Main theorem on Schönflies-singular planar Stewart Gough platforms

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[1] Singular configurations of SGPs

The geometry of a Stewart Gough Platform (SGP) is given by the six base anchor points

$\mathbf{M}_i := (A_i, B_i, 0)^T$ in the fixed space Σ_0

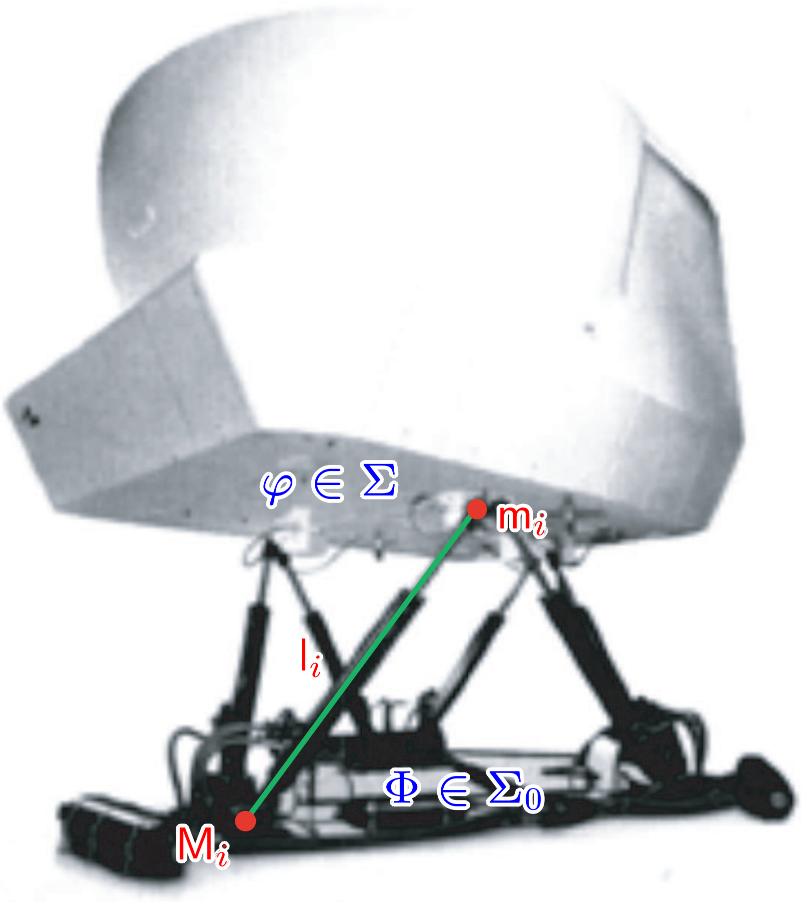
and by the six platform anchor points

$\mathbf{m}_i := (a_i, b_i, 0)^T$ in the moving space Σ.

$\Phi \in \Sigma_0$ denotes the carrier plane of the M_i's.

$\varphi \in \Sigma$ denotes the carrier plane of the m_i's.

Theorem Merlet [1992]
A SGP is singular iff the carrier lines l_i of the six legs belong to a linear line complex.



[1] Schönflies-singular SGPs

The Schönflies motion group X(a) consists of three linearly independent translations and all rotations about the infinity of axes with direction a.

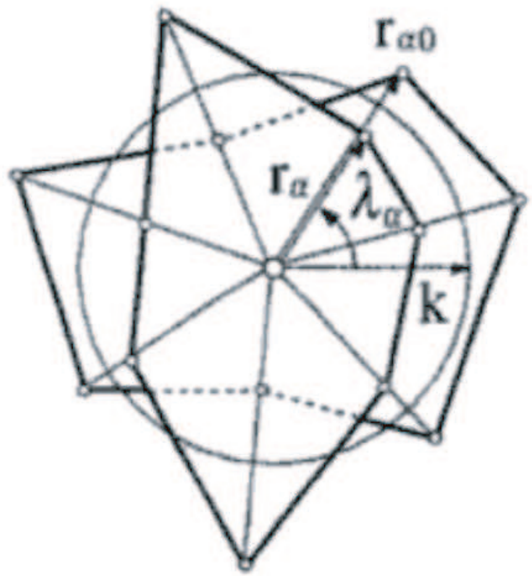
Definition **Schönflies-singular SGP**
A SGP is called Schönflies-singular (or more precisely X(a)-singular) if there exists a Schönflies group X(a) such that the manipulator is singular for all transformations from X(a) (applied to the moving part of the SGP).

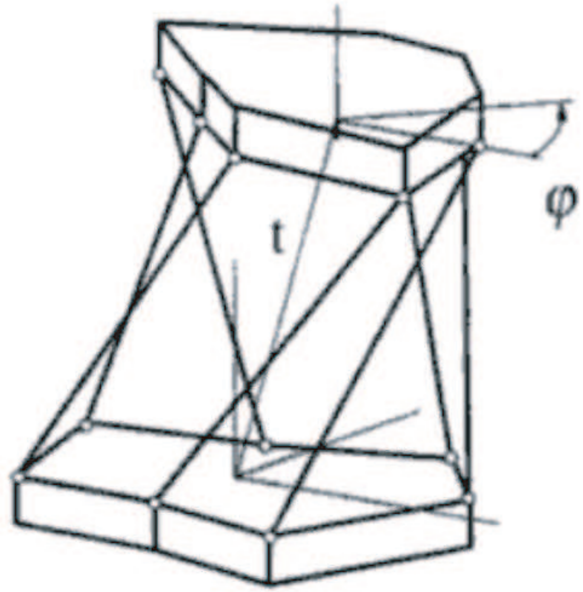
Every Schönflies-singular manipulator belongs to one of these cases:

1. $\alpha \neq \beta$: (a) $\alpha = \pi/2$, $\beta \in [0, \pi/2[$ (b) $\alpha, \beta \in [0, \pi/2[$
2. $\alpha = \beta$: (a) $\alpha = \pi/2$ (b) $\alpha \in]0, \pi/2[$ (c) $\alpha = 0$

with $\alpha := \angle(\mathsf{a}, \Phi) \in [0, \pi/2]$ and $\beta := \angle(\mathsf{a}, \varphi) \in [0, \pi/2]$.

[2] Review on Schönflies-singular SGPs





Wohlhart [2000] presented the *polygon platform*, i.e. the anchor points in Φ and φ are related by an inversion.

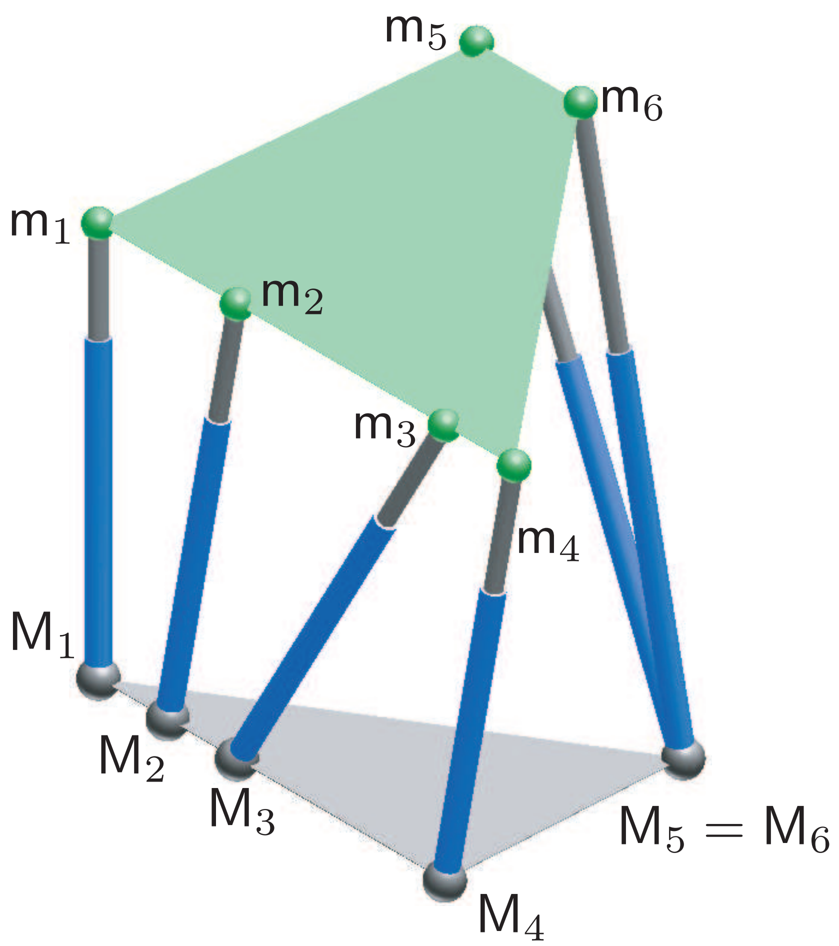
This manipulator is X(a)-singular of case (2a). Moreover it even possesses a Schönflies self-motion.

[2] Schönflies-singular SGPs of case (1a)

According to **Nawratil** **[2010]** the solution set of case (1a) can be characterized as:

Theorem **Nawratil** **[2010]**
A non-architecturally singular planar SGP is X(a)-singular, where a is orthogonal to Φ and orthogonal to the x-axis of the moving frame iff $rk(\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{Bb}, \mathbf{a}, \mathbf{b}, \mathbf{Ab}) = 4$ holds with

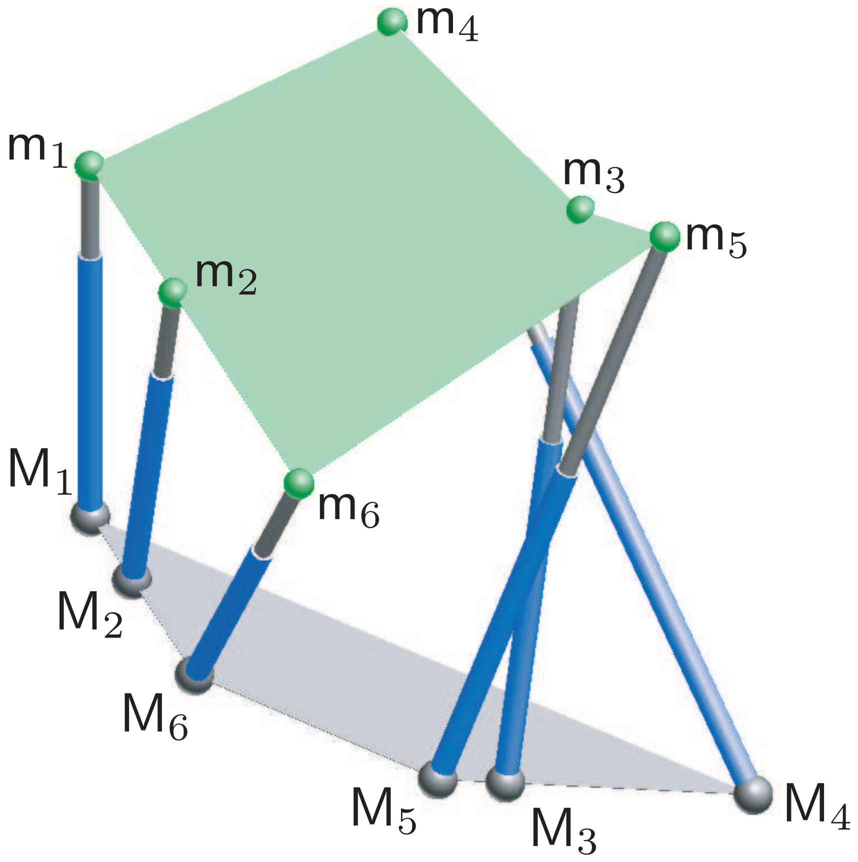
$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_6 \end{bmatrix}, \quad \mathbf{Xy} = \begin{bmatrix} X_1 y_1 \\ X_2 y_2 \\ \vdots \\ X_6 y_6 \end{bmatrix}.$$



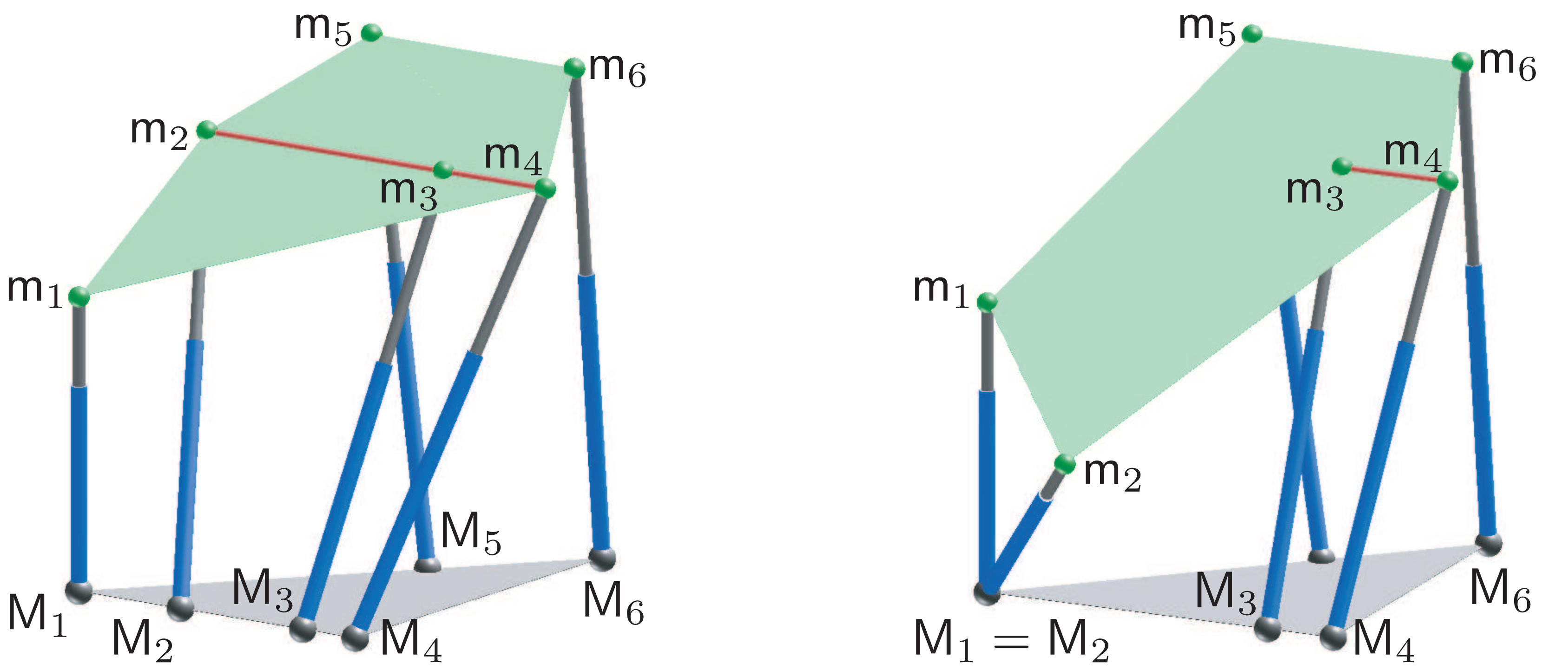
[2] Schönflies-singular SGPs of case (1a)

The given rank condition possesses the following geometric interpretation:

Corrolary Nawratil [2010]
Given are 2 sets of points $\{\mathrm{M}_i\}$ and $\{\mathrm{m}_i\}$ $(i = 1, \ldots, 6)$ in Φ resp. φ. Then the non-architecturally singular SGP, where a is orthogonal to Φ, is X(a)-singular iff $\{\mathrm{M}_i, \mathrm{m}_i\}$ are 3-fold conjugate pairs of points with respect to a 2-dimensional linear manifold of correlations, which map the ideal line of Φ onto the ideal point of the intersection line of φ and Φ.

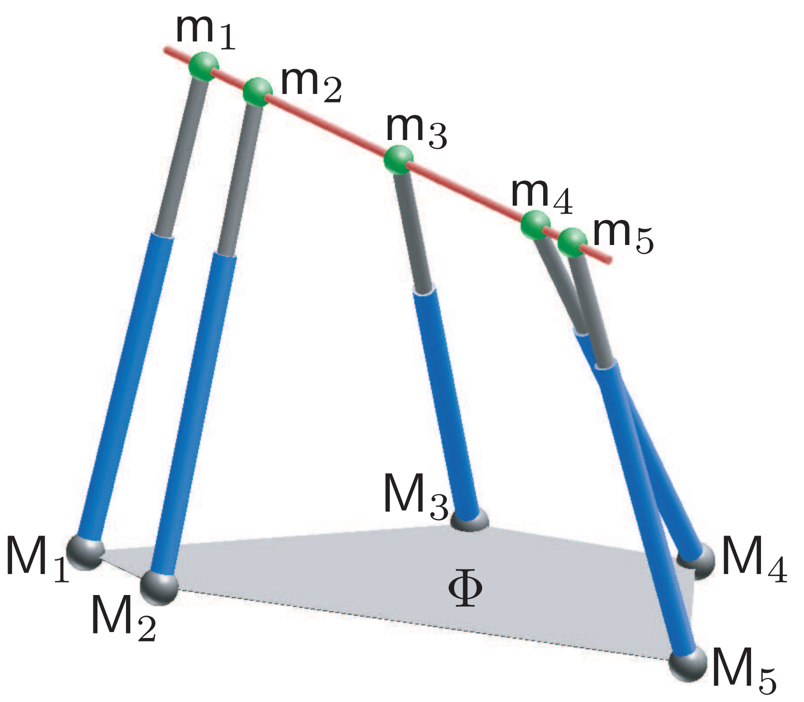


[2] Schönflies-singular SGPs of case (1a)



Theorem Nawratil [2010]
Manipulators of the solution set of case (1a) have a quadratic singularity surface.

[2] Schönflies-singular SGPs of case (1a)



Degenerated cases

The 5 legs $l_1, \ldots, l_5$ belong in any configuration with $[m_1, \ldots, m_5] \parallel \Phi$ to a congruence of lines. These designs also imply *non-planar* Schönflies-singular SGPs.

[3] Preparatory work for the main theorem

Plücker coordinates of l_i can be written as $(\mathbf{l}_i, \widehat{\mathbf{l}}_i) := (\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t} - H\mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$

$$\text{with} \quad \mathbf{R} := (r_{ij}) = \begin{pmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2+e_0e_3) & 2(e_1e_3-e_0e_2) \\ 2(e_1e_2-e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3+e_0e_1) \\ 2(e_1e_3+e_0e_2) & 2(e_2e_3-e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{pmatrix},$$

$\mathbf{t} := (t_1, t_2, t_3)^T$ and the homogenizing factor $H := e_0^2 + e_1^2 + e_2^2 + e_3^2$.

Remark: The group SO_3 is parametrized by Euler Parameters (e_0, e_1, e_2, e_3).

l_i belong to a linear line complex $\iff$ $Q := det(\mathbf{Q}) = 0$ with $\mathbf{Q} := \begin{pmatrix} \mathbf{l}_1 & \widehat{\mathbf{l}}_1 \\ \dots & \dots \\ \mathbf{l}_6 & \widehat{\mathbf{l}}_6 \end{pmatrix}$

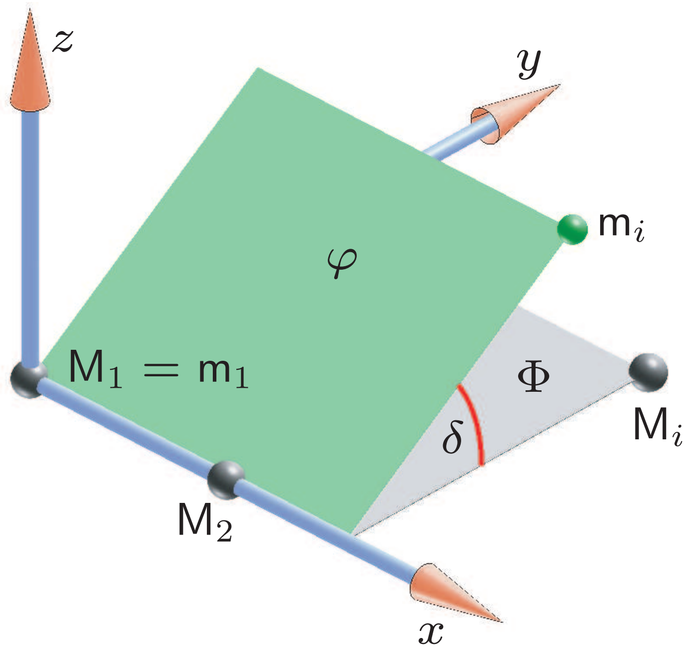
[4] Main theorem for the general case

Theorem
Any X(a)-singular planar SGP with no 4 collinear anchor points, where $\alpha \neq \beta$ and a not orthogonal to Φ or φ is architecturally singular.

W.l.o.g. we can assume $\alpha > \beta$ ($\Rightarrow \Phi \nparallel$ a). We can rotate φ about a such that the common line s of Φ and φ is parallel to $[\mathrm{M}_1, \mathrm{M}_2]$.

This yields the following coordinatization:

$\mathbf{M}_i = (A_i, B_i, 0)$, $\mathbf{m}_i = (a_i, b_i \cos\delta, b_i \sin\delta)$ with $A_1 = B_1 = B_2 = a_1 = b_1 = 0$, $\sin\delta \neq 0$.



[4] Outline of the proof

As no four anchor points are collinear we can apply the elementary matrix manipulations of **Karger [2003]** to $\mathbf{Q} \quad \Longrightarrow \quad (\mathbf{l}_6, \widehat{\mathbf{l}}_6) := (v_1, v_2, v_3, 0, -w_3, w_2)$ with

$$v_i := r_{i1}K_1 + (r_{i3}\sin\delta + r_{i2}\cos\delta)K_2, \quad w_j := r_{j1}K_3 + (r_{j3}\sin\delta + r_{j2}\cos\delta)K_4$$

and

$$K_1 := |\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{a}|, \qquad K_3 := |\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Aa}|,$$
$$K_2 := |\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{b}|, \qquad K_4 := |\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Ab}|.$$

$K_1 = K_2 = K_3 = K_4 = 0 \Longrightarrow$ SGP is architecturally singular

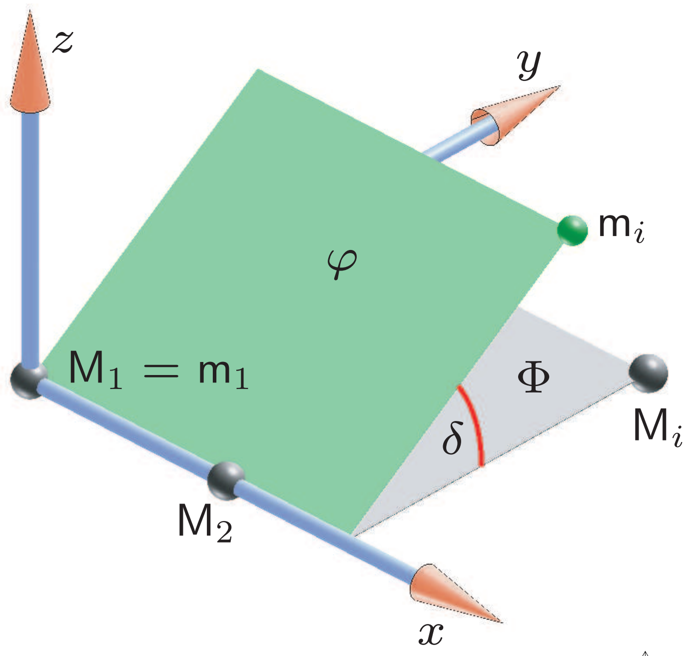
We show that Q is fulfilled identically for all transformations from a Schönflies group X(a) with a $\nparallel \Phi$ ($\Rightarrow e_3 \neq 0$) if and only if $K_1 = K_2 = K_3 = K_4 = 0$ holds.

[4] Outline of the proof

Lemma Mick and Röschel [1998]
If the connecting lines l_i of $M_i \in \Phi$ and $m_i \in \varphi$ of two intersecting planes Φ and φ belong to a linear line complex, the property remains unchanged under rotations of the planes about their intersection line s.

Due to this Lemma the manipulator is also X(s)-singular. This property already implies $K_1 = K_2 = 0$ (cf. presented paper).

Now we go back to the general case. We prove by contradiction that for all transformations of a X(a) motion with a $\nparallel \Phi$ the vanishing of $Q(K_3, K_4)$ implies $K_3 = K_4 = 0$. $\square$



[5] Main theorem for the special case

Theorem
Any X(a)-singular planar SGP with 4 collinear anchor points, where $\alpha \neq \beta$ and a not orthogonal to Φ or φ is architecturally singular.

In order to prove this theorem efficiently, we need a good choice for the coordinate systems in Σ and Σ_0. Geometric considerations yield following coordinatization:

W.l.o.g. we can assume that the four collinear anchor points are on the platform φ $\Longrightarrow$ $\mathrm{m}_1, \ldots, \mathrm{m}_4$ are situated on the line g.

Now we have to distinguish two cases:

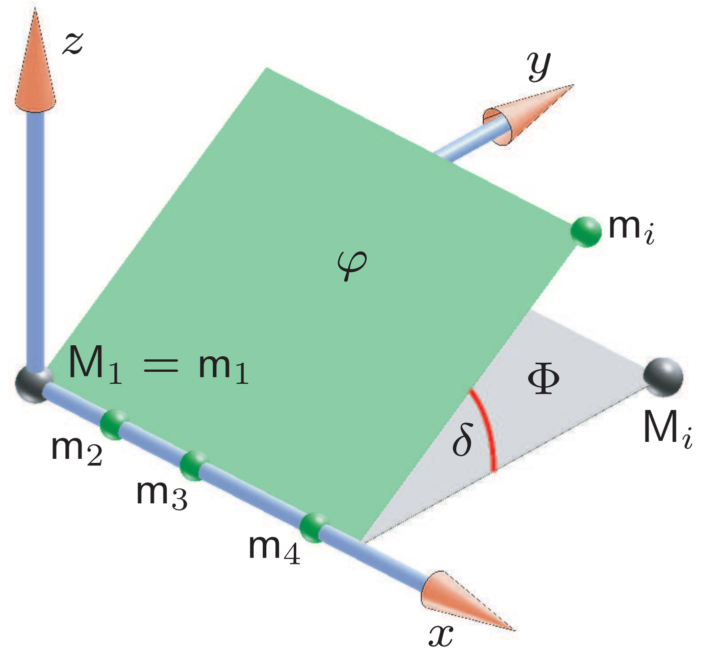
- $\gamma \geq \alpha$,
- $\gamma < \alpha$ with $\gamma := \angle(\mathrm{g}, \mathrm{a}) \in [0, \pi/2]$.

[5] Coordinatization for $\gamma \geq \alpha$

We translate φ and Φ such that $\mathsf{M}_1 = \mathsf{m}_1 \in \mathsf{a}$ holds. Due to $\gamma \geq \alpha$ there exists at least one position by rotating φ about a such that $\mathsf{g} \in \Phi$ holds.

This is the initial configuration of the following coordinatization:

$\mathbf{M}_i = (A_i, B_i, 0)$, $\mathbf{m}_i = (a_i, b_i \cos\delta, b_i \sin\delta)$ with $A_1 = B_1 = a_1 = b_1 = b_2 = b_3 = b_4 = 0$ and $\sin\delta \neq 0$.



Proof:

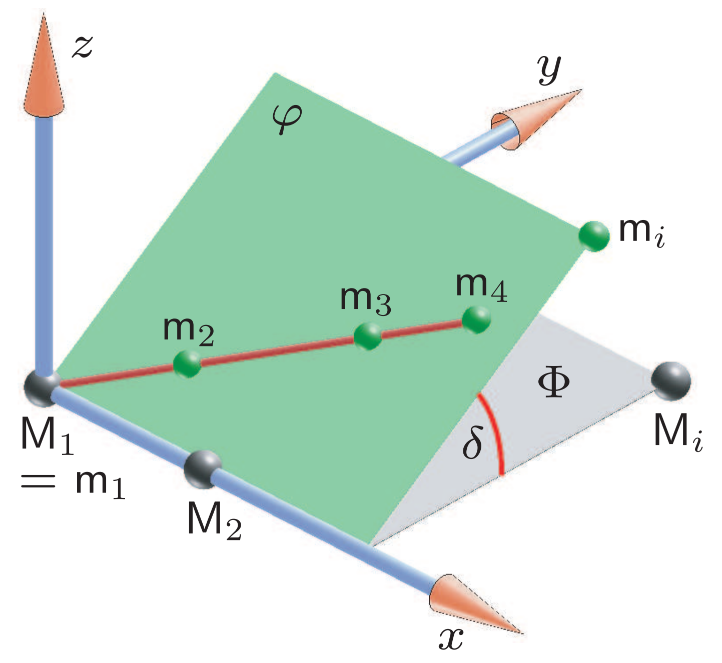
The proof is done by contradiction. A detailed case study is given in the presented paper. □

[5] Coordinatization for $\gamma < \alpha$

We translate φ and Φ such that $\mathsf{M}_1 = \mathsf{m}_1 \in \mathsf{a}$ holds. Due to $\gamma < \alpha$ there are two positions by rotating φ about a such that $[\mathsf{M}_1, \mathsf{M}_2] \in \varphi$ holds.

This is the initial configuration of the following coordinatization:

$\mathbf{M}_i = (A_i, B_i, 0)$, $\mathbf{m}_i = (a_i, b_i \cos\delta, b_i \sin\delta)$ with $A_1 = B_1 = B_2 = a_1 = b_1 = 0$ and $a_i = b_i a_2 / b_2$ for $i = 3, 4$ and $b_2 \sin\delta \neq 0$.



Proof:

The proof is done by contradiction. For the detailed case study see **Nawratil [2009]**. □

[6] Conclusions and future research

Main Theorem
X(a)-singular planar Stewart Gough platforms with $\alpha \neq \beta$ and where a is not orthogonal to Φ or φ are necessarily architecturally singular.

Mick and Röschel [1998] proved that a planar SGP is architecturally singular iff it is singular with respect to a special 5-parametric set of displacements.

We can improve this statement even to 4-parametric sets of displacements, namely the Schönflies motion groups for which the main theorem holds.

Can this statement further be improved to an even 3-dim Lie subgroup of SE(3)?

Due to known results, we can restrict to SO(3) and H(d) $\rtimes \mathbb{R}^2$ where the axis d of H is not orthogonal to Φ or φ, $\angle(\Phi, \mathrm{d}) \neq \angle(\varphi, \mathrm{d})$ and the pitch $p \in [0, \infty[$.

[6] Conclusions and future research

Due to the main theorem, the manipulators of the solution set of case (1a) are the only non-architecturally singular planar SGPs with $\alpha \neq \beta$ which are X(a)-singular.

Schönflies-singular planar Stewart Gough platforms with $\alpha = \beta$ are determined in:

Nawratil G. Special cases of Schönflies-singular planar Stewart Gough platforms, New Trends in Mechanisms Science (D. Pisla et al. eds.), to appear.

Paper presentation: Third European Conf. on Mechanism Science, September 14-18 2010, Cluj-Napoca Romania.

The determination of the whole set $\mathcal{S}$ of non-planar Schönflies-singular SGPs remains open. The degenerated planar cases of (1a) imply two manipulators of $\mathcal{S}$.

[7] References

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