mathbf{x} \cdot \mathbf{y}$ and for the squared norm $\|\mathbf{x}\|^2$ we also use $\mathbf{x}^2$. For representing points in projective space $\mathbb{P}^3 \supset \mathbb{R}^3$ we use homogeneous coordinates $(x_0, x_1, x_2, x_3)\mathbb{R}$, which are determined only up to a common factor. Assuming $x_0 = 0$ to be the ideal plane, the conversion between Cartesian and homogeneous coordinates is $x = x_1/x_0$, $y = x_2/x_0$, and $z = x_3/x_0$.

1.2. Contribution

The main result being proved is that any rational ruled surface F admits a rational polar representation $\mathbf{f}(u,v) = r(u,v)\mathbf{k}(u,v)$ with a rational radius function $r(u,v) = \|\mathbf{f}(u,v)\|$ and a particular rational parameterization $\mathbf{k}(u,v)$ of the unit sphere S^2. This implies that the conchoid surface G of F with respect to any point O in $\mathbb{R}^3$ and distance d admits a rational parameterization.

The construction involves finding a suitable rational parameterization $\mathbf{k}(u,v)$ of S^2 for a given rational ruled surface F together with the determination of the rational radius function $r(u,v)$. In the general case this amounts in parameterizing a rational one-parameter family of conics in $\mathbb{P}^2$, see

Section 2. The construction of surfaces with rational conchoid surfaces is related to rational offset surfaces. The dual approach to these surfaces uses a rational parameterization of the unit sphere together with a rational function determining the distance of tangent planes from the origin, see [7]. A Laguerre geometric approach to rational offset surfaces is discussed in [3, 5].

We investigate the general conchoid construction of rational ruled surfaces as well as several particular cases where the parameterization problem turns out to be simpler. Ruled surfaces generated by rational motions of a line are discussed in Section 2.2 which includes rational cylinders and rotational ruled surfaces. Geometric considerations and arguments go along with this discussion. Further we give some examples in Section 3 to illustrate the results.

1.3. The conchoid of a line in $\mathbb{R}^2$

To introduce to the subject of conchoid curves and surfaces, we deal at first with two simple examples, the conchoid of a line in $\mathbb{R}^2$ and the conchoid of a plane in $\mathbb{R}^3$, see Figure 1. The first example gives the conchoid of Nicomedes and has even been studied by the ancient Greeks. We introduce to basic techniques to construct parameterizations of conchoids.

Since the conchoid construction is invariant with respect to rotations and central similarities, we assume that the line C whose conchoid we want to construct, is given by $y = 1$. Let $\mathbf{k}(u) = (\cos u, \sin u)$ be a parameterization of the unit circle, then C is obtained by

$$\mathbf{c}(u) = \frac{1}{\sin u}(\cos u, \sin u) = \left(\frac{\cos u}{\sin u}, 1\right), \text{ with } r = \frac{1}{\sin u}. \tag{4}$$

Increasing $r(u)$ by a constant d leads to a trigonometric parameterization of the conchoid D of C,

$$\mathbf{d}(u) = \frac{1 + d\sin u}{\sin u}(\cos u, \sin u). \tag{5}$$

Substituting trigonometric functions by rational functions converts (4) and (5) into rational parameterizations of C and D. By eliminating the parameter u we find that D is an algebraic curve of order four,

$$D : y^2(x^2 + y^2) - 2y(x^2 + y^2) + x^2 + y^2(1 - d^2) = 0.$$

Since this equation does not contain constant and linear terms in x and y, the origin $O = (0, 0)$ is a double point of D. Moreover the leading term is $y^2(x^2 + y^2)$, thus D is circular which means that D passes through the ideal points $(0, 1, \mathrm{i})\mathbb{R}$ and $(0, 1, -\mathrm{i})\mathbb{R}$, besides the ideal double point at $(0, 1, 0)\mathbb{R}$.

1.4. The conchoid of a plane $\mathbb{R}^3$

Analogously to the conchoid curve of a line in $\mathbb{R}^2$ we construct the conchoid surface of a plane F in $\mathbb{R}^3$. Applying a rotation and a scaling with center $O = (0, 0, 0)$, we assume $F : z = 1$. Considering the parameterization $\mathbf{k}(u, v) = (\cos u \cos v, \sin u \cos v, \sin v)$ of the unit sphere, the plane $F : z = 1$ is represented by the radius function $r(u, v) = 1/\sin v$. Thus the conchoid surface G of the plane F admits the trigonometric parameterization

$$\mathbf{g}(u, v) = (r(u, v) + d)\mathbf{k}(u, v) = \frac{1 + d\sin v}{\sin v}(\cos u \cos v, \sin u \cos v, \sin v), \tag{6}$$

and an implicit equation of G reads

$$G : z^2(x^2 + y^2 + z^2) - 2z(x^2 + y^2 + z^2) + x^2 + y^2 + z^2(1 - d^2) = 0.$$

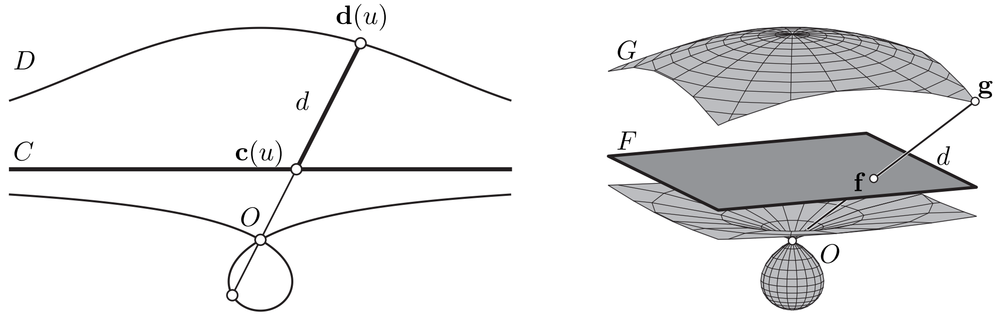


Figure 1: Left: Conchoid of a line in $\mathbb{R}^2$. Right: Conchoid of a plane in $\mathbb{R}^3$.

The intersection of G with the ideal plane is $z^2(x^2 + y^2 + z^2) = 0$ and consists of the ideal conic $x^2 + y^2 + z^2 = 0$ and the doubly counted line $z^2 = 0$. The conchoid G is a surface of rotation of degree four with z as axis. The horizontal intersections are circles with centers on z and G can be generated by rotating the conchoid of a line around z.

Considering a plane F in general position with respect to the coordinate system, the unknown radius function of F can also be obtained by inserting the parameterization $r(u,v)\mathbf{k}(u,v)$ into the linear equation $F : \mathbf{a}\cdot\mathbf{x}+a_0 = 0$, with $a_0 \neq 0$ and $\mathbf{a} = (a_1, a_2, a_3)$ as normal vector of F. This results in $r(u,v) = -a_0/(\mathbf{a}\cdot\mathbf{k})$. Planes passing through the origin ($a_0 = 0$) have to be excluded here, since these planes are not in bijective correspondence to the bundle of lines with vertex $O = (0,0,0)$.

Considering a quadric surface F passing through the origin O, the presented method to determine $r(u,v)$ works too. Let F be given by the implicit equation $F : \mathbf{x}\cdot A\cdot\mathbf{x}+\mathbf{b}\cdot\mathbf{x} = 0$. Inserting a rational polar representation $r(u,v)\mathbf{k}(u,v)$ with an unknown function $r(u,v)$ into F gives a rational solution $r(u,v) = -(\mathbf{b}\cdot\mathbf{k})/(\mathbf{k}\cdot A\cdot\mathbf{k})$ besides the trivial solution $r = 0$. This argumentation can be extended to monomial algebraic surfaces of degree n with an $(n-1)$-fold point at O and it is proved easily that their conchoid surfaces are rational. Note that in this whole paragraph the focus is not taken in generic position but as a point on the quadric in the first case and as the singularity in the second case. Changing the position of the focus (e.g. out of the quadric) results in a different behavior.

We summarize the presented parameterization technique: to construct a parameterization of the conchoid surface G of a given surface F with respect to the origin O and a specified distance d, we have to determine a polar representation $r(u,v)\mathbf{k}(u,v)$, where $\mathbf{k}(u,v)$ is a suitable parameterization of S^2. If both $r(u,v)$ and $\mathbf{k}(u,v)$ are rational functions, the surface F has a rational polar representation and its conchoid surfaces G are rational. The following sections contain a detailed discussion for rational ruled surfaces F and their conchoids.

2. The conchoid surfaces of rational ruled surfaces

A ruled surface F carries a one-parameter family of straight lines, thus admits a parametric representation $\mathbf{f}(u,v) = \mathbf{c}(u) + v\,\mathbf{e}(u)$, where $\mathbf{c}(u)$ is called *directrix curve* and $\mathbf{e}(u)$ is a direction vector field of F's generating lines. The directrix curve $\mathbf{c}$ on F can be replaced with any other curve $\mathbf{h}(u) = \mathbf{c}(u) + v(u)\mathbf{e}(u)$ different from F's generating lines.

Let $\mathbf{f}_u = \dot{\mathbf{c}} + v\dot{\mathbf{e}}$ and $\mathbf{f}_v = \mathbf{e}$ be the partial derivatives of $\mathbf{f}$. The normal vector

$$\mathbf{n}(u,v) = \dot{\mathbf{c}}(u) \times \mathbf{e}(u) + v\dot{\mathbf{e}}(u) \times \mathbf{e}(u) = \mathbf{n}_1(u) + v\mathbf{n}_2(u)$$

is a linear combination of the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Given a fixed generating line L of F, the normal vectors along L may be linearly independent or not. In the previous case L is called *skew generating line*, and in case of linear dependece L is called *torsal generating line* of F. A ruled surface carrying only a finite number of torsal generators is called *skew ruled surface.* If all generating lines of F are torsal, F is a *developable ruled surface.* The latter family consists of cylinders, cones and tangent surfaces of space curves or combinations of those.

The construction of a conchoid surface of a parameterized ruled surface F relies on the polar representation $\mathbf{f}(u,v) = r(u,v)\mathbf{k}(u,v)$. We consider the mapping

$$\sigma : \mathbf{f}(u,v) \to \mathbf{k}(u,v) = \sigma(\mathbf{f}(u,v)), \tag{7}$$

and denote $\sigma(\mathbf{f}(u,v))$ as *spherical part* of F and $r(u,v)$ as *radius function.* We may write $\sigma(F)$ instead of $\sigma(\mathbf{f}(u,v))$.

The spherical part $\sigma(F)$ of a ruled surface F (see Fig.2, right) consists of a one parameter family of great circles in S^2 being the v-lines of $\mathbf{k}(u,v)$. Considering the parameterization $\mathbf{f} = \mathbf{c} + v\mathbf{e}$ of F, the circles are the intersections of S^2 with planes $E : \mathbf{x} \cdot (\mathbf{c} \times \mathbf{e}) = 0$. Typically the great circles $S^2 \cap E$ envelope some curve which needs not to be real or may degenerate to single points.

2.1. General construction

Let a rational ruled surface F be given by $\mathbf{f}(u,v) = \mathbf{c}(u) + v\mathbf{e}(u)$, with rational directrix $\mathbf{c}(u)$ and rational direction vectors $\mathbf{e}(u)$. Typically it is difficult to define F as zero set of an implicit function. In order to find a rational polar representation $r(u,v)\mathbf{k}(u,v)$ of F, we investigate the squared length

$$\|\mathbf{f}(u,v)\|^2 = \|\mathbf{c}(u)\|^2 + 2v\mathbf{c}(u) \cdot \mathbf{e}(u) + v^2\|\mathbf{e}(u)\|^2. \tag{8}$$

We prove that there exists a reparameterization of F such that $\|\mathbf{f}(u,v)\|$ is a rational function. At first a curve $\mathbf{h}(u) = \mathbf{c}(u) + v(u)\mathbf{e}(u)$ of F with rational radius function $\|\mathbf{h}(u)\|$ is constructed. In a second step we show how to extend this result to a rational polar representation of F. The constructive proof that any rational ruled surface F carries a curve $\mathbf{h}(u)$ with rational radius function $\|\mathbf{h}(u)\|$ is mainly based on the following lemma.

Lemma 1. *Let*

$$a(u) : (x_0, x_1, x_2) \cdot A(u) \cdot (x_0, x_1, x_2)^T = 0, \tag{9}$$

be a one parameter family of conics in $\mathbb{P}^2(\mathbb{R})$, with a symmetric matrix $A(u)$ in $\mathbb{R}^{3\times 3}$ with real rational entries ($a_{ij}(u) \in \mathbb{R}(u)$), and $(x_0, x_1, x_2)\mathbb{R}$ are homogeneous coordinates in $\mathbb{P}^2(\mathbb{R})$. If for all but finitely many $u \in \mathbb{R}$ the quadratic curve $a(u)$ contains more than one real point, then there exists real rational functions $y_0(u), y_1(u)$ and $y_2(u)$ which satisfy (9) *identically.*

Proof: We give a short outline. The complete proof can be found in [6, 8] and further applications of this property to rational surfaces with families of conics can be found in [5, 9, 10]. One way to construct a rational curve $\mathbf{y}(u)\mathbb{R} = (y_0, y_1, y_2)(u)\mathbb{R}$ which satisfies (9) identically, is to apply a coordinate transformation such that the conics $a(u)$ are represented in diagonal form

$$a(u) : L(u)x_0^2 + M(u)x_1^2 + N(u)x_2^2 = 0. \tag{10}$$

We assume that the coefficients L, M and N are polynomials without common zeros. Consider a zero $u^\star$ of L. Equation (10) factorizes into two real or conjugate complex linear equations in x_1 and x_2, depending on the signs of $M(u^\star)$ and $N(u^\star)$. Performing these factorizations for all zeros

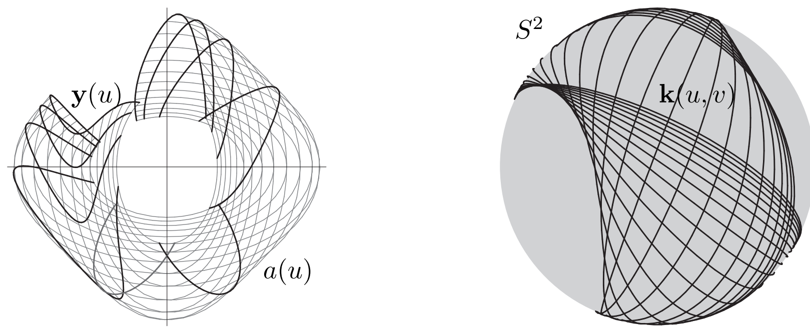


Figure 2: Left: The family of conics $a(u)$ and different solutions $\mathbf{y}(u)$. Right: Spherical part $\mathbf{k}(u,v)$ of a ruled surface.

of the polynomials L, M and N, we obtain a system of necessary linear equations for a solution of (10). Typically we have to take into account also one quadratic equation to determine a real curve $\mathbf{y}(u) \subset \mathbb{P}^2(\mathbb{R})$ satisfying (10) identically. $\square$

What does this mean for the equation (8), the squared norm of a ruled surface parameterization? Performing the substitutions $v = x_2/x_1$ and $\|\mathbf{f}\| = x_0/x_1$ in equation (8) results in a one-parameter family of conics

$$a(u) : x_1^2\mathbf{c}(u)^2 + 2x_1x_2\mathbf{c}(u)\cdot\mathbf{e}(u) + x_2^2\mathbf{e}(u)^2 - x_0^2 = 0, \tag{11}$$

with rational coefficients. According to the Cauchy-Schwarz inequality $\mathbf{c}^2\mathbf{e}^2 - (\mathbf{c}\cdot\mathbf{e})^2 \geq 0$, the quadratic curves $a(u)$ are real for all but finitely many $u \in \mathbb{R}$. According to Lemma 1 there exists a rational curve $\mathbf{y}(u)\mathbb{R} = (y_0, y_1, y_2)(u)\mathbb{R}$ with $\mathbf{y}(u) \in a(u)$. For an illustration see Fig.2, left. This proves

Lemma 2. *Let $\mathbf{f}(u,v) = \mathbf{c}(u) + v\mathbf{e}(u)$ be a rational parameterization of a rational ruled surface F in $\mathbb{R}^3$. Then there exists a rational solution $\mathbf{y}(u) = (y_0, y_1, y_2)(u)$ of (11) and consequently a rational curve $\mathbf{h}(u) = \mathbf{f}(u, v(u))$ in F with rational distance $\|\mathbf{h}(u)\|$ from the origin $O = (0,0,0)$, with*

$$\mathbf{h}(u) = \mathbf{c}(u) + v(u)\mathbf{e}(u), \text{ with } v(u) = \frac{y_2(u)}{y_1(u)}, \text{ and } \|\mathbf{h}(u)\| = \frac{y_0(u)}{y_1(u)}. \tag{12}$$

Applying a stereographic projection, the family of conics $a(u)$ from (11) admits a rational parameterization $\mathbf{z}(u,t)$ with the property that the t-lines of $\mathbf{z}(u,t)$ are the conics $a(u)$. Substituting $v = z_2(u,t)/z_1(u,t)$ in $\mathbf{f}(u,v) = \mathbf{c}(u) + v\mathbf{e}(u)$ results in a rational parameterization of F

$$\mathbf{f}(u,t) = \mathbf{c}(u) + \frac{z_2(u,t)}{z_1(u,t)}\,\mathbf{e}(u), \text{ with } \|\mathbf{f}(u,t)\| = \frac{z_0(u,t)}{z_1(u,t)} = r(u,t). \tag{13}$$

Rational polar representations of F and its conchoids G with respect to O and distance d read

$$\begin{aligned} \mathbf{f}(u,t) &= r(u,t)\mathbf{k}(u,t) = \frac{1}{z_1}(z_1\mathbf{c} + z_2\mathbf{e}), \text{ with } r = \frac{z_0}{z_1}, \text{ and } \mathbf{k} = \frac{1}{z_0}(z_1\mathbf{c} + z_2\mathbf{e}), \\ \mathbf{g}(u,v) &= (r(u,t)+d)\mathbf{k}(u,t) = \frac{z_0 + dz_1}{z_0z_1}\left(z_1\mathbf{c} + z_2\mathbf{e}\right). \end{aligned} \tag{14}$$

Theorem 3. *Let F be a rational ruled surface in $\mathbb{R}^3$. Then there exists a rational polar representation $\mathbf{f}(u,v) = r(u,v)\mathbf{k}(u,v)$ in terms of a rational distance function $r(u,v)$ and a rational parameterization $\mathbf{k}(u,v)$ with $\|\mathbf{k}\| = 1$. The straight lines on F are represented by the v-lines of $\mathbf{f}(u,v)$. Consequently, the conchoid surfaces G of F admit rational parameterizations* (14).

Distinguished directrix curve. To represent the family of conics (11) by a diagonal matrix $A(u)$ means to represent the ruled surface $\mathbf{f}(u,v) = \mathbf{p}(u) + v\mathbf{e}(u)$ by the foot-point curve $\mathbf{p}(u)$ of F with respect to $O = (0,0,0)$. Starting from a general directrix curve $\mathbf{c}(u)$, the foot-point curve $\mathbf{p}(u) = \mathbf{c}(u) + v(u)\mathbf{e}(u)$ is determined by the condition $\mathbf{p}\cdot\mathbf{e} = 0$. This leads to $v(u) = -(\mathbf{c}\cdot\mathbf{e})/\mathbf{e}^2$, and we obtain

$$\mathbf{p}(u) = \mathbf{c}(u) - \frac{\mathbf{c}(u)\cdot\mathbf{e}(u)}{\|\mathbf{e}(u)\|^2}\mathbf{e}(u). \tag{15}$$

According to this distinguished directrix curve $\mathbf{p}(u)$, the equation (11) in $\mathbb{P}^2$ simplifies to

$$a(u) : \mathbf{p}(u)^2 x_1^2 + \mathbf{e}(u)^2 x_2^2 - x_0^2 = 0. \tag{16}$$

Introducing affine coordinates $x = x_1/x_0$ and $y = x_2/x_0$ we obtain a family of real ellipses centered at $(0,0)$ and common axes x and y. The lengths of the major and minor axes are the reciprocal values of $\|\mathbf{p}(u)\|$ and $\|\mathbf{e}(u)\|$, which are in general not rational.

Typically the construction of the parameterization $\mathbf{z}(u,t)$ of the family of conics $a(u)$ in equation (16) requires some computational effort. There are a couple of geometrically distinguished cases where this computation simplifies or is elementary. In particular, if the squared norm of $\mathbf{e}(u)$ or $\mathbf{p}(u)$ is constant, the family of conics (16) passes through two fixed points.

2.2. The conchoid surfaces of ruled surfaces whose direction vectors have rational length

Let F be a rational ruled surface with foot-point curve $\mathbf{p}(u)$ with respect to the origin and directions vectors $\mathbf{e}(u)$ with rational length. Without loss of generality we may assume $\|\mathbf{e}(u)\| = 1$. Consequently the squared norm of the parameterization $\mathbf{f}(u,v) = \mathbf{p}(u) + v\mathbf{e}(u)$ simplifies to $\|\mathbf{f}\|^2 = \|\mathbf{p}\|^2 + v^2$. Using the substitutions $v = x_2/x_1$ and $\|\mathbf{f}\| = x_0/x_1$, the corresponding family of conics (16) reads

$$a(u) : \mathbf{p}(u)^2 x_1^2 + x_2^2 = x_0^2. \tag{17}$$

Obviously these conics share the common points $(1,0,1)\mathbb{R}$ and $(1,0,-1)\mathbb{R}$. Applying a stereographic projection to this family of conics $a(u)$ leads to a rational parameterization of (17),

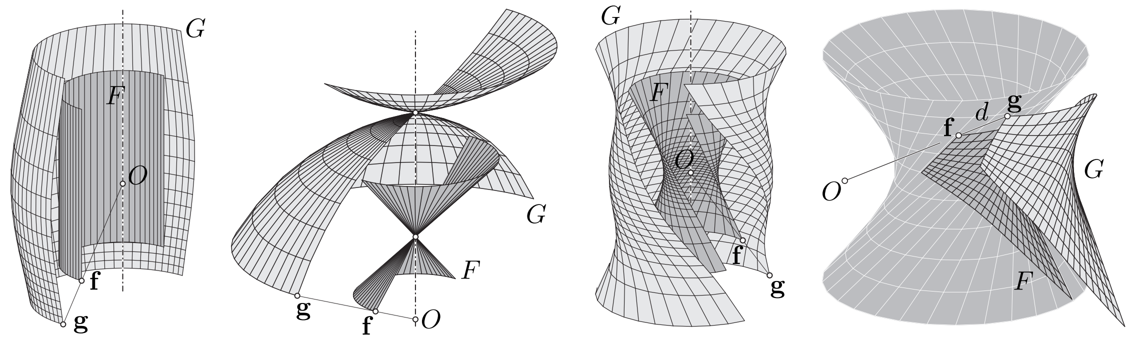
$$\mathbf{z}(u,t) = (1 + \mathbf{p}(u)^2 t^2, 2t, 1 - \mathbf{p}(u)^2 t^2).$$

Consequently the ruled surface F admits the rational parameterization

$$\mathbf{f}(u,t) = \mathbf{p}(u) + \frac{z_2(u,t)}{z_1(u,t)}\mathbf{e}, \text{ with } \|\mathbf{f}(u,t)\| = \frac{z_0(u,t)}{z_1(u,t)}. \tag{18}$$

Using (14) one obtains a rational polar representation of F and its conchoid surfaces G. In the remainder of this section we discuss some examples.

- Let F be a rational cylinder. We may assume that F's direction vector is $\mathbf{e} = (0,0,1)$. The cross section curve $\mathbf{p}(u) = (p_1, p_2, 0)(u)$ is the foot-point curve of F with respect to O.



(a) Rotational cylinder (b) Rotational cone (c) Rotational hyperboloid (d) Rotational hyperboloid

Figure 3: Rotational ruled surfaces and their conchoids.

- Let F be a rotational ruled surface. We may assume that F's rotational axis is parallel to the z-axis of the coordinate system, but different from it, and that its direction vector is $\mathbf{e}(u) = (\cos\alpha\cos u, \cos\alpha\sin u, \sin\alpha)$, with $\alpha = \text{const}$. Besides the trivial cases $\alpha = 0$ where F is a plane and $\alpha = \pi/2$ where F is a rotational cylinder, F is a one sheet hyperboloid. An example is illustrated in Fig. 3(d).

- More general examples are obtained by applying a rational motion to a line. The direction vector $\mathbf{e}(u)$ defines a rational curve in the unit sphere. A well known example, the Plücker conoid, is discussed in Sect 3.1.

- Other special cases of rational ruled surfaces F occur if the norms $\|\mathbf{p}(u)\|$ and $\|\mathbf{e}(u)\|$ in equation (16) are both rational. We may assume $\|\mathbf{e}\| = 1$ and we denote $\|\mathbf{p}(u)\| = \alpha(u)$. Thus there exists a spherical rational curve $\mathbf{a}(u)$ with $\|\mathbf{a}(u)\| = 1$ and $\mathbf{p}(u) = \alpha(u)\mathbf{a}(u)$. The spherical part $\sigma(F)$ consists of great circles being contained in planes spanned by the *rational orthogonal unit vectors* $\mathbf{a}(u)$ and $\mathbf{e}(u)$ and $\sigma(F)$ admits the parameterization

  $$\mathbf{k}(u,t) = \mathbf{a}(u)\cos t + \mathbf{e}(u)\sin t.$$

  To determine the radius function $r(u,t)$ of $\mathbf{f}(u,t) = r(u,t)\mathbf{k}(u,t)$, the parameterization $\mathbf{z}(u,t) = (\alpha(u), \cos t, \alpha(u)\sin t)$ of the conics $a(u)$ from (16) leads to $r(u,t) = z_0/z_1$.

  This case occurs when computing conchoid surfaces G of rotational ruled surfaces F with respect to a point O on the rotational axis. Examples are illustrated in Fig.3(a), 3(b), and 3(c). It is evident that the conchoid surface G of F is a rotational surface. The generating curve of the conchoid G is the conchoid curve with respect to O of a generating line of F.

2.3. The conchoid surfaces of rational cones

Let F be a rational cone with vertex $\mathbf{v} = (0,0,1)$ and directrix curve $\mathbf{c}(u) = (c_1(u), c_2(u), 0)$. For dealing with the general case we assume that $O \notin F$. Then F is parameterized by

$$\mathbf{f}(u,v) = \mathbf{v} + v(\mathbf{c}(u) - \mathbf{v}) = \mathbf{v} + v\mathbf{e}(u),$$

with $\mathbf{e}(u) = (c_1(u), c_2(u), -1)$. With respect to these choices the squared length of $\mathbf{f}(u,v)$ is $\|\mathbf{f}(u,v)\|^2 = 1 - 2v + \mathbf{e}(u)^2 v^2$. According to (11) the family of conics reads

$$a(u) : x_1^2 - 2x_1x_2 + x_2^2\mathbf{e}(u)^2 = x_0^2. \tag{19}$$

The conics $a(u)$ share two common points $(1,-1,0)\mathbb{R}$ and $(1,1,0)\mathbb{R}$, and a stereographic projection results in their rational parameterization $\mathbf{z}(u,t) = (1 - 2t + \mathbf{e}(u)^2t^2, 1 - \mathbf{e}(u)^2t^2, 2t(1-t))$. Substituting $v = z_2/z_1$ in $\mathbf{f}(u,v)$ gives the rational parameterization

$$\mathbf{f}(u,t) = \mathbf{v} + \frac{2t(1-t)}{1-\mathbf{e}(u)^2t^2}\mathbf{e}(u), \text{ with } \|\mathbf{f}(u,t)\| = \frac{z_0(u,t)}{z_1(u,t)} = \frac{1-2t+\mathbf{e}(u)^2t^2}{1-\mathbf{e}(u)^2t^2}. \tag{20}$$

Rational polar representations of F and its conchoid surfaces G are obtained with (14).

3. Examples

This section shows two examples in detail. The conchoid construction of a famous cubic ruled surface, the Plücker conoid, which is projectively equivalent to the Whitney umbrella, and the conchoid of a hyperbolic paraboloid. The necessary computational steps are outlined. To obtain a diagonal normal form of the family of conics determined by the squared distance $\|\mathbf{f}(u,v)\|^2$ we might use the method presented in Section 2.1. These examples show another method where the conics $a(u)$ are represented in a coordinate system which is based on the vertices of a polar triangle as base points.

3.1. Plücker conoid

The Plücker conoid F is an algebraic ruled surface of order three, also called cylindroid and projectively equivalent to the Whitney umbrella. A trigonometric parameterization with the double line as z-axis reads $(0,0,\sin 2u) + v(\cos u, \sin u, 0)$. It can be generated in the following way. Rotate the x-axis around z and superimpose this rotation by the translation $(0,0,\sin 2u)$ in z-direction. An implicit equation of F is $z(x^2+y^2) = 2xy$.

Since the z-axis is a double line of F, the origin is a double point and the computation of the conchoid with respect to O is trivial. Thus we apply a translation by $(0,1,2)$. A rational parameterization of the translated surface which is again denoted by F is given by

$$\mathbf{f}(u,v) = \left(\frac{-(u^2-1)v}{u^2+1}, \frac{u^2+2vu+1}{u^2+1}, \frac{2(u^4-2u^3+2u^2+2u+1)}{(u^2+1)^2}\right) \tag{21}$$

The squared length of $\mathbf{f}(u,v)$ corresponds to the family of conics

$$a(u) : -(u^2+1)^4x_0^2 + \alpha(u))x_1^2 + 4u(u^2+1)^3x_1x_2 + (u^2+1)^4x_2^2 = 0, \tag{22}$$

with $\alpha(u) = 5(u^2+1)^4 + 16u(u^2-1)((u^2+1)^2 + u(u^2-1))$. It's obvious that these conics share the vertices $(\pm 1, 0, 1)\mathbb{R}$.

A rational solution (z_0, z_1, z_2) of (22) is computed by stereographic projection of the line $(0,1,t)\mathbb{R}$ onto the conics from (22) with center $(1,0,1)\mathbb{R}$. This leads to the reparameterization along the generating lines of F,

$$v(u,t) = \frac{z_2}{z_1} = -\frac{\alpha(u)t^2 + (u^2+1)^2(4tu-1)}{2(u^2+1)^3t}. \tag{23}$$

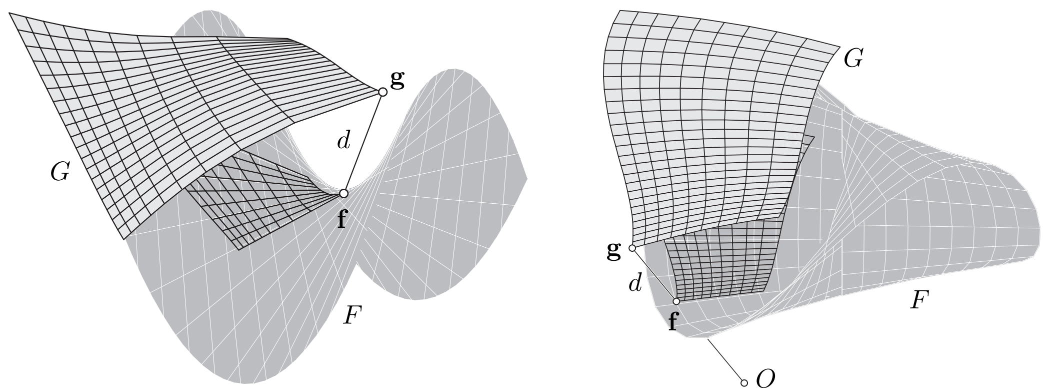


Figure 4: Conchoids G of a hyperbolic paraboloid (left) and of a Plücker conoid (right).

The surface F has the rational radius function

$$\|\mathbf{f}(u,t)\| = \frac{z_0}{z_1} = -\frac{\alpha(u)t^2 + (u^2+1)^2}{2(u^2+1)^3 t}. \tag{24}$$

Substituting (23) in (21) leads to a rational parameterization of $\mathbf{f}(u,t)$ and a rational parameterization of the conchoid G with respect to O and distance $d = 1$

$$\mathbf{g}(u,t) = \begin{pmatrix} \beta(u,t)(u^2-1)(\alpha(u)t^2 + (u^2+1)^2(4tu-1)) \\ \beta(u,t)(-\alpha(u)t^2 u + (u^4-1)^2 t + (u^2+1)u) \\ 4t\beta(u,t)(u^2+1)^2((u^2+1)^2 - 2(u^2+2)u) \end{pmatrix}$$

with

$$\beta(u,t) = \frac{\alpha(u)t^2 + 2(u^2+1)^3 t + (u^2+1)^2}{2t(u^2+1)^2(\alpha(u)(u^2+1)^2t^2 + (u^2+1)^4)}.$$

3.2. Hyperbolic paraboloid

A hyperbolic paraboloid can be given by $\mathbf{f}(u,v) = (u, v, uv+1)$, and the squared length of $\mathbf{f}(u,v)$ reads $\|\mathbf{f}(u,v)\|^2 = (u^2+1)v^2 + 2uv + (u^2+1)$. For the corresponding one parameter family of conics $a(u)$ we get

$$a(u) : -x_0^2 + (u^2+1)x_1^2 + 2ux_1x_2 + (u^2+1)x_2^2 = 0.$$

By the rational transformation $x_0 = \overline{x}_0$, $x_1 = \overline{x}_1 + \overline{x}_2$, $x_2 = \overline{x}_1 - \overline{x}_2$, the conics are transformed into the normal form, where we use again x_i instead of $\overline{x}_i$,

$$a(u) = -x_0^2 + 2(u^2+u+1)x_1^2 + 2(u^2-u+1)x_2^2 = 0. \tag{25}$$

According to Lemma 1 there exists a curve

$$\mathbf{y}(u) = \frac{1}{5}\left((2\sqrt{2} - 4\sqrt{3})u^2 + (6\sqrt{2} - 2\sqrt{3})u - 2\sqrt{2} - 6\sqrt{3}, (2\sqrt{6}+2)u + 5, 2\sqrt{6}+3\right).$$

following the conics $a(u)$. Stereographic projection applied to each conic $a(u)$ finally leads to a parameterization $\mathbf{z}(u,t)$ of $a(u)$ and the desired reparameterization of F. The center of the stereographic projection is $\mathbf{y}(u)$ and the line which is projected to $a(u)$ is chosen by $\mathbf{q}(t) = (0,1,t)\mathbb{R}$. This leads to the rational polar representation of the hyperbolic paraboloid F,

$$\mathbf{f}(u,t) = (u, v(u,t), uv(u,t) + 1),$$

with

$$v(u,t) = \frac{b - 2t + bt^2 + u(-c + 2at - at^2) + u^2(c - 4t - at^2) + u^3(-1 - 2t + t^2)}{-1 - 2bt + t^2 + 2ut(2+t) + 2u^2t(-a+t) + u^3(-1 + 2t + t^2)},$$

and the constant factors $a = 1 - \sqrt{6}$, $b = 2 - \sqrt{6}$ and $c = 3 - \sqrt{6}$. For the rational radius function of $\mathbf{f}(u,t)$ we obtain

$$\|\mathbf{f}(u,t)\| = \frac{\sqrt{2}(1 + t^2 + u(-1 + t^2) + u^2(1 + t^2))(4 - \sqrt{6} + \sqrt{6}u + 2u^2)}{2(-1 - 2bt + t^2 + 2ut(2+t) + 2u^2t(-a+t) + u^3(-1 + 2t + t^2))}.$$

4. Conclusion

We have discussed the well known conchoid construction applied to rational ruled surfaces. The main result says that the conchoid of a rational ruled surface is a rational surface independent on the position of the focus point. The construction basically relies on the parameterization of a rational one-parameter family of conics. Special cases like cylinders, cones and rotational ruled surfaces are discussed and several examples are given to illustrate the method.

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