On the set of oriented line-elements: point-models, metrics and applications

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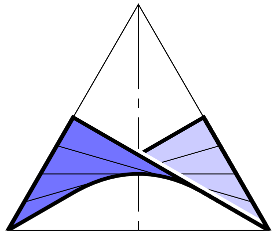

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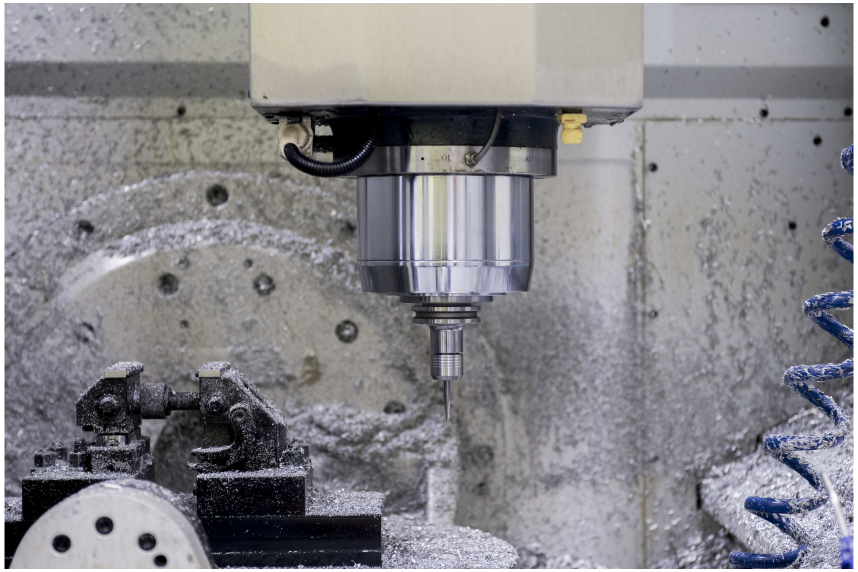
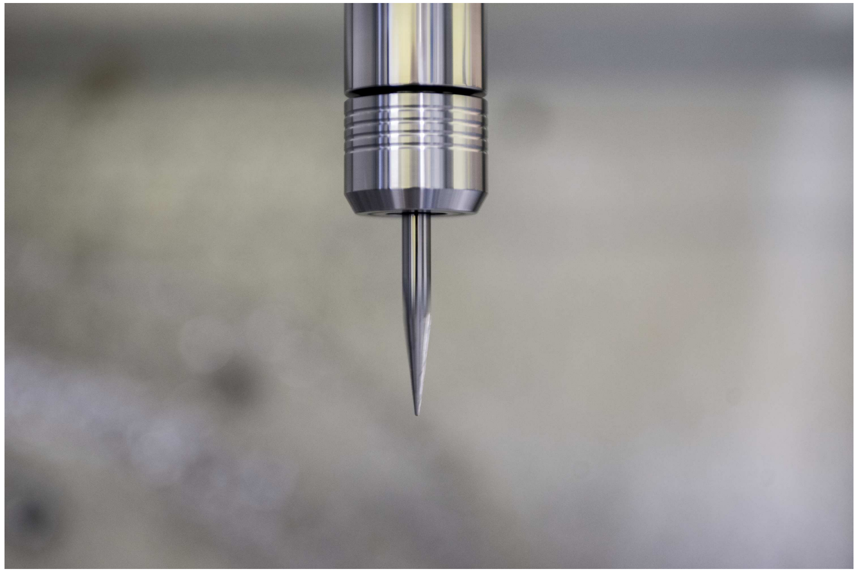
Introduction & Motivation



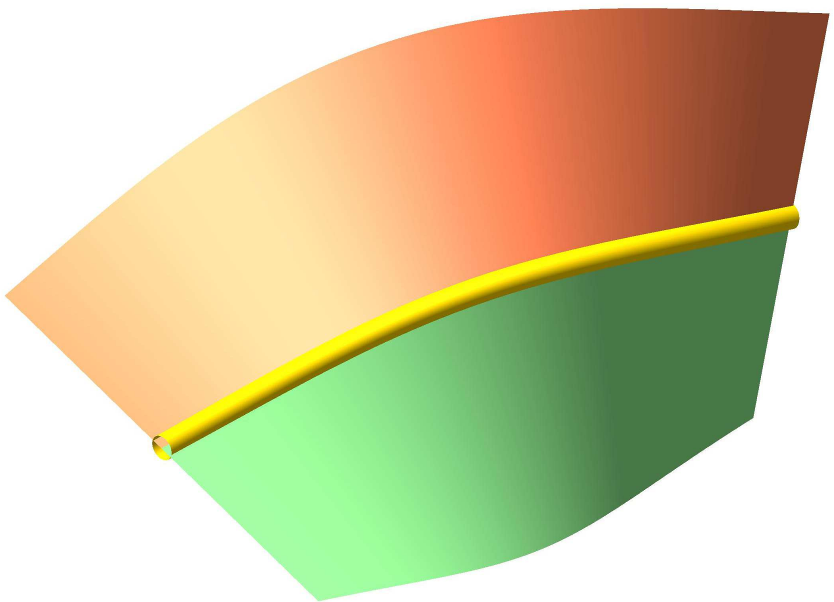


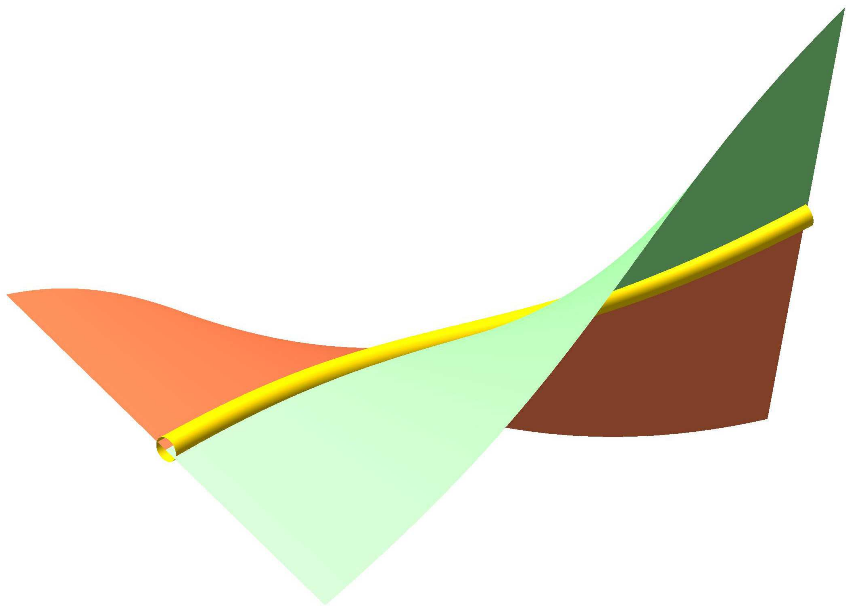
Both pictures by courtesy of F. Rist (TU Vienna, Department for 3D Design and Model Making).

For a large number of applications in robotics the *end-effector* has a rotational symmetry; e.g. milling, spot-welding, laser or water-jet engraving/cutting, etc.

Introduction & Motivation

For the determination of an axial symmetric task the rotation axis a of the tool is of importance as well as the location of the *tool tip* A. In addition, the orientation of the line a has to be taken into account ($\Rightarrow \vec{a}$).

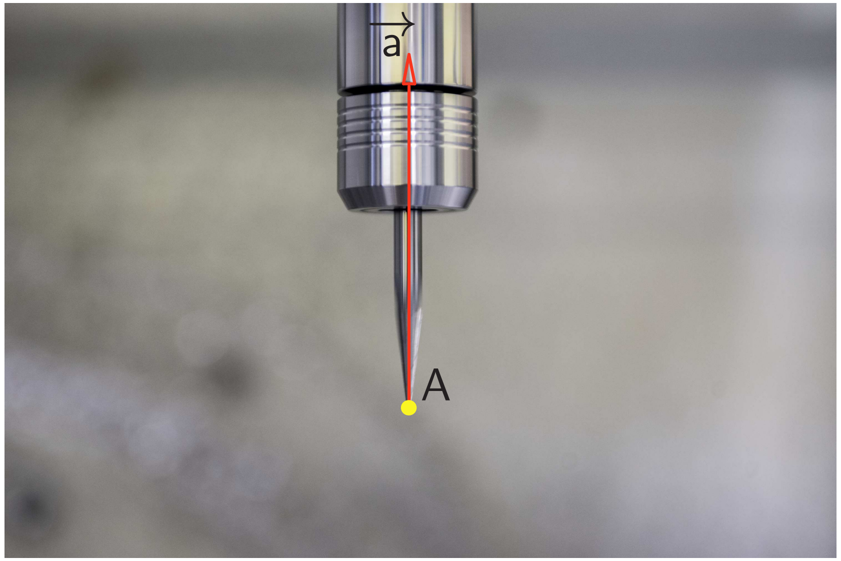




Introduction & Motivation

The two geometric objects A and $\vec{a}$ can be combined to a so-called oriented line-element $(A, \vec{a})$, which is also known as:

- oriented pointed line (e.g. Selig [1])
- point-line (e.g. Zhang & Ting [2])
- point dirigé (e.g. De Saussure [3])



In the Euclidean plane two oriented line-elements can be transformed uniquely into each other by a planar displacement. Therefore the set of oriented line-elements of $\mathbb{R}^2$ is isomorphic to the group of planar Euclidean displacements.

The kinematic mapping of Blaschke [8] and Grünwald [9] implies a point-model for the set of oriented line-elements of $\mathbb{R}^2$.

Introduction & Motivation

We are interested in point-models for the set $\overrightarrow{\mathcal{L}}$ of oriented line-elements of $\mathbb{R}^3$, which can be used for the motion design based on well-known methods for curves. This approach is a standard technique for designing Euclidean motions [10–13].

Therefore our point-model $\mathcal{P}$ should have the follow three properties:

- P1 The point-model $\mathcal{P}$ is an algebraic variety.
- P2 The underlying kinematic mapping $\overrightarrow{\mathcal{L}} \to \mathcal{P}$ is a bijection.
- P3 A change of the moving and the fixed frame implies a linear transformation of the point-model $\mathcal{P}$.

Remark: Due to the demand P3 linear curve design algorithms remain invariant under the choice of the fixed and moving frame. $\diamond$

Outline of the Talk

1. **Point-models possessing P1–P3**
   (a) Point-model based on rigid-body motions
   (b) Point-models based on representations
2. **Metric aspects**
3. **Application examples**
   (a) Interpolation by variational motion design
   (b) Motion design by De Casteljau's algorithm
   (c) Closeness to singularities in robotics

1(a) Point-model based on rigid-body motions

We consider the set $\mathcal{D}$ of $\textsc{Euclidean}$ displacements SE(3), which map one oriented line-element $(\mathrm{B}, \overrightarrow{\mathrm{b}})$ into another one $(\mathrm{A}, \overrightarrow{\mathrm{a}})$. Clearly, $\mathcal{D}$ is a 1-dimensional set.

Remark: According to [17,18] it is an incompletely specified displacement. $\diamond$

It is well-known [15] that $\mathcal{D}$ corresponds to a line in the $\textsc{Study}$ quadric. Therefore we can compute the $\textsc{Grassmann}$ coordinates of these lines, which imply the following point-model (for details see [Naw]):

Theorem 1. There exists a bijection between $\overrightarrow{\mathcal{L}}$ and all real points of the 15-dimensional projective space $\mathbb{P}^{15}$ located on the 5-dimensional variety of degree 20, which is sliced along a hyperplane.

1(b) Point-models based on representations

ODEHNAL, POTTMANN, WALLNER [35] studied unoriented line-elements of $\mathbb{R}^3$. Their result can be adapted for oriented ones by adding a normalization condition:

Theorem 2. There exists a bijection between $\overrightarrow{\mathcal{L}}$ and all real points $(\mathbf{a}, \widehat{\mathbf{a}}, a)$ of the 7-dimensional space $\mathbb{R}^7$ located on the 5-dimensional quartic variety given by:

$$\langle \mathbf{a}, \mathbf{a} \rangle = 1, \quad \langle \mathbf{a}, \widehat{\mathbf{a}} \rangle = 0.$$

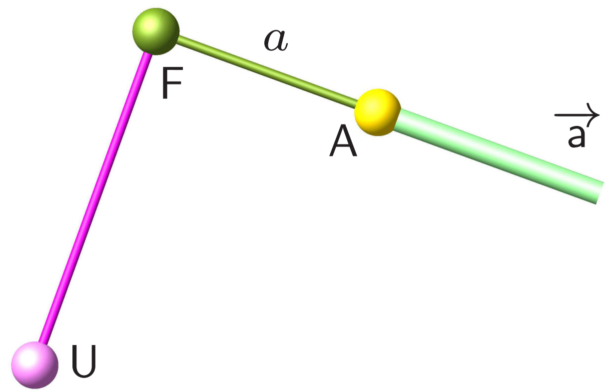
$\mathbf{a}$ ... direction vector of the oriented line $\overrightarrow{\mathsf{a}}$

$\widehat{\mathbf{a}}$ ... moment vector $\mathbf{A} \times \mathbf{a}$ of $\overrightarrow{\mathsf{a}}$

$\mathbf{A}$ ... position vector of A $\in$ a

a ... oriented distance $\overline{\mathsf{FA}}$ w.r.t. $\overrightarrow{\mathsf{a}}$

F ... pedal point of a w.r.t origin U



1(b) Point-models based on representations

Based on CLIFFORD algebras [1,15,30], oriented line-elements are just represented by combining

- points (grade 4 elements) with
- oriented lines (grade 2 elements)

under the side condition that the point is located on the oriented line.

This is similar to the approach of ODEHNAL [31] taken for characterizing unoriented line-elements of $\mathbb{P}^3$. Therefore, these two approaches imply the same point-model:

Theorem 3. There exists a bijection between $\overrightarrow{\mathcal{L}}$ and all real points $(\mathbf{a}, \widehat{\mathbf{a}}, \mathbf{A})$ of the 9-dimensional space $\mathbb{R}^9$ located on the 5-dimensional variety of degree 10 given by

$$\langle \mathbf{a}, \mathbf{a} \rangle = 1, \quad \langle \mathbf{a}, \widehat{\mathbf{a}} \rangle = 0, \quad \langle \mathbf{A}, \widehat{\mathbf{a}} \rangle = 0, \quad \mathbf{A} \times \mathbf{a} = \widehat{\mathbf{a}}.$$

1(b) Point-models based on representations

The most intuitive approach for representing an oriented line-element is just to combine the point coordinates $\mathbf{A}$ and the unit-direction-vector $\mathbf{a}$ of $\overrightarrow{\mathrm{a}}$.

Theorem 4. There exists a bijection between $\overrightarrow{\mathcal{L}}$ and all real points $(\mathbf{a}, \mathbf{A})$ of the 6-dimensional space $\mathbb{R}^6$ located on the singular quadric

$$\langle \mathbf{a}, \mathbf{a} \rangle = 1.$$

- Zhang & Ting [2] represented oriented line-elements by $(\mathbf{a}, \widehat{\mathbf{a}} + a\mathbf{a})$.
- Combebiac [32] used the description $(\mathbf{a}, \widehat{\mathbf{a}} + \mathbf{A})$.

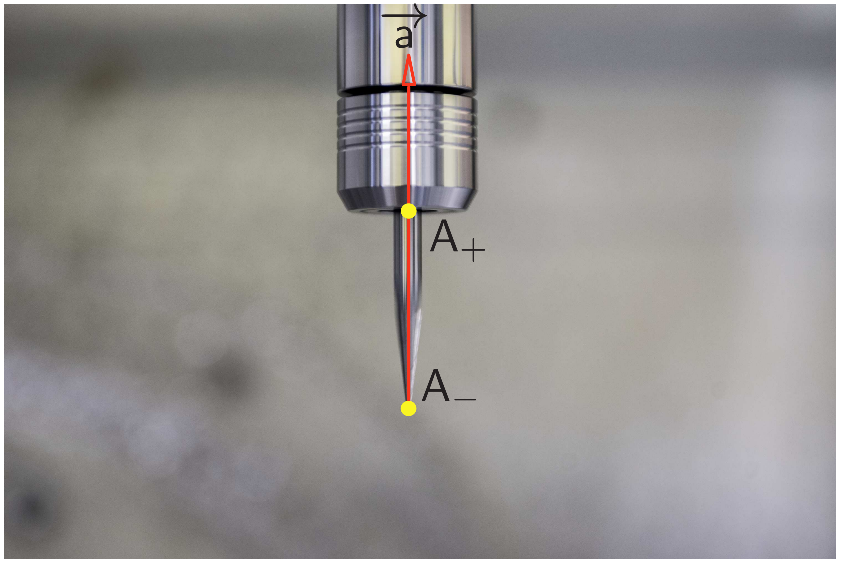
These two representations have also the singular quadric of Theorem 4 as point-model. But the transform between these three point-models is in all three cases a non-linear one.

1(b) Point-models based on representations

We represent an oriented line-element by an oriented line-segment with a constant length d given by an ordered pair $(\mathsf{A}_-, \mathsf{A}_+)$ of points with $\overline{\mathsf{A}_-\mathsf{A}_+} = d$. From the applicational point of view two possibilities are reasonable:

1. If the rotational end-effector has a second remarkable point beside the tool tip A, then these two points can be regarded as A_+ and A_-.

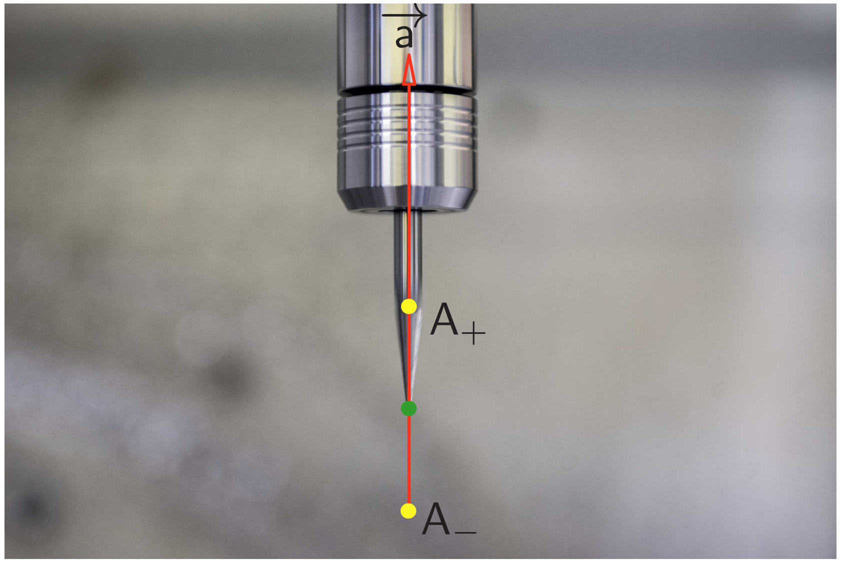
   Example: If the end-effector is a miller, then the second endpoint can be considered as A_+.



1(b) Point-models based on representations

2. One can select A_- and A_+ in a way on $\vec{\mathsf{a}}$ that A is their midpoint. In this case we still have the free choice of d.

   Remark: In the remainder of the talk we assume this point of view. $\diamond$



CHEN & POTTMANN [35] represented a line-segment by their endpoints $(\mathbf{A}_-, \mathbf{A}_+)$. This implies the following point-model:

Theorem 5. There exists a bijection between $\vec{\mathcal{L}}$ and all real points $(\mathbf{A}_-, \mathbf{A}_+)$ of the 6-dimensional space $\mathbb{R}^6$ located on the singular hyperquadric

$$\Omega: \quad < \mathbf{A}_- - \mathbf{A}_+, \mathbf{A}_- - \mathbf{A}_+\rangle = d^2.$$

2 Metric aspects

It is desirable for path planning in robotics (e.g. approximation, interpolation, optimization, . . .) to have a metric f on $\overrightarrow{\mathcal{L}}$.

One can come up with the idea to base a distance measure on EUCLIDEAN displacements transforming $(\mathsf{B}, \overrightarrow{\mathsf{b}}) \mapsto (\mathsf{A}, \overrightarrow{\mathsf{a}})$. But distance metrics on SE(3) are quite problematic as they depend on the choice of length and angle scales (cf. [37]).

Instead of a distance metric on SE(3) one can consider the distance between two poses of the same rigid body, which yields *object dependent metrics*.

This interpretation suggests to consider an oriented line-element as an oriented line-segment with a constant length d.

2 Metric of Kazerounian & Rastegar

The metric proposed by KAZEROUNIAN & RASTEGAR [38] modified for line-segments $(\mathbf{A}_-, \mathbf{A}_+)$ and $(\mathbf{B}_-, \mathbf{B}_+)$ equals

$$f_1 = \sqrt{\text{mean of the squared distances of corresponding points over the entire line-segment}}$$

$$f_1^2 = \tfrac{1}{3}\left[(\mathbf{A}_- - \mathbf{B}_-)^2 + (\mathbf{A}_+ - \mathbf{B}_+)^2 + (\mathbf{A}_- - \mathbf{B}_-)(\mathbf{A}_+ - \mathbf{B}_+)\right].$$

This metric can also be extended to the ambient space $\mathbb{R}^6$ ($\widehat{=}$ line-segments of different lengths) of the point-model Ω according to CHEN & POTTMANN [35]. Note that f_1 implies a EUCLIDEAN metric in the ambient space $\mathbb{R}^6$.

Remark: This metric has been used on [39] for optimizing 5-axis machining. $\diamond$

2 Metric of Pottmann, Hofer, Ravani

Basic idea: One samples a number n of points $\mathsf{X}_1, \ldots, \mathsf{X}_n$ from the surface of the moving object and defines the squared distance between two of its poses by the sum of the squared distances of the n corresponding point pairs (cf. [40]).

Remark: As this distance strongly depends on the number n of points we suggest to divide the sum by n. ◇

As in our case the rigid body is only 1-dimensional, its boundary is just given by the two end points $(\mathbf{A}_-, \mathbf{A}_+)$ and $(\mathbf{B}_-, \mathbf{B}_+)$, respectively, which yields:

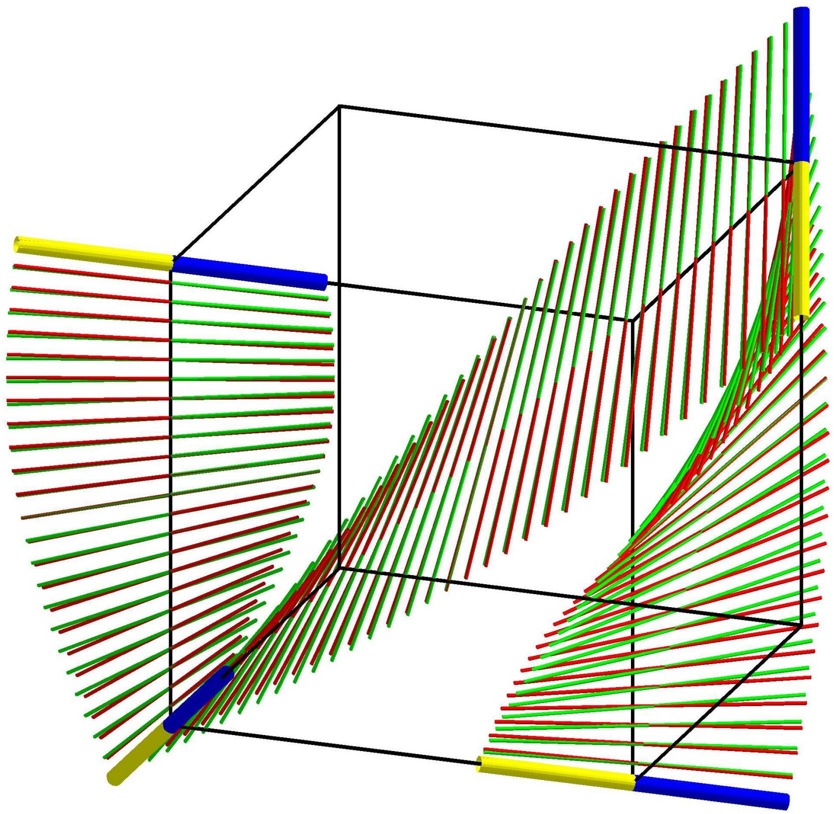
$$f_2^2 = \tfrac{1}{2}\left[(\mathbf{A}_- - \mathbf{B}_-)^2 + (\mathbf{A}_+ - \mathbf{B}_+)^2\right].$$

This metric can also be extended from Ω to the ambient space $\mathbb{R}^6$. $\mathbb{R}^6$ equipped with f_2 is again a Euclidean space.

3(a) Interpolation by variational motion design

The variational motion design algorithm of [40] can be adapted to the path planing of oriented line-elements [NAW], based on the object depended metrics in the ambient space $\mathbb{R}^6$ of Ω.

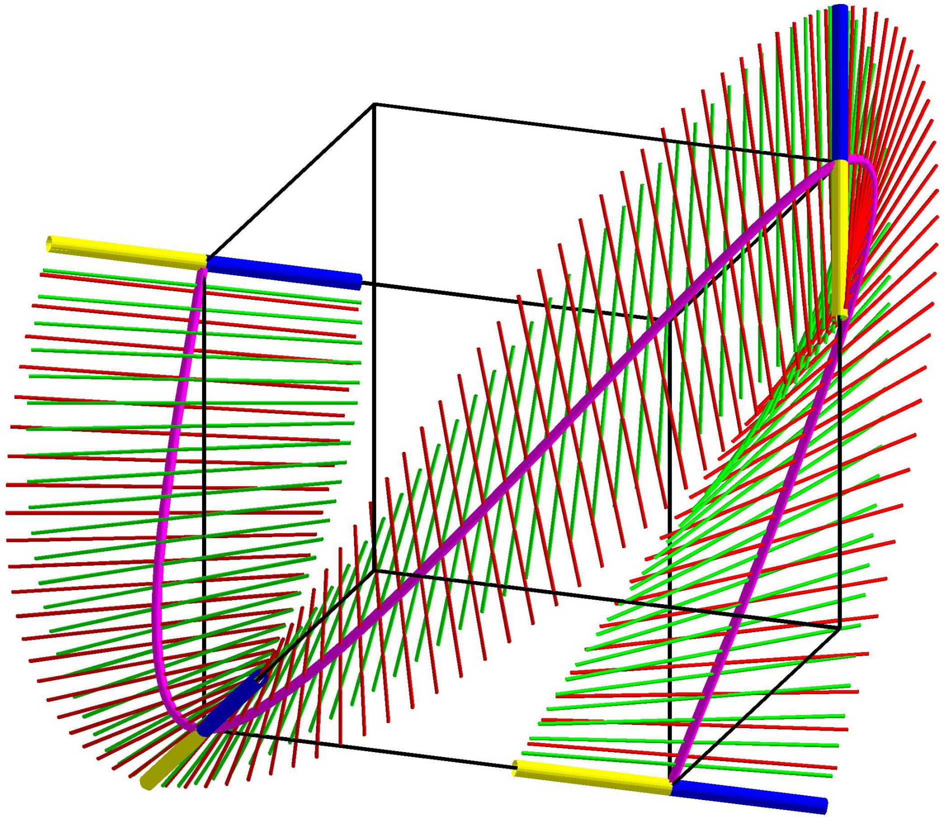
The corresponding points of the four given poses in $\mathbb{R}^6$ are interpolated by three line-segments. Their projection onto Ω is illustrated in red. The geodesic motion is displayed in green. In both cases the barycenter of the line-segment moves along a straight line between two given poses.



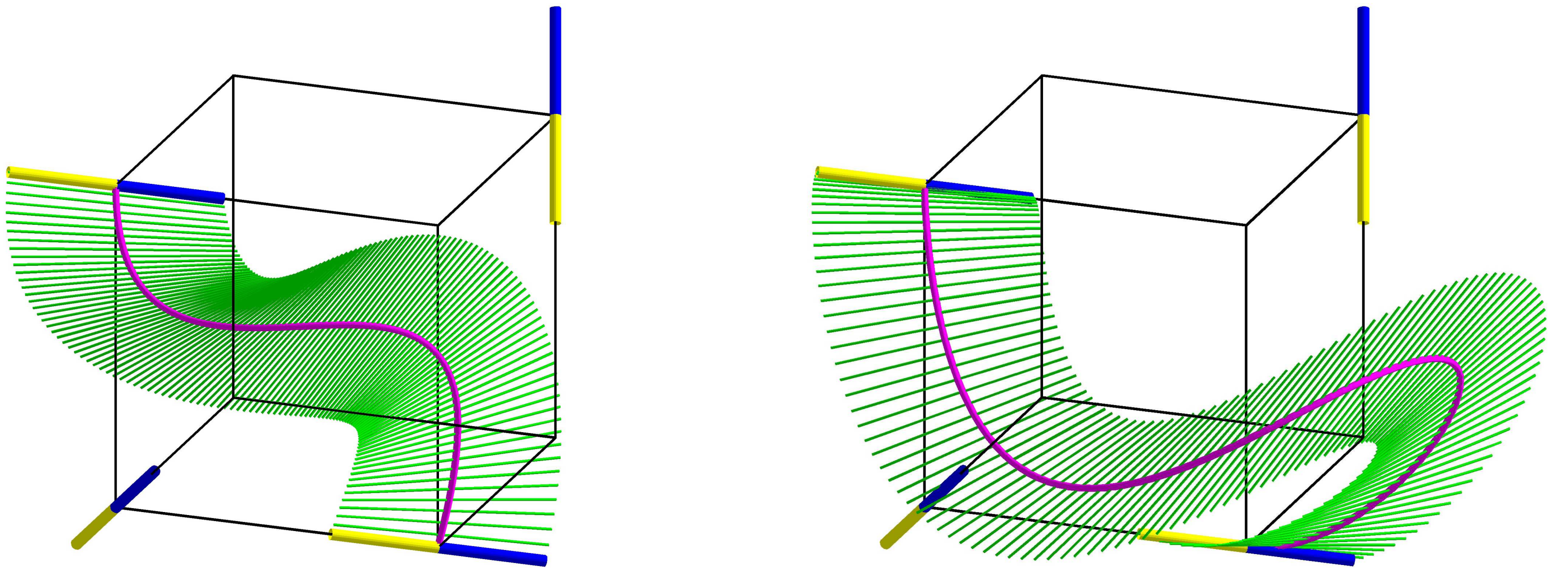
3(a) Interpolation by variational motion design

The interpolant with minimal bending energy E_b is displayed in red. The barycenter moves along a cubic C^2 spline (cf. [35,50]), which is illustrated as magenta-colored curve.

Moreover the minimizer of $E_b + 0.05E_g$ is illustrated in green where E_g denotes the energy-functional of the geodesic motion.

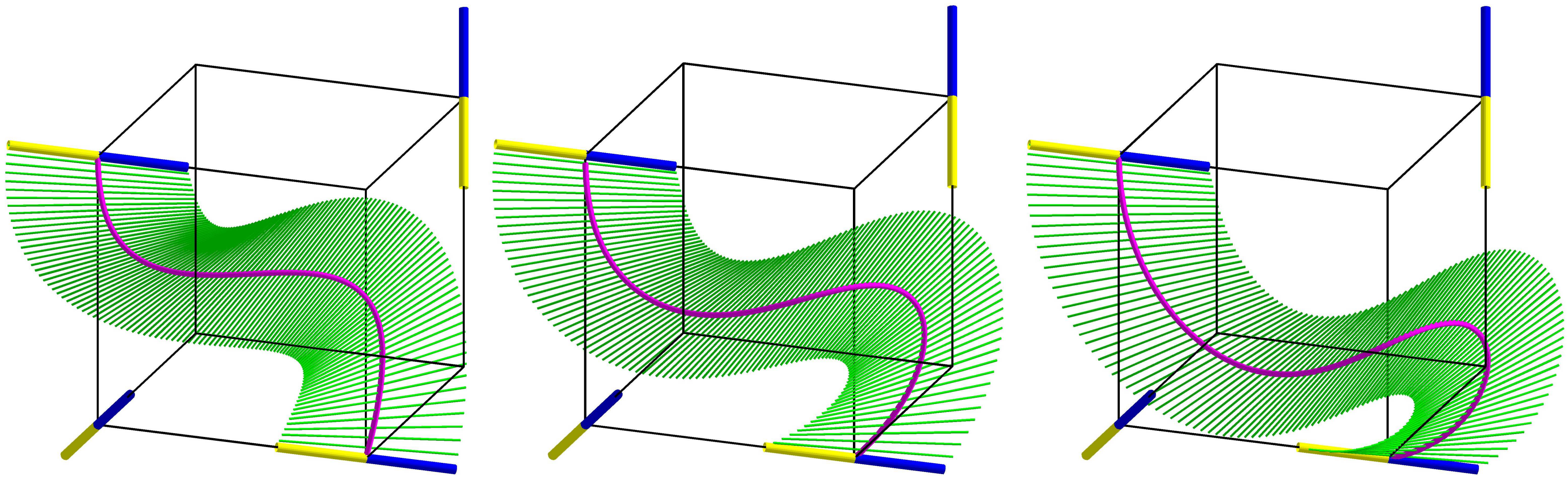


3(b) Motion design by De Casteljau's algorithm



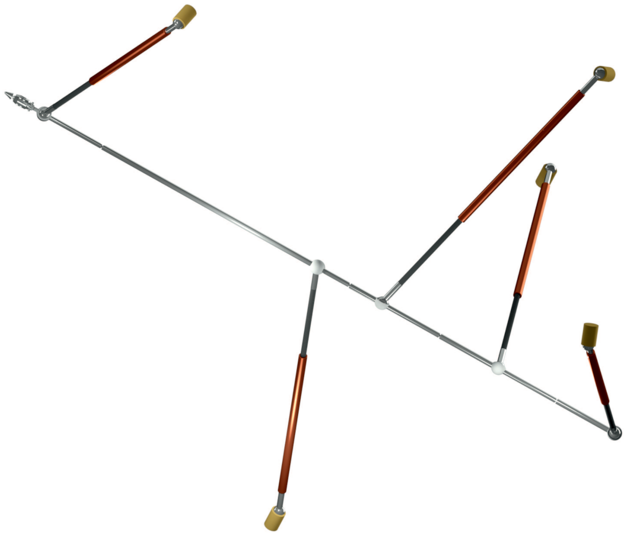
Projection algorithms: The Bézier curve is constructed in the ambient space of the point-model and then projected back onto it. Left: Point-model Ω in the ambient space $\mathbb{R}^6$. Right: Point-model is the 5-dimensional quartic variety of Theorem 2 and its ambient space is $\mathbb{R}^7$.

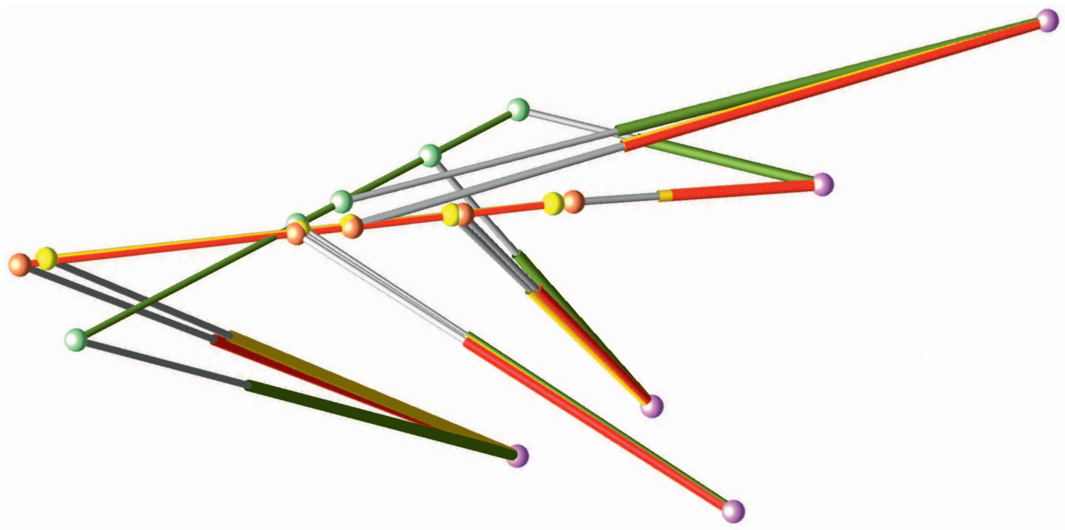
3(b) Motion design by De Casteljau's algorithm



Geodesic algorithms: The basic idea is to replace the straight line of the control polygon in the ambient space by their analog on the point-model; i.e. by geodesics. The result depends on the underlying geodesic motions [NAW].

3(c) Closeness to singularities in robotics





Left: Sketch of a linear pentapod; i.e. a pentapod with a linear platform. Right: A linear pentapod in the (green) given configuration and the (red) closest singular configuration (cf. [Raz]). The yellow configuration is the closest singularity under similarity transformations of the platform.

References

All references refer to the list of publications given in the article:

[Naw] Nawratil, G.: Point-models for the set of oriented line-elements – a survey. Mechanism and Machine Theory 111 118–134 (2017)

Moreover the following work has been cited:

[Raz] Rasoulzadeh, A. and Nawratil, G.: Rational Parametrization of Linear Pentapod's Singularity Variety and the Distance to it. Computational Kinematics (S. Zeghloul et al. eds.), Springer (2017) [Extended version on arXiv:1701.09107]

Thank you for the attention!