Fundamentals of Quaternionic Kinematics in Euclidean 4-Space

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1. Motivation

The elegance of the quaternion based analytical treatment of kinematics in Euclidean spaces of dimension 2 and 3 was pointed out and used by various authors:

Blaschke W. Kinematik und Quaternionen. DVW, Berlin (1960)

Müller H.R. Sphärische Kinematik. DVW, Berlin (1962)

Ströher W. Sphärische und Räumliche Kinematik. unpublished book (1973)

The quaternionic approach does not only yield a more compact notation in comparison with matrices, but it also provides an easier access to the geometry of motions.

Motivated by this circumstance, we want to extend this quaternionic kinematic to the Euclidean 4-space E^4 in the tradition of the above cited works.

2. Review on Kinematic Mappings

A kinematic mapping of SE(n) is a bijective mapping between the group of displacements of E^n and a set of points in a certain projective space. For $n = 3$, this mapping can be constructed by the usage of unit dual quaternions:

Quaternions: $\mathbf{Q} := q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = q_0 + \mathbf{q}$ with $q_0, \ldots, q_3 \in \mathbb{R}$ is an element of the skew field of quaternions $\mathbb{H}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the so-called quaternion units.

The conjugated quaternion to $\mathbf{Q}$ is given by $\widetilde{\mathbf{Q}} := q_0 - \mathbf{q}$.

$\mathbf{Q}$ is called pure quaternion for $\mathbf{Q} = \mathbf{q}$ and unit quaternion for $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

We embed the points X of E^3 with Cartesian coordinates (x_1, x_2, x_3) into the set of pure quaternions by the following mapping:

$$\iota_3 : \mathbb{R}^3 \to \mathbb{H} \quad \text{with} \quad (x_1, x_2, x_3) \mapsto \mathbf{x} := x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}.$$

2a. Study Mapping of SE(3)

Dual Quaternions: An element $\mathbf{E} + \varepsilon\mathbf{T}$ of $\mathbb{H} + \varepsilon\mathbb{H}$ is called dual quaternion, where ε is the dual unit with the property $\varepsilon^2 = 0$ and $\varepsilon \neq 0$.

It is called unit dual quaternion, if $\mathbf{E}$ is an unit quaternion and following condition holds:

$$e_0 t_0 + e_1 t_1 + e_2 t_2 + e_3 t_3 = 0.$$

The mapping of points $\mathsf{X} \in E^3$ to $\mathsf{X}' \in E^3$ induced by any element of SE(3) can be written as follows by using ι_3 (e.g. Husty et al [2]):

$$\mathbf{x} \mapsto \mathbf{x}' \quad \text{with} \quad \mathbf{x}' := \mathbf{E}\mathbf{x}\widetilde{\mathbf{E}} + (\mathbf{T}\widetilde{\mathbf{E}} - \mathbf{E}\widetilde{\mathbf{T}}).$$

Moreover this mapping is an element of SE(3) for any unit dual quaternion $\mathbf{E} + \varepsilon\mathbf{T}$.

2a. Study Mapping of SE(3)

As both unit dual quaternions $\pm(\mathbf{E}+\varepsilon\mathbf{T})$ correspond to the same Euclidean motion of E^3, we consider the homogeneous 8-tuple $(e_0 : \ldots : e_3 : t_0 : \ldots : t_3)$.

These so-called Study parameters can be interpreted as a point of a projective 7-dimensional space P^7. Therefore there is a bijection between SE(3) and all real points of P^7 located on the so-called Study quadric

$$\Phi : \quad e_0t_0 + e_1t_1 + e_2t_2 + e_3t_3 = 0,$$

which is sliced along the 3-dimensional generator-space $e_0 = e_1 = e_2 = e_3 = 0$, as the corresponding quaternion $\mathbf{E}$ cannot be normalized.

Remark: Restricting the Study mapping to planar Euclidean displacements within a plane yields the so-called Blaschke-Grünwald Mapping of SE(2). $\diamond$

2b. Klawitter-Hagemann Mapping of SE(n)

Based on Clifford algebras, Klawitter & Hagemann [3] presented an unified concept for constructing kinematic mappings for certain Cayley-Klein geometries.

Especially for E^2 and E^3, they demonstrated that their approach yields the Blaschke-Grünwald mapping and the Study mapping (see also Selig [6]).

The Study parameters (resp. Blaschke-Grünwald parameters) are isomorphic to the Spin group of the Clifford Algebra with signature $(3_+, 0_-, 1_0)$ (resp. $(2_+, 0_-, 1_0)$).

The Spin group of the Clifford Algebra with signature $(4_+, 0_-, 1_0)$ implies a mapping between displacements of SE(4) and points of P^{15} with homogeneous coordinates $(a_0 : \ldots : a_7 : c_0 : \ldots : c_7)$ located in the intersection of nine quadrics R_i $(i = 1, \ldots, 9)$, which is additionally sliced along the quadric N:

2b. Klawitter-Hagemann Mapping of SE(4)

$$R_1 : a_2c_6 - a_3c_5 + a_4c_0 - c_1c_4 = 0, \qquad R_2 : a_5c_0 - c_1c_7 + c_2c_5 - c_3c_6 = 0,$$
$$R_3 : a_1c_5 - a_2c_7 + a_7c_0 - c_3c_4 = 0, \qquad R_4 : a_1c_6 - a_3c_7 + a_6c_0 - c_2c_4 = 0,$$
$$R_5 : a_0c_0 - a_1c_1 + a_2c_2 - a_3c_3 = 0, \qquad R_6 : a_0c_7 - a_1a_5 - a_6c_3 + a_7c_2 = 0,$$
$$R_7 : a_0c_4 - a_1a_4 + a_2a_6 - a_3a_7 = 0, \qquad R_8 : a_0c_6 - a_3a_5 - a_4c_2 + a_6c_1 = 0,$$
$$R_9 : a_0c_5 - a_2a_5 - a_4c_3 + a_7c_1 = 0, \qquad N : a_0^2 + \ldots + a_7^2 + c_0^2 + \ldots + c_7^2 = 0.$$

Computation of the Hilbert-polynomial shows that the kinematic image of SE(4) is a 10-dimensional variety $R_1 \cap R_2 \cap \ldots \cap R_9 \in P^{15}$ of degree 12, which is sliced along $N = 0$.

Therefore we are interested in a simplified kinematic mapping of SE(4).

2c. Quaternionic Kinematic Mapping of SE(4)

We embed the points X of E^4 with Cartesian coordinates (x_0, x_1, x_2, x_3) into the set of quaternions by the mapping:

$$\iota_4 : \mathbb{R}^4 \to \mathbb{H} \quad \text{with} \quad (x_0, x_1, x_2, x_3) \mapsto \mathbf{X} := x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}.$$

Theorem 1. The mapping of points X $\in E^4$ to X$' \in E^4$ induced by any element of SE(4) can be written as follows by using ι_4:

$$\mathbf{X} \mapsto \mathbf{X}' \quad \textit{with} \quad \mathbf{X}' := \mathbf{E}\mathbf{X}\widetilde{\mathbf{F}} - 2\mathbf{E}\widetilde{\mathbf{T}}.$$

Moreover this mapping is an element of SE(4) for any triple of quaternions $\mathbf{E}, \mathbf{F}, \mathbf{T}$, where $\mathbf{E}$ and $\mathbf{F}$ are unit-quaternions.

As both triples of quaternions $\pm(\mathbf{E}, \mathbf{F}, \mathbf{T})$, where $\mathbf{E}$ and $\mathbf{F}$ are unit quaternions, correspond to the same Euclidean motion of E^4, we consider the homogeneous 12-tuple $(\mathbf{E} : \mathbf{F} : \mathbf{T})$, which can be interpreted as a point of P^{11}.

2c. Quaternionic Kinematic Mapping of SE(4)

Therefore there is a bijection between SE(4) and all real points of P^{11} located on the quadric

$$\Xi: \quad (e_0^2+e_1^2+e_2^2+e_3^2)-(f_0^2+f_1^2+f_2^2+f_3^2)=0,$$

and is sliced along the 7-dimensional space $e_0=e_1=e_2=e_3=0$, as the corresponding quaternion $\mathbf{E}$ cannot be normalized. The real intersection of this 7-space and Ξ equals the 3-dimensional generator-space U of Ξ with:

$$\mathrm{U}: \quad e_0=e_1=e_2=e_3=f_0=f_1=f_2=f_3=0.$$

Remark: Note that Ξ expresses the fact that $\mathbf{F}$ is also normalized if $\mathbf{E}$ is. $\diamond$

Resume: There is a bijection between elements of SE(4) and real points of $\Xi \setminus$ U. The Study parameters and subsequently the Blaschke-Grünwald parameters can be obtained from $(\mathbf{E}:\mathbf{F}:\mathbf{T})$.

3. Representation of Displacements

The composition $(\mathbf{E}, \mathbf{F}, \mathbf{T})$ of two displacements $(\mathbf{E}_i, \mathbf{F}_i, \mathbf{T}_i)$ for $i = 1, 2$ corresponds to the multiplication $\underline{\mathbf{D}} = \underline{\mathbf{D}}_2\underline{\mathbf{D}}_1$ of lower triangular 2×2 quaternionic matrices (cf. WILKER [7]) with

$$\underline{\mathbf{D}} = \begin{pmatrix} \mathbf{E} & \mathbf{O} \\ \mathbf{T} & \mathbf{F} \end{pmatrix} \quad \text{and} \quad \underline{\mathbf{D}}_i = \begin{pmatrix} \mathbf{E}_i & \mathbf{O} \\ \mathbf{T}_i & \mathbf{F}_i \end{pmatrix}.$$

Remark: This map from SE(4) to the group of lower triangular 2×2 quaternionic matrices with unit-quaternions in the diagonal is a representation (e.g. [1]). ◇

This motivates us to embed a point $\mathsf{X} \in E^4$ into the set of 2×2 quaternionic matrices in a way that its multiplication with such matrices gives the point coordinates of $\mathsf{X}' \in E^4$; i.e. an analogue to the 3-dimensional case, where we can embed the points into the set of dual unit-quaternions in a way that:

$$1 + \varepsilon\mathbf{x}' = (\mathbf{E} + \varepsilon\mathbf{T})\,(1 + \varepsilon\mathbf{x})\left(\widetilde{\mathbf{E}} - \varepsilon\widetilde{\mathbf{T}}\right) \quad \text{holds.}$$

3a. Representation of Point Displacements

By introducing the following notation

$$\underline{\mathbf{X}} = \begin{pmatrix} -1 & \mathbf{X} \\ \mathbf{O} & 1 \end{pmatrix}, \quad \underline{\mathbf{X}}' = \begin{pmatrix} -1 & \mathbf{X}' \\ \mathbf{O} & 1 \end{pmatrix}, \quad \underline{\widetilde{\mathbf{D}}}^T = \begin{pmatrix} \widetilde{\mathbf{E}} & \widetilde{\mathbf{T}} \\ \mathbf{O} & \widetilde{\mathbf{F}} \end{pmatrix}, \quad \underline{\widetilde{\mathbf{D}}}^{-T} = \begin{pmatrix} \mathbf{E} & -\mathbf{E}\widetilde{\mathbf{T}}\mathbf{F} \\ \mathbf{O} & \mathbf{F} \end{pmatrix},$$

this can be done as follows:

Theorem 2. The mapping of points $\mathsf{X} \in E^4$ to $\mathsf{X}' \in E^4$ induced by any element of SE(4) can be written as follows:

$$\underline{\mathbf{X}} \mapsto \underline{\mathbf{X}}' \quad with \quad \underline{\mathbf{X}}' := \underline{\widetilde{\mathbf{D}}}^{-T} \underline{\mathbf{X}}\, \underline{\widetilde{\mathbf{D}}}^{T}.$$

Proof: Straightforward computation. □

In the following we show that the displacement of oriented lines, planes and hyperplanes in E^4 can also be written by 2×2 quaternionic matrices.

3b. Representation of Hyperplane Displacements

All points X $\in E^4$ with coordinates (x_0, x_1, x_2, x_3) located in a hyperplane fulfill a linear equation, which can be written in the Hesse normal form as

$$x_0 w_0 + x_1 w_1 + x_2 w_2 + x_3 w_3 + w = 0 \quad \text{with} \quad w_0^2 + w_1^2 + w_2^2 + w_3^2 = 1.$$

Thus a hyperplane can be fixed by a unit-quaternion $\mathbf{W} = w_0 + w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ and a real number w.

Therefore $-w$ gives the oriented distance of the footpoint on the hyperplane to the origin with respect to the direction of $\mathbf{W}$. Applying a rotation about the origin the footpoint has still the distance $-w$, but now in direction of $\mathbf{EW\widetilde{F}}$. This distance is only changed by the component of the translational vector, which is orthogonal to the rotated hyperplane; i.e. $\langle -2\mathbf{E\widetilde{T}}, \mathbf{EW\widetilde{F}} \rangle$.

3b. Representation of Hyperplane Displacements

Summed up we have: $\quad \mathbf{W} \mapsto \mathbf{E}\mathbf{W}\widetilde{\mathbf{F}}$ and $w \mapsto w + \langle 2\mathbf{E}\widetilde{\mathbf{T}}, \mathbf{E}\mathbf{W}\widetilde{\mathbf{F}} \rangle$.

Having in mind that $(\mathbf{W}, w)$ also assigns an orientation we can state the following theorem under consideration of the notation:

$$\underline{\mathbf{W}} = \begin{pmatrix} \mathbf{O} & \mathbf{W} \\ \widetilde{\mathbf{W}} & w \end{pmatrix}, \qquad \underline{\mathbf{W}}' = \begin{pmatrix} \mathbf{O} & \mathbf{W}' \\ \widetilde{\mathbf{W}}' & w' \end{pmatrix}.$$

Theorem 3. The mapping of oriented hyperplanes $(\mathbf{W}, w)$ of E^4 to oriented hyperplanes $(\mathbf{W}', w')$ of E^4 induced by any element of SE(4) can be written as follows:

$$\underline{\mathbf{W}} \mapsto \underline{\mathbf{W}}' \quad \textit{with} \quad \underline{\mathbf{W}}' := \underline{\mathbf{D}}\,\underline{\mathbf{W}}\,\underline{\widetilde{\mathbf{D}}}^T.$$

Proof: Straightforward computation under consideration of

$$\langle 2\mathbf{E}\widetilde{\mathbf{T}}, \mathbf{E}\mathbf{W}\widetilde{\mathbf{F}} \rangle = \mathbf{F}\widetilde{\mathbf{W}\widetilde{\mathbf{T}}} + \mathbf{T}\mathbf{W}\widetilde{\mathbf{F}}. \qquad \square$$

3c. Representation of Line Displacements

We characterize an oriented line by its footpoint $\mathbf{C}$ and by its direction, which can be written as a unit-quaternion $\mathbf{Y}$. Clearly this direction is transformed by an arbitrary displacement into $\mathbf{Y}' = \mathbf{E}\mathbf{Y}\widetilde{\mathbf{F}}$.

Now it only remains to calculate the footpoint $\mathbf{C}'$ of the displaced line, which is composed of the rotated footpoint $\mathbf{E}\mathbf{C}\widetilde{\mathbf{F}}$ plus the component of the translational vector orthogonal to $\mathbf{Y}'$. Thus we get:

$$\mathbf{C}' = \mathbf{E}\mathbf{C}\widetilde{\mathbf{F}} - \mathbf{E}\widetilde{\mathbf{T}} + \mathbf{E}\mathbf{Y}\widetilde{\mathbf{F}}\mathbf{T}\mathbf{Y}\widetilde{\mathbf{F}}.$$

Due to the last term we do not represent the line by the pair $(\mathbf{Y}, \mathbf{C})$, but by $(\mathbf{Y}, \widetilde{\mathbf{Y}}\mathbf{C})$ as the following holds:

$$\widetilde{\mathbf{Y}}'\mathbf{C}' = \mathbf{F}\widetilde{\mathbf{Y}}\mathbf{C}\widetilde{\mathbf{F}} - \mathbf{F}\widetilde{\mathbf{Y}}\widetilde{\mathbf{T}} + \mathbf{T}\mathbf{Y}\widetilde{\mathbf{F}}.$$

3c. Representation of Line Displacements

Remark: Note that in the 3-dimensional case $\widetilde{\mathbf{Y}}\mathbf{C}$ equals the moment vector. Thus $(\mathbf{Y}, \widetilde{\mathbf{Y}}\mathbf{C})$ is the 4-dimensional analogue of the spear coordinates of E^3. As $\widetilde{\mathbf{Y}}\mathbf{C}$ is a pure quaternion, the spear coordinates $(\mathbf{Y}, \widetilde{\mathbf{Y}}\mathbf{C})$ of E^4 have 7 entries. $\diamond$

The following notation is needed for the formulation of the next theorem:

$$\underline{\mathbf{Y}} = \begin{pmatrix} \mathbf{O} & \mathbf{Y} \\ -\widetilde{\mathbf{Y}} & \widetilde{\mathbf{Y}}\mathbf{C} \end{pmatrix}, \qquad \underline{\mathbf{Y}}' = \begin{pmatrix} \mathbf{O} & \mathbf{Y}' \\ -\widetilde{\mathbf{Y}}' & \widetilde{\mathbf{Y}}'\mathbf{C}' \end{pmatrix}.$$

Theorem 4. The mapping of oriented lines $(\mathbf{Y}, \widetilde{\mathbf{Y}}\mathbf{C})$ of E^4 to oriented lines $(\mathbf{Y}', \widetilde{\mathbf{Y}}'\mathbf{C}')$ of E^4 induced by any element of SE(4) can be written as follows:

$$\underline{\mathbf{Y}} \mapsto \underline{\mathbf{Y}}' \quad \textit{with} \quad \underline{\mathbf{Y}}' := \underline{\mathbf{D}}\,\underline{\mathbf{Y}}\,\underline{\widetilde{\mathbf{D}}}^T.$$

Proof: Straightforward computation. $\square$

3d. Representation of Plane Displacements

We describe a finite plane by a finite point $\mathbf{X}$ and two unit-vectors, which are orthogonal to each other. Based on the corresponding unit-quaternions $\mathbf{Y}$ and $\mathbf{Z}$ we can compute the oriented Plücker coordinates $(\bar{\mathbf{l}}, \widehat{\mathbf{l}})$ of the planes ideal line oriented from the ideal point in direction $\mathbf{Y}$ to the ideal point in direction $\mathbf{Z}$ as

$$\bar{\mathbf{l}} := \tfrac{1}{2}(\mathbf{Z}\widetilde{\mathbf{Y}} + \widetilde{\mathbf{Y}}\mathbf{Z}), \quad \widehat{\mathbf{l}} := \tfrac{1}{2}(\mathbf{Z}\widetilde{\mathbf{Y}} - \widetilde{\mathbf{Y}}\mathbf{Z})$$

with respect to the ideal 3-space according to Müller [4].

Moreover the ideal 3-space is the elliptic space described in [4], where oriented lines can alternatively be described by their left and right direction vectors

$$\mathbf{l}_+ = \bar{\mathbf{l}} + \widehat{\mathbf{l}} = \mathbf{Z}\widetilde{\mathbf{Y}}, \qquad \mathbf{l}_- = \bar{\mathbf{l}} - \widehat{\mathbf{l}} = \widetilde{\mathbf{Y}}\mathbf{Z},$$

which are transformed by: $\quad \mathbf{l}_+ \mapsto \mathbf{l}'_+ = \mathbf{E}\mathbf{l}_+\widetilde{\mathbf{E}}, \qquad \mathbf{l}_- \mapsto \mathbf{l}'_- = \mathbf{F}\mathbf{l}_-\widetilde{\mathbf{F}}.$

3d. Representation of Plane Displacements

Clearly one can compute the 10 Grassmann coordinates of the plane, which can abstractly be represented by the quaternionic triple $(\bar{\mathbf{l}} : \widehat{\mathbf{l}} : \mathbf{L})$ with

$$\mathbf{L} := \tfrac{1}{2}(\mathbf{l}_{+}\mathbf{X} - \mathbf{X}\mathbf{l}_{-}).$$

In order to avoid the loss of the information on the plane's orientation we can use normalized Grassmann coordinates, where the normalization is done with respect to the Plücker coordinates of the ideal line. As already $l_{01}^2 + l_{02}^2 + l_{03}^2 + l_{23}^2 + l_{31}^2 + l_{12}^2 = 1$ holds, the normalized Grassmann coordinates of the plane can be written as $(\bar{\mathbf{l}}, \widehat{\mathbf{l}}, \mathbf{L})$.

Under a displacement the quaternion $\mathbf{L}$ is transformed to $\mathbf{L}'$ with:

$$\mathbf{L}' = \tfrac{1}{2}(\mathbf{l}'_{+}\mathbf{X}' - \mathbf{X}'\mathbf{l}'_{-}) = \tfrac{1}{2}(\mathbf{E}\mathbf{Z}\widetilde{\mathbf{Y}}\mathbf{X}\widetilde{\mathbf{F}} - \mathbf{E}\mathbf{X}\widetilde{\mathbf{Y}}\mathbf{Z}\widetilde{\mathbf{F}}) - \mathbf{E}\mathbf{Z}\widetilde{\mathbf{Y}}\widetilde{\mathbf{T}} + \mathbf{E}\widetilde{\mathbf{T}}\mathbf{F}\widetilde{\mathbf{Y}}\mathbf{Z}\widetilde{\mathbf{F}}.$$

3d. Representation of Plane Displacements

Instead of the triple $(\bar{\mathbf{l}}, \widehat{\mathbf{l}}, \mathbf{L})$ we can also use the representation $(\mathbf{l}_+, \mathbf{l}_-, \mathbf{L})$. This is the most suitable form for our purpose as we can state the following theorem under consideration of the notation:

$$\underline{\mathbf{L}} = \begin{pmatrix} -\mathbf{l}_+ & \mathbf{L} \\ \mathbf{O} & -\mathbf{l}_- \end{pmatrix}, \qquad \underline{\mathbf{L}}' = \begin{pmatrix} -\mathbf{l}'_+ & \mathbf{L}' \\ \mathbf{O} & -\mathbf{l}'_- \end{pmatrix}.$$

Theorem 5. The mapping of oriented planes $(\mathbf{l}_+, \mathbf{l}_-, \mathbf{L})$ of E^4 to oriented planes $(\mathbf{l}'_+, \mathbf{l}'_-, \mathbf{L}')$ of E^4 induced by any element of SE(4) can be written as follows:

$$\underline{\mathbf{L}} \mapsto \underline{\mathbf{L}}' \quad with \quad \underline{\mathbf{L}}' := \underline{\widetilde{\mathbf{D}}}^{-T} \underline{\mathbf{L}}\, \underline{\widetilde{\mathbf{D}}}^{T}.$$

Proof: Straightforward computation. □

4. Instantaneous Kinematics

We consider a constrained motion $(\mathbf{E}_\tau, \mathbf{F}_\tau, \mathbf{T}_\tau)$ in dependency of the time τ:

$$\mathbf{X}_\tau^\oplus = \mathbf{E}_\tau \mathbf{X} \widetilde{\mathbf{F}}_\tau - 2\mathbf{E}_\tau \widetilde{\mathbf{T}}_\tau \quad \Longleftrightarrow \quad \underline{\mathbf{X}}_\tau^\oplus = \underline{\widetilde{\mathbf{D}}}_\tau^{-T} \underline{\mathbf{X}}\, \underline{\widetilde{\mathbf{D}}}_\tau^{T}$$

$\mathbf{X}$ ... coordinates of X w.r.t. the moving frame $\mathcal{C}$

$\mathbf{X}_\tau^\oplus$ ... coordinates of X w.r.t. the fixed frame $\mathcal{C}^\oplus$ in dependency of the time τ

We change the fixed frame from the old $\mathcal{C}^\oplus$ into the new one $\mathcal{C}^\otimes$ in a way that at the time instance $\tau = *$ the moving frame $\mathcal{C}$ and $\mathcal{C}^\otimes$ coincide. This is achieved by:

$$\underline{\mathbf{X}}_\tau^\otimes = \underline{\widetilde{\mathbf{D}}}_*^{T} \underline{\mathbf{X}}_\tau^\oplus \underline{\widetilde{\mathbf{D}}}_*^{-T} = \underline{\widetilde{\mathbf{B}}}_\tau^{-T} \underline{\mathbf{X}}\, \underline{\widetilde{\mathbf{B}}}_\tau^{T} \quad \Longleftrightarrow \quad \mathbf{X}_\tau^\otimes = \mathbf{G}_\tau \mathbf{X} \widetilde{\mathbf{H}}_\tau - 2\mathbf{G}_\tau \widetilde{\mathbf{U}}_\tau$$

$$\text{with} \quad \underline{\mathbf{B}}_\tau = \begin{pmatrix} \mathbf{G}_\tau & \mathbf{O} \\ \mathbf{U}_\tau & \mathbf{H}_\tau \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{E}}_* \mathbf{E}_\tau & \mathbf{O} \\ -\widetilde{\mathbf{F}}_* \mathbf{T}_* \widetilde{\mathbf{E}}_* \mathbf{E}_\tau + \widetilde{\mathbf{F}}_* \mathbf{T}_\tau & \widetilde{\mathbf{F}}_* \mathbf{F}_\tau \end{pmatrix}.$$

4. Instantaneous Kinematics

The time derivative of the normalizing condition $\mathbf{G}_\tau\widetilde{\mathbf{G}}_\tau = 1$ and the equation $\mathbf{G}_\tau\widetilde{\mathbf{G}}_\tau - \mathbf{H}_\tau\widetilde{\mathbf{H}}_\tau = 0$ of Ξ with respect to τ yields:

$$\dot{\mathbf{G}}_\tau\widetilde{\mathbf{G}}_\tau + \mathbf{G}_\tau\dot{\widetilde{\mathbf{G}}}_\tau = 0 \quad \text{and} \quad \dot{\mathbf{G}}_\tau\widetilde{\mathbf{G}}_\tau + \mathbf{G}_\tau\dot{\widetilde{\mathbf{G}}}_\tau - \dot{\mathbf{H}}_\tau\widetilde{\mathbf{H}}_\tau - \mathbf{H}_\tau\dot{\widetilde{\mathbf{H}}}_\tau = 0.$$

Evaluation of these formulas at $\tau = *$ implies $\dot{\mathbf{G}}_* = \dot{\mathbf{g}}_*$ and $\dot{\mathbf{H}}_* = \dot{\mathbf{h}}_*$.

Moreover by the differentiation of $\underline{\mathbf{X}}_\tau^\otimes$ and $\mathbf{X}_\tau^\otimes$, respectively, with respect to τ and its evaluation at $\tau = *$ yields:

$$\dot{\underline{\mathbf{X}}}_*^\otimes = \underline{\dot{\widetilde{\mathbf{B}}}}_*^{-T}\underline{\mathbf{X}} + \underline{\mathbf{X}}\,\underline{\dot{\widetilde{\mathbf{B}}}}_*^{T} \quad \Longleftrightarrow \quad \dot{\mathbf{X}}_*^\otimes = \dot{\mathbf{g}}_*\mathbf{X} - \mathbf{X}\dot{\mathbf{h}}_* - 2\dot{\widetilde{\mathbf{U}}}_*.$$

It can easily be checked that the affine mapping $\mathbf{X} \mapsto \dot{\mathbf{X}}_*^\otimes$ is singular if and only if $\dot{\mathbf{g}}_*\dot{\mathbf{g}}_* - \dot{\mathbf{h}}_*\dot{\mathbf{h}}_* = 0$ holds, which implies the following notation:

4. Instantaneous Screw

Definition 1. The triple $(\dot{\mathbf{g}}_*, \dot{\mathbf{h}}_*, \dot{\mathbf{U}}_*)$ is called the instantaneous screw $\$_*^{\otimes}$ of the motion $(\mathbf{G}_\tau, \mathbf{H}_\tau, \mathbf{U}_\tau)$ at time instance $\tau = *$ with respect to the fixed system $\mathcal{C}^{\otimes}$. $\$_*^{\otimes}$ is called singular if $\dot{\mathbf{g}}_*\dot{\mathbf{g}}_* - \dot{\mathbf{h}}_*\dot{\mathbf{h}}_* = 0$ holds; otherwise regular.

Remark: Note that in the singular case $\$_*^{\otimes}$ is located on Ξ. $\diamond$

Direct computation shows that $\$_*^{\otimes}$ is transformed into $\$_*^{\oplus} = (\dot{\mathbf{g}}_*^{\oplus}, \dot{\mathbf{h}}_*^{\oplus}, \dot{\mathbf{U}}_*^{\oplus})$ by:

$$\underline{\$}_*^{\oplus} = \underline{\widetilde{\mathbf{D}}}_*^{-T}\, \underline{\$}_*^{\otimes}\, \underline{\widetilde{\mathbf{D}}}_*^{T} \quad \text{with} \quad \underline{\$}_*^{\oplus} = \begin{pmatrix} -\dot{\mathbf{g}}_*^{\oplus} & \dot{\widetilde{\mathbf{U}}}_*^{\oplus} \\ \mathbf{O} & \dot{\mathbf{h}}_*^{\oplus} \end{pmatrix}, \quad \underline{\$}_*^{\otimes} = \begin{pmatrix} -\dot{\mathbf{g}}_* & \dot{\widetilde{\mathbf{U}}}_* \\ \mathbf{O} & \dot{\mathbf{h}}_* \end{pmatrix}.$$

Theorem 6. The mapping of an instantaneous screw $\$$ of E^4 to an instantaneous screw $\$'$ of E^4 induced by any element of SE(4) can be written as follows:

$$\underline{\$} \mapsto \underline{\$}' \quad \textit{with} \quad \underline{\$}' := \underline{\widetilde{\mathbf{D}}}^{-T}\, \underline{\$}\, \underline{\widetilde{\mathbf{D}}}^{T}.$$

4. Geometric Parameters of $\$_*^{\otimes}$

The quaternionic formulation also provides easy access to the geometric parameters of $\$_*^{\otimes} = (\dot{\mathbf{g}}_*, \dot{\mathbf{h}}_*, \dot{\mathbf{U}}_*)$, which is demonstrated for the most general case:

$\$_*^{\otimes}$ regular and $\dot{\mathbf{g}}_* \neq \mathbf{o} \neq \dot{\mathbf{h}}_*$: Instantaneously we have a double-rotation with angular velocities:

$$\omega_1 = \|\dot{\mathbf{g}}_*\| + \|\dot{\mathbf{h}}_*\|, \qquad \omega_2 = \|\dot{\mathbf{g}}_*\| - \|\dot{\mathbf{h}}_*\|,$$

about the oriented total-orthogonal planes ε_1 and ε_2 given in the form $(\mathbf{l}_+, \mathbf{l}_-, \mathbf{L})$:

$$\varepsilon_1 : \left(\dot{\mathbf{g}}_0, \dot{\mathbf{h}}_0, \tfrac{1}{2}(\dot{\mathbf{g}}_0\mathbf{P} - \mathbf{P}\dot{\mathbf{h}}_0)\right), \qquad \varepsilon_2 : \left(\dot{\mathbf{g}}_0, -\dot{\mathbf{h}}_0, \tfrac{1}{2}(\dot{\mathbf{g}}_0\mathbf{P} + \mathbf{P}\dot{\mathbf{h}}_0)\right),$$

with the velocity pole $\mathbf{P} = 2\dfrac{\dot{\mathbf{g}}_*\dot{\widetilde{\mathbf{U}}}_* + \dot{\widetilde{\mathbf{U}}}_*\dot{\mathbf{h}}_*}{\dot{\mathbf{g}}_*\dot{\mathbf{g}}_* - \dot{\mathbf{h}}_*\dot{\mathbf{h}}_*}$ and $\dot{\mathbf{g}}_0 = \dfrac{\dot{\mathbf{g}}_*}{\|\dot{\mathbf{g}}_*\|}$, $\dot{\mathbf{h}}_0 = \dfrac{\dot{\mathbf{h}}_*}{\|\dot{\mathbf{h}}_*\|}$.

5. Conclusion and Outlook

Based on the quaternionic kinematic mapping of SE(4) we showed that the displacement of basic geometric elements in E^4 can be treated in a unified way using the compact notation of 2×2 quaternionic matrices:

- oriented hyperplanes, oriented lines: $\underline{*} \mapsto \underline{\mathbf{D}}\,\underline{*}\,\underline{\widetilde{\mathbf{D}}}^T$
- points, oriented planes, instantaneous screws: $\underline{*} \mapsto \underline{\widetilde{\mathbf{D}}}^{-T}\,\underline{*}\,\underline{\widetilde{\mathbf{D}}}^T$

Note that the algebra of 2×2 quaternionic matrices is isomorphic to the Clifford algebra $(1_+, 3_-, 0_0)$, which shows again the difference to the Klawitter-Hagemann construction based on the Spin group of the Clifford algebra $(4_+, 0_-, 1_0)$.

The presented quaternionic approach is also well suited for studying equiform kinematics of E^4 and its relation to the geometry of line-elements (cf. [5]).

5. References

The presented results are contained in:

NAWRATIL G. Fundamentals of quaternionic kinematics in Euclidean 4-space. submitted (2015)

The list of referred publications:

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[3] KLAWITTER D., HAGEMANN M. Kinematic mappings for Cayley-Klein geometries via Clifford algebras. Beiträge zur Algebra und Geometrie **54** (2) 737–761 (2013)

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