Parallel manipulators in terms of dual Cayley-Klein parameters

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Introduction & Motivation

A planar displacement of the EUCLIDEAN plane can be written as:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix}$$

$(x_0, y_0)^T$ ... coordinates of a point P with respect to the fixed frame

$(x, y)^T$ ... coordinates of a point P with respect to the moving frame

$(m, n)^T$ ... translation vector

φ ... angle of rotation

By interpreting the EUCLIDEAN plane as GAUSSIAN plane it can be rewritten as:

$$\underbrace{x_0 + y_0 i}_{p_0} = e^{i\varphi}(\underbrace{x + yi}_{p}) + (\underbrace{m + ni}_{\tau})$$

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In order to make this formulation algebraic, we replace EULER's formula $e^{i\varphi}$ by the complex number κ, which has to fulfill the normalizing condition $\kappa\overline{\kappa} = 1$; i.e.

$$p_0 = \kappa p + \tau \quad \text{with} \quad \kappa\overline{\kappa} = 1$$

$(p_0, \overline{p}_0) \ldots$ isotropic coordinates of a point P with respect to the fixed frame

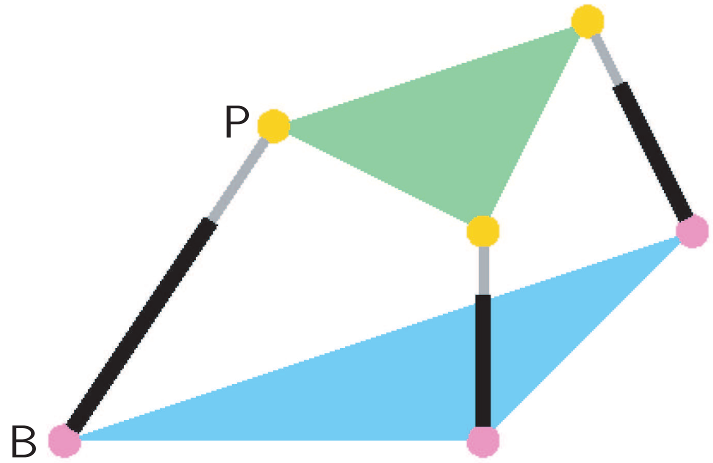
$(p, \overline{p}) \ldots$ isotropic coordinates of a point P with respect to the moving frame

$(\tau, \overline{\tau}) \ldots$ isotropic coordinates of the translation vector

Remark: A historical overview on planar kinematics based on isotropic coordinates is given by WAMPLER [21], who has also done most of the recent work on this topic (cf. [22] and all self-references therein). Further references and historical remarks are given by WUNDERLICH [23] under the nomenclature *minimal coordinates*. ◇

Introduction & Motivation

Based on this algebraic formulation we can derive the basic equation for planar parallel manipulators with RPR legs; namely the condition that a point P of the moving system is located on a circle with radius R centered at the point B with fixed coordinates $(u_0, v_0)^T$:



3-dof RPR planar parallel manipulator

$$(\kappa p + \tau - b_0)(\overline{\kappa}\,\overline{p} + \overline{\tau} - \overline{b}_0) - R^2 = 0$$

$(b_0, \overline{b}_0) \ldots$ isotropic coordinates of B with respect to the fixed system

Expanding this so-called circle condition shows that it is very compact formulation having 10 terms, but it is inhomogeneous quadratic in $\kappa, \overline{\kappa}, \tau, \overline{\tau}$.

Introduction & Motivation

A lot of recent publications [2,7,9,11,19] use a formulation of the circle condition in terms of Blaschke-Grünwald (BG) parameters, which has 26 terms. A motive for doing this is that one ends up with a homogenous quadratic equation in the BG parameters, thus methods of projective algebraic geometry can be applied.

This gives reason to ask for a formulation, which has both benefits (compactness and homogeneity). We present such a formulation as a special case of an approach taken for spatial kinematics. In detail the talk is structured as follows:

1. **Spherical displacements**
2. **Spatial displacements**
3. **Planar displacements**
4. **Application to parallel manipulators**

1. Spherical displacements: Quaternions

$\mathfrak{Q} := q_0 + q_1\mathfrak{i} + q_2\mathfrak{j} + q_3\mathfrak{k}$ ... quaternion with $q_0, \ldots, q_3 \in \mathbb{R}$

$\widetilde{\mathfrak{Q}} := q_0 - q_1\mathfrak{i} - q_2\mathfrak{j} - q_3\mathfrak{k}$ ... conjugated quaternion to $\mathfrak{Q}$

$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$ ... $\mathfrak{Q}$ is a unit-quaternion

It is well-known (e.g. [10]) that spherical displacements of points in the Euclidean 3-space can be expressed by unit-quaternions $\mathfrak{E} = e_0 + e_1\mathfrak{i} + e_2\mathfrak{j} + e_3\mathfrak{k}$ as follows:

$$\underbrace{x_0\mathfrak{i} + y_0\mathfrak{j} + z_0\mathfrak{k}}_{\mathfrak{P}_0} = \mathfrak{E} \circ (\underbrace{x\mathfrak{i} + y\mathfrak{j} + z\mathfrak{k}}_{\mathfrak{P}}) \circ \widetilde{\mathfrak{E}},$$

$\circ$ ... quaternion multiplication

$\mathfrak{P}$ ... embedding of point P with moving coordinates $(x, y, z)^T$ into $\mathbb{H}$

$\mathfrak{P}_0$ ... embedding of point P with fixed coordinates $(x_0, y_0, z_0)^T$ into $\mathbb{H}$

1. Spherical displacements: SU(2)

By using unitary 2×2 matrices for the representation of quaternions:

$$1 \widehat{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{i} \widehat{=} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathfrak{j} \widehat{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathfrak{k} \widehat{=} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

the spherical displacement of points based on EULER parameters $(e_0, \ldots, e_3)\mathbb{R}$; i.e.

$$\mathfrak{P}_0 = \mathfrak{E} \circ \mathfrak{P} \circ \widetilde{\mathfrak{E}} \quad \text{with} \quad e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

can be rewritten as

$$\begin{pmatrix} -z_0 i & -\overline{p}_0 i \\ -p_0 i & z_0 i \end{pmatrix} = \begin{pmatrix} e_0 - e_3 i & -e_2 - e_1 i \\ e_2 - e_1 i & e_0 + e_3 i \end{pmatrix} \begin{pmatrix} -z i & -\overline{p} i \\ -p i & z i \end{pmatrix} \begin{pmatrix} e_0 + e_3 i & e_2 + e_1 i \\ -e_2 + e_1 i & e_0 - e_3 i \end{pmatrix}$$

Remark: The group SU(2) of unitary 2×2 matrices with determinant 1 is isomorphic to the group of unit-quaternions. $\diamond$

1. Spherical displacements: CK parameters

By multiplying both sides by the complex unit i and by introducing the Cayley-Klein (CK) parameters $\alpha := e_0 + e_3$i and $\beta := e_2 + e_1 i$ the following representation [17] is obtained:

$$\mathbf{P}_0 = \mathbf{E}\mathbf{P}\mathbf{E}^* \quad \text{with} \quad \alpha\overline{\alpha} + \beta\overline{\beta} = 1$$

and

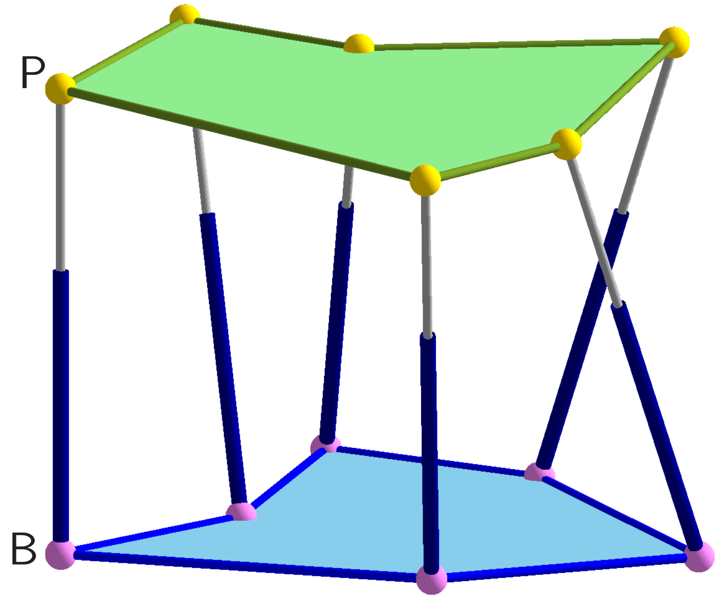
$$\mathbf{P}_0 = \begin{pmatrix} z_0 & \overline{p}_0 \\ p_0 & -z_0 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} z & \overline{p} \\ p & -z \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix}, \quad \mathbf{E}^* = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}.$$

Remarks:
- The upper index * denotes the transposed conjugate of a matrix.
- A historical overview and a detailed list of references on CK parameters is given in the recently published work [17].
- Note that we can compute $\|\mathbf{p}\|^2$ simply as $-\det \mathbf{P}$. ◇

2. Principle of Transference

Due to the "Principle of Transference", which dates back to KOTELNIKOV [12] and STUDY [20], the CK formulation of spherical displacements of points can also be applied to the spatial displacements of oriented lines by substituting complex numbers by dual complex numbers.

Up to the author's knowledge the resulting dual CK parameters have only be used for this purpose [3,17].



SG manipulator

For the study of STEWART-GOUGH (SG) platforms we are interested in the description of displacements of points in EUCLIDEAN 3-space in terms of dual CK parameters.

2. Spatial displacements: Dual quaternions

$$\varepsilon \ldots \text{dual unit with } \varepsilon^2 = 0$$

$$\mathfrak{E} + \varepsilon\mathfrak{T} \ldots \text{dual quaternion with } \mathfrak{T} = t_0 + t_1\mathfrak{i} + t_2\mathfrak{j} + t_3\mathfrak{k}$$

$\mathfrak{E} + \varepsilon\mathfrak{T} \ldots$ dual unit-quaternion iff

- $\mathfrak{E}$ is an unit-quaternion and
- the Study condition holds: $e_0t_0 + e_1t_1 + e_2t_2 + e_3t_3 = 0$.

Remark: $(e_0, \ldots, e_3, t_0, \ldots, t_3)\mathbb{R}$ are the so-called Study parameters. ◇

It is well-known (e.g. [10]) that displacements of points in the Euclidean 3-space can be expressed by dual unit-quaternions $\mathfrak{E} + \varepsilon\mathfrak{T}$ as follows:

$$1 + \varepsilon\mathfrak{P}_0 = (\mathfrak{E} + \varepsilon\mathfrak{T}) \circ (1 + \varepsilon\mathfrak{P}) \circ (\widetilde{\mathfrak{E}} - \varepsilon\widetilde{\mathfrak{T}}) = 1 + \varepsilon(\mathfrak{E} \circ \mathfrak{P} \circ \widetilde{\mathfrak{E}} + \mathfrak{T} \circ \widetilde{\mathfrak{E}} - \mathfrak{E} \circ \widetilde{\mathfrak{T}})$$

2. Spatial displacements: Complex matrices

A straightforward translation into terms of complex 2×2 matrices yields:

$$(\mathbf{I}i + \varepsilon\mathbf{P}_0) = (\mathbf{E} + \varepsilon\mathbf{T})(\mathbf{I}i + \varepsilon\mathbf{P})(\mathbf{E}^* - \varepsilon\mathbf{T}^*) = \mathbf{I}i + \varepsilon(\mathbf{EPE}^* + i\mathbf{TE}^* - i\mathbf{ET}^*)$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \overline{\gamma} & -\delta \\ \overline{\delta} & \gamma \end{pmatrix}, \quad \text{with} \quad \gamma = t_0 + t_3 i, \quad \delta = t_2 + t_1 i.$$

In order that our later obtained symbolic expressions are free of the complex unit i ($\Rightarrow$ real polynomials) we make the following redefinition:

$$\mathbf{S} := i\mathbf{T} = \begin{pmatrix} \lambda & \overline{\mu} \\ \mu & -\overline{\lambda} \end{pmatrix} \quad \text{and} \quad \mathbf{S}^* := -i\mathbf{T}^* = \begin{pmatrix} \overline{\lambda} & \overline{\mu} \\ \mu & -\lambda \end{pmatrix}$$

$$\text{with} \quad \lambda = t_3 + t_0 i \quad \text{and} \quad \mu = t_1 + t_2 i.$$

2. Spatial displacements: Dual CK parameters

Theorem 1. Any spatial displacement of points P can be written as:

$$\mathbf{P} \mapsto \mathbf{P}_0 = \mathbf{EPE}^* + \mathbf{SE}^* + \mathbf{ES}^*,$$

where the four involved parameters $\alpha, \beta, \lambda, \mu \in \mathbb{C}$ fulfill the normalizing condition $\Phi = 1$ with

$$\Phi := \alpha\overline{\alpha} + \beta\overline{\beta}$$

and the analogue of the STUDY condition, which is given by $\Psi = 0$ with

$$\Psi := (\alpha\lambda - \overline{\alpha}\overline{\lambda}) + (\beta\mu - \overline{\beta}\overline{\mu}).$$

Moreover, the above mapping is a spatial displacement of points for each quadruple $\alpha, \beta, \lambda, \mu \in \mathbb{C}$ fulfilling $\Phi = 1$ and $\Psi = 0$.

Remark: The dual complex numbers $\alpha + \varepsilon\gamma$ and $\beta + \varepsilon\mu$ are the dual CK parameters for the spatial displacements of points. $\diamond$

3. Planar displacements

Restricting the STUDY parameters to planar EUCLIDEAN displacements within the plane $x_3 = 0$ implies $e_1 = e_2 = t_0 = t_3 = 0$ (cf. [10]) , thus one ends up with the homogenous 4-tuple $(e_0, e_3, t_1, t_2)\mathbb{R}$, which are the BG parameters [1,6].

For the dual CK parameters this implies $\beta = 0$ and $\lambda = 0$, which yields:

Corollary 1. Any planar displacement of points P can be written as:

$$p \mapsto p_0 = \alpha(\alpha p + 2\mu) \quad \text{with} \quad \alpha, \mu \in \mathbb{C} \quad \text{and} \quad \alpha\overline{\alpha} = 1.$$

Moreover, this mapping is a planar displacement of points for each bituple $\alpha, \mu \in \mathbb{C}$ fulfilling $\alpha\overline{\alpha} = 1$.

Remark: $\kappa, \tau \in \mathbb{C}$ and $\alpha, \mu \in \mathbb{C}$ are related by: $\kappa = \alpha^2$ and $\tau = 2\alpha\mu$. $\diamond$

4. Application to parallel manipulators

We consider $\overline{\alpha}, \overline{\beta}, \overline{\lambda}, \overline{\mu}$ as independent variables (uncoupled from $\alpha, \beta, \lambda, \mu$). Under this assumption STUDY's kinematic mapping (e.g. [16]) can be reformulated as:

Corollary 2. There is a bijection between SE(3) and 8-tuples of complex numbers $(\alpha, \beta, \lambda, \mu, \overline{\alpha}, \overline{\beta}, \overline{\lambda}, \overline{\mu})\mathbb{R}$ fulfilling $\Psi = 0$ with $(\alpha, \beta, \overline{\alpha}, \overline{\beta}) \neq (0, 0, 0, 0)$ and the condition that the quadruple $(\overline{\alpha}, \overline{\beta}, \overline{\lambda}, \overline{\mu})$ is the conjugate quadruple of $(\alpha, \beta, \lambda, \mu)$.

Based on this result the sphere condition, that the platform point P is located on a sphere with radius R centered in the base point B with fixed coordinates $\mathbf{b}_0 = (u_0, v_0, w_0)^T$, can be computed as:

$$\Phi^2 R^2 + \det(\mathbf{P}_0 - \mathbf{B}_0) = 0 \quad \text{with} \quad \mathbf{B}_0 := \begin{pmatrix} w_0 & \overline{b}_0 \\ b_0 & -w_0 \end{pmatrix}.$$

4. Application to parallel manipulators

Doing the corresponding tricky summation of HUSTY [8] by adding Ψ^2 to the left hand-side, shows that Φ factors out. The remaining quadratic factor Σ, is homogenous in $(\alpha, \beta, \lambda, \mu, \overline{\alpha}, \overline{\beta}, \overline{\lambda}, \overline{\mu})\mathbb{R}$ and has only 38 terms in contrast to its formulation based on STUDY parameters, which has 80 terms (cf. [8]).

By setting $z = w_0 = \beta = \overline{\beta} = \lambda = \overline{\lambda} = 0$ we get from $\Sigma = 0$ the circle condition:

$$(\alpha p + 2\mu - \overline{\alpha} b_0)(\overline{\alpha}\,\overline{p} + 2\overline{\mu} - \alpha \overline{b}_0) - \alpha\overline{\alpha}R^2 = 0.$$

which has both benefits:

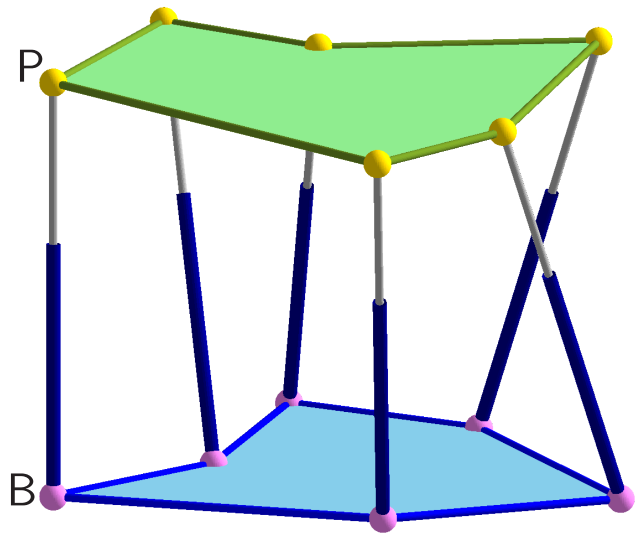
- compactness of the isotropic formulation and
- homogeneity of the approach based on BG parameters.

4. Symbolic direct kinematics computation

An example for the beneficial effects of the reduction of terms is the symbolic elimination process in the direct kinematics of SG platforms:

It is well-known [8] that the differences of two sphere conditions are only linear in the translational parameters. Therefore the system of five equations

$$\Psi = \Sigma_5 - \Sigma_1 = \Sigma_4 - \Sigma_1 = \Sigma_3 - \Sigma_1 = \Sigma_2 - \Sigma_1 = 0$$



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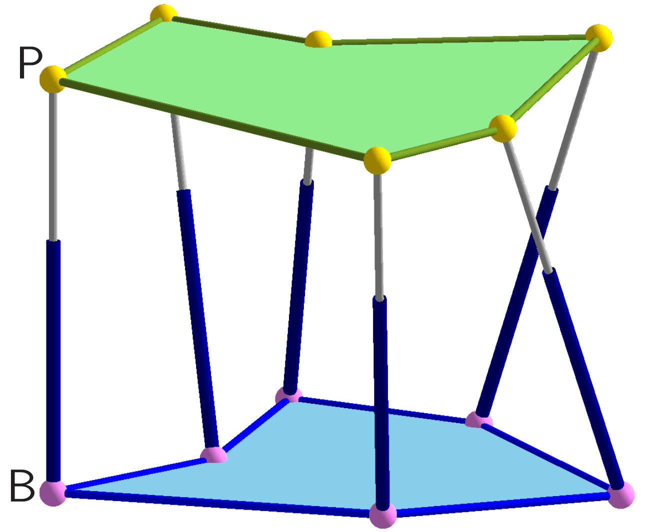
linear in $\lambda, \mu, \overline{\lambda}, \overline{\mu}$ can only have a non-trivial solution if the determinant of the 5×5 coefficient matrix vanishes. This determinant splits up into Φ and a factor with $53\,280$ terms, which is homogenous of degree 4 in $\alpha, \beta, \overline{\alpha}, \overline{\beta}$. In contrast, the corresponding quartic expression based on Study parameter has $258\,720$ terms [5].

4. Symbolic singularity locus expression

The compactness of the proposed formulation passes on to the symbolic expression of the singularity loci of SG platforms:

The singularity locus expression in terms of dual CK parameters has $542\,496$ terms if the platform and base anchor points are chosen as follows with respect to the moving and fixed frame:

- the first anchor point is located in the origin,
- the second one on the x-axis and
- the third one in the xy-plane.



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This singularity equation reformulated in Study parameters has $1\,748\,184$ terms.

Conclusion & References

- We discussed the transformation of points in terms of dual CK parameters.
- They imply a very compact symbolic expression of the sphere condition and the singularity loci of SG platforms.
- They cannot only be restricted to planar motions, but they can also be extended for kinematics in EUCLIDEAN 4-space according to [16].
- They are of interest for the determination of SG platforms with self-motions [5], but maybe they are also beneficial for the symbolic study of other mechanisms.

All references refer to the list of publications given in the presented paper:

NAWRATIL, G.: Parallel manipulators in terms of dual CAYLEY-KLEIN parameters.
Computational Kinematics (S. Zeghloul et al. eds.), Springer (2017)