Selbstbewegliche Stewart-Gough-Manipulatoren mit projektiv gekoppelter ebener Plattform und Basis

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[A] Self-motions of planar projective Stewart Gough platforms, Latest Advances in Robot Kinematics (J. Lenarcic, M. Husty eds.), 27–34, Springer (2012)

[B] On elliptic self-motions of planar projective Stewart Gough platforms, Transactions of the Canadian Society for Mechanical Engineering, under review

[C] All planar projective Stewart Gough platforms with self-motions, Technical Report No. 223, Geometry Preprint Series, TU Vienna (2012)

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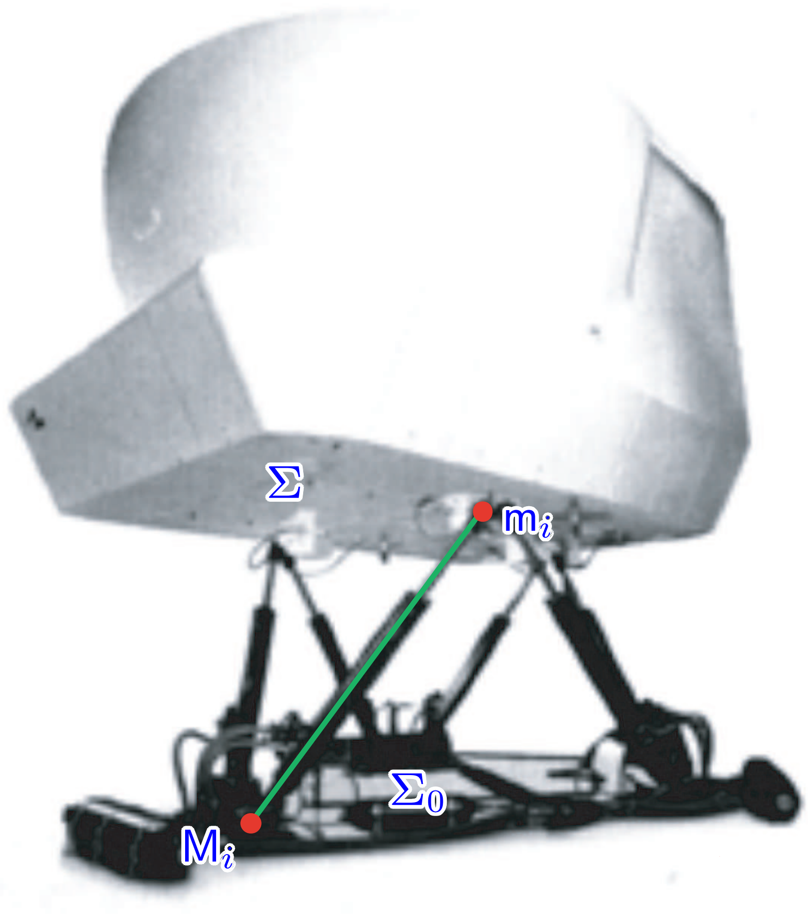
1. What is a self-motion of a SGP?

The geometry of a SGP is given by the six base anchor points $\mathrm{M}_i \in \Sigma_0$ and by the six platform points $\mathrm{m}_i \in \Sigma$ for $i = 1, \ldots, 6$.

A SGP is called planar, if $\mathrm{M}_1, \ldots, \mathrm{M}_6$ are coplanar and $\mathrm{m}_1, \ldots, \mathrm{m}_6$ are coplanar. The carrier planes are denoted by π_{M} resp. π_{m}.

M_i and m_i are connected with a S<u>P</u>S leg.

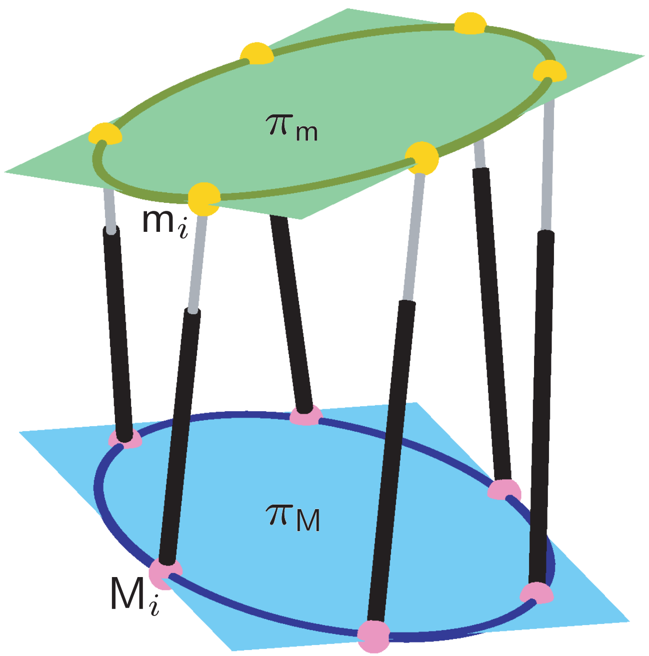
If all <u>P</u>-joints are locked, a SGP is in general rigid. But, under particular conditions, the manipulator can perform an n-parametric motion ($n > 0$), which is called self-motion.



1. Planar projective SGPs

Definition 1
A planar SGP is called projective if M_i and m_i are related by a non-singular projectivity κ; i.e. $\mathsf{m}_i\kappa = \mathsf{M}_i$ for $i = 1, \ldots, 6$.

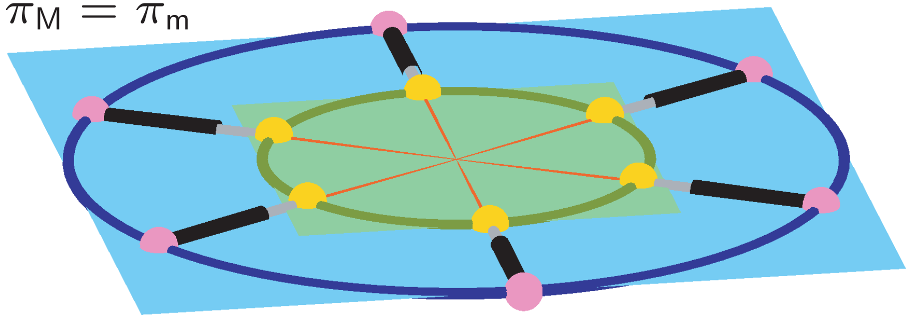
Theorem 1
A SGP is singular (infinitesimal flexible, shaky), if and only if, the carrier lines of the six SPS legs belong to a linear line complex.



A planar projective SGP is singular in every possible configuration (= architecturally singular), if and only if, one set of anchor points is located on a conic section (e.g. Chasles [1]).

1. Related result

As architecturally singular SGPs are redundant, they possess self-motions in each pose (over $\mathbb{C}$).
Therefore, we are only interested in non-architecturally singular planar projective SGPs with self-motions.



Theorem 2 (Proof was given by Karger [2])
A singular configuration of a non-architecturally singular planar projective SGP does not depend on the distribution of the anchor points in the platform and the base, but only on the mutual position of the planes π_{M} and π_{m} and on κ.
The configuration is singular, if and only if, either one of the legs can be replaced by a leg of zero length or two legs can be replaced by aligned legs.

2. Basic results

Lemma 1 (Proof was given in [A])
A two-parametric set of additional legs $\overline{mM}$ with $m\kappa = M$ can be attached to planar projective SGPs without changing the direct kinematics and singularity surface.

Remark: Due to Lemma 1, it is clear why a singular configuration does not depend on the distribution of the anchor points in π_M and π_m (cf. Theorem 2). $\diamond$

Theorem 3
A self-motion of non-architecturally singular planar projective SGPs can only be:

1. a spherical self-motion with rotation center $m\kappa = m$,
2. a Schönflies self-motion, where the direction of the rotation axis is parallel to the planes π_M and π_m,
3. an elliptic self-motion.

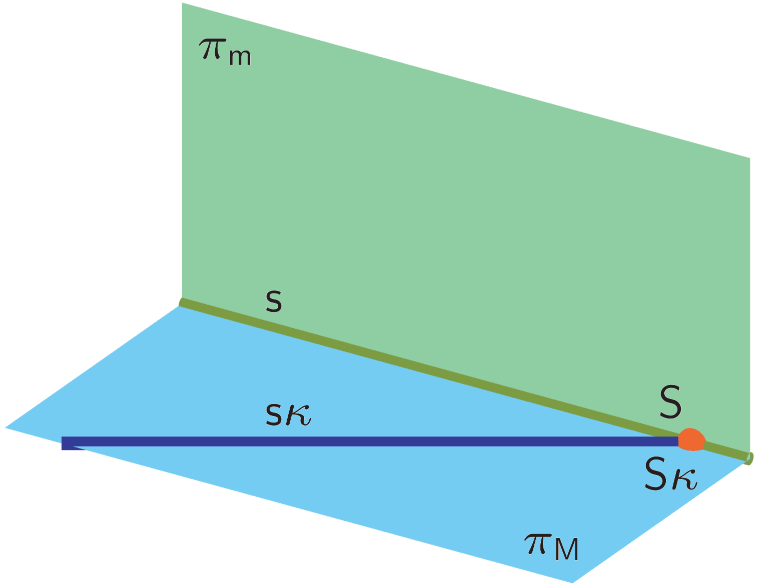
2. Basic results

Proof of Theorem 3

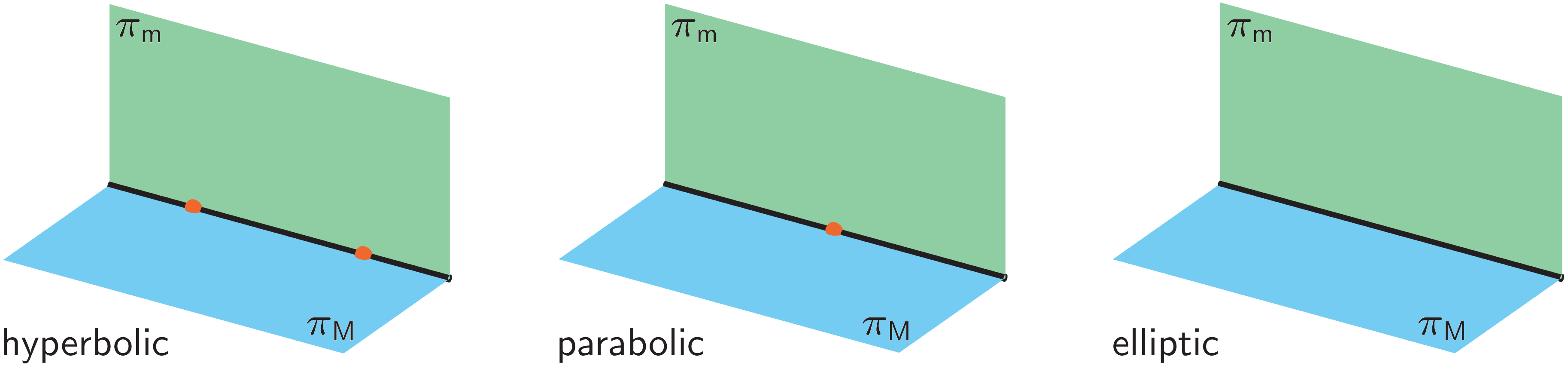
As in any pose of a self-motion of a planar projective SGP, the manipulator has to be in a singular configuration, we can apply Theorem 2. Therefore, the manipulator is singular, if and only if, one of the following cases hold:

a) π_M and π_m coincide:
$\Rightarrow \exists$ real fixed point $\Rightarrow$ case 1 or 2.

b) $S = S\kappa$: This real fixed point ($\Rightarrow$ case 1 or 2) is the intersection point of the line $s := (\pi_M, \pi_m)$ and $s\kappa$.



2. Basic results



c) $s = s\kappa$. If the restriction of κ to s is the identity, hyperbolic or parabolic, we also get at least one real fixed point ($\Rightarrow$ case 1 or 2). The elliptic case yields:

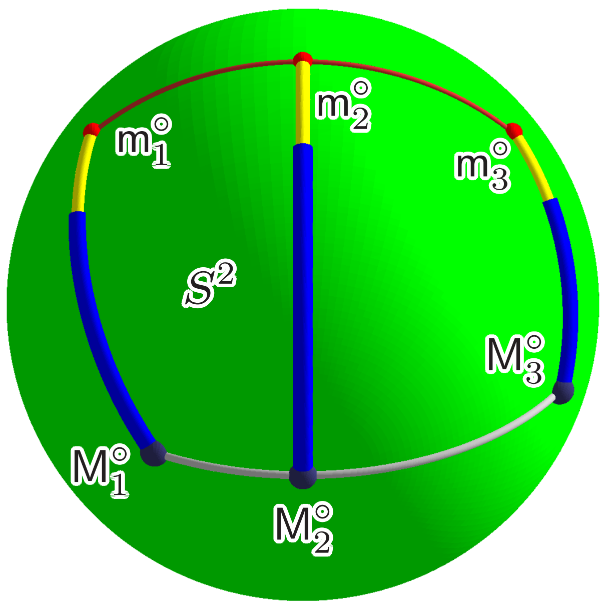
Definition 2
A self-motion of a non-architecturally singular planar projective SGP is called elliptic, if in each pose of this motion s exists with $s = s\kappa$ and where the projectivity from s onto itself is elliptic. $\square$

3. Spherical self-motions

If a planar projective SGP has a spherical self-motion about mκ = m, the spherical image of this manipulator, with respect to the unit sphere S^2 centered in mκ = m, has to have a self-motion as well.

Therefore, the problem reduces to the determination of non-degenerated spherical 3-dof R$\underline{\text{P}}$R manipulators with self-motions, where the base points M_1°, M_2°, M_3° and the platform points m_1°, m_2°, m_3° are located on great circles.

Due to Nawratil [3], there exists only one solution (after relabeling of anchor points and interchange of platform and base).

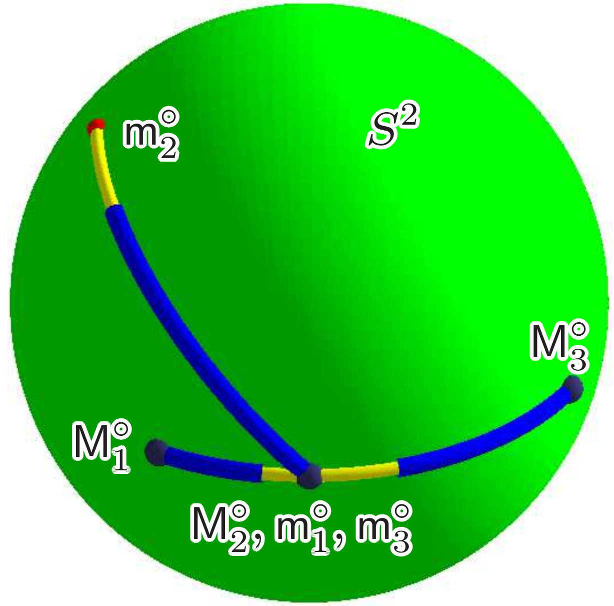


3. Spherical self-motions

This 3-dof R$\underline{\text{P}}$R manipulator has a pure rotational self-motion around the axis $\mathsf{a} := [\mathsf{m}\kappa = \mathsf{m}, \mathsf{m}_1^\circ = \mathsf{m}_3^\circ = \mathsf{M}_2^\circ]$.

Therefore, we can only add an additional leg $\overline{\mathsf{m}_4^\circ \mathsf{M}_4^\circ}$ without restricting the self-motion if $\mathsf{m}_4^\circ = \mathsf{m}_1^\circ$ or $\mathsf{M}_4^\circ = \mathsf{M}_2^\circ$ holds.

Therefore, κ has to map all platform anchor points $\notin \mathsf{a}$ on points of a $\Rightarrow$ κ is singular $\Rightarrow$

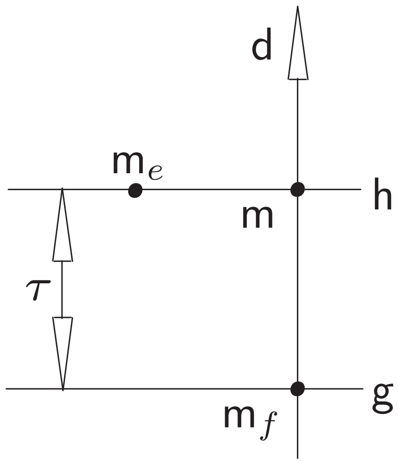


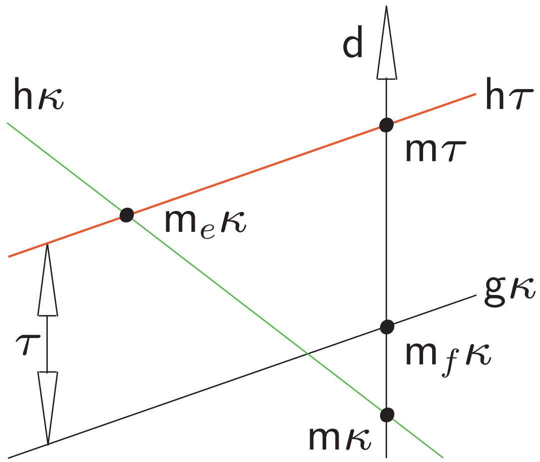
Theorem 4
Non-architecturally singular planar projective SGPs do not have spherical self-motions with rotation center $\mathsf{m}\kappa = \mathsf{m}$.

4. Schönflies self-motions

The Schönflies motion group consists of all translations combined with all rotations about a fixed direction d, which in our case is parallel to π_{M} and π_{m}.

It is well known (e.g. Husty and Karger [4]), that platform points, being on lines parallel to d, have congruent trajectories in a Schönflies motion. Therefore, every leg can be translated in direction d without changing this motion.





4. Case $\mathsf{h}\kappa \neq \mathsf{h}\tau$

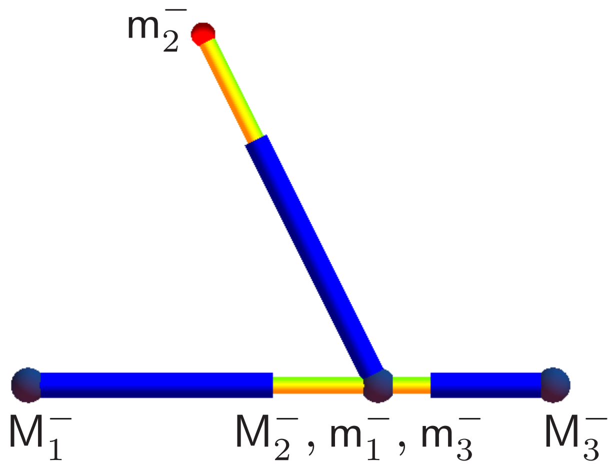
Now, every point $\mathsf{m} \in \mathsf{h}$ (with exception of m_e) can only rotate about the line $[\mathsf{m}\tau, \mathsf{m}\kappa] \parallel \mathsf{d}$. Therefore, the platform cannot move in direction d during the self-motion and the problem reduces to the following planar one:

Determine all non-degenerated 3-dof R$\underline{\text{P}}$R manipulators with self-motions, where the platform points m_1^-, m_2^-, m_3^- and base points M_1^-, M_2^-, M_3^- are collinear.

It is well known, that only two solutions exist:

- Planar analogue of the spherical self-motion:

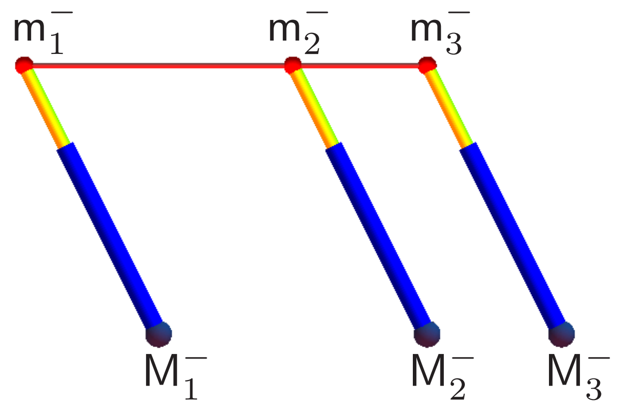
  The same arguments as in the spherical case yield again a contradiction.



4. Case h$\kappa \neq$ hτ

- <u>Circular translation:</u>

  If we choose the y-axis of the moving and the fixed frame in direction of d, the matrix $\mathbf{P}$ of the projectivity κ can be written as:



$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ p_{21} & 1 & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \quad \text{with} \quad p_{33} \in \mathbb{R} \setminus \{0,1\} \quad \text{and} \quad p_{21}, p_{31}, p_{32} \in \mathbb{R}. \tag{1}$$

As ideal points are mapped onto ideal points, κ is an affinity.

Remark: For the proof of the matrix $\mathbf{P}$ see [A]. ◇

4. Case $h\kappa = h\tau$

For this case, it can also be proven (cf. [A]) that κ has to be an affinity with the following matrix $\mathbf{P}$:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ p_{21} & p_{22} & 0 \\ p_{31} & p_{32} & 1 \end{pmatrix} \quad \text{with} \quad p_{22} \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad p_{21}, p_{31}, p_{32} \in \mathbb{R}. \qquad (2)$$

Theorem 5

A non-architecturally singular planar projective SGP can only have a Schönflies self-motion with the direction d of the rotation axis parallel to π_M and π_m, if it belongs to the subset of planar affine SGPs.

Moreover, if we choose the y-axis of the moving and the fixed frame in direction of d, the affinity κ has to be of the form given in (1) or (2).

4. Planar affine SGPs with self-motions

Theorem 6
Assume a non-architecturally singular planar affine SGP is determined by $\mathbf{M}_i = \mathbf{a} + \mathbf{A}\mathbf{m}_i$. Then, this manipulator has a self-motion, if and only if, the singular values s_1 and s_2 of $\mathbf{A}$ with $0 < s_1 \leq s_2$ fulfill $s_1 \leq 1 \leq s_2$.

Proof of Theorem 6
First of all, we prove that planar affine SGPs cannot have elliptic self-motions: If $\mathsf{s} = \mathsf{s}\varkappa$ is not the ideal line, then at least the ideal point of $\mathsf{s} = \mathsf{s}\varkappa$ is a fixed point. Therefore, $\mathsf{s} = \mathsf{s}\varkappa$ has to be the ideal line during the whole elliptic self-motion. Hence, the self-motion is a Schönflies motion with d orthogonal to $\pi_{\mathsf{M}} \parallel \pi_{\mathsf{m}}$.

As all points of π_{m} have to run on spherical paths, this Schönflies motion can only be the Borel Bricard motion due to Husty and Karger [4]. Therefore, the corresponding points of π_{m} and π_{M} have to be related by an inversion ($\neq$ projectivity).

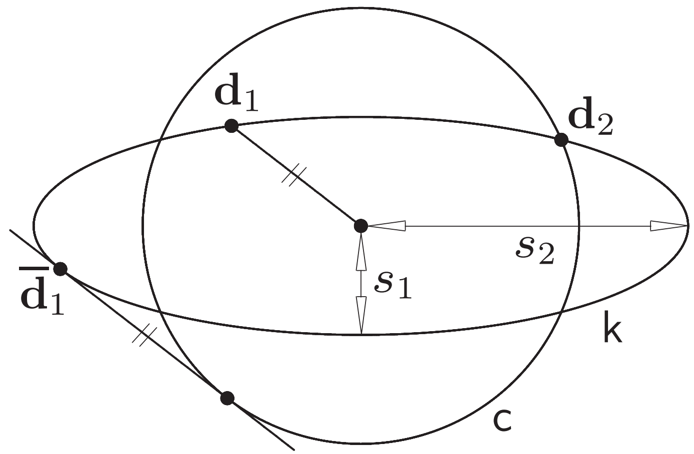
4. Planar affine SGPs with self-motions

Therefore, planar affine SGPs with self-motions have to be of type (1) or (2). We consider the image of the unit vectors $\mathbf{c} = (\cos\varphi, \sin\varphi) \in \pi_\mathsf{m}$ for $\varphi \in [0, 2\pi]$. Clearly, the tie points of the vectors $\mathbf{Ac}$ are located on an ellipse k.

Now, it can easily be seen (cf. [A]), that the necessary and sufficient condition for an affinity of type:

(1) is that k and c have a common tangent,

(2) is that k and c have a common point.



Clearly, we only get real common points and tangents of k and the unit circle c, if the singular values $0 < s_1 \leq s_2$ of $\mathbf{A}$ fulfill $s_1 \leq 1 \leq s_2$. $\square$

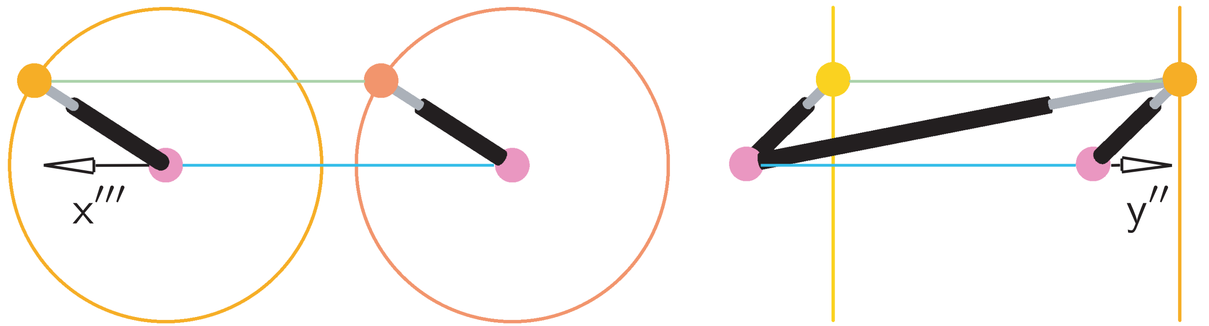
4. Example

$$\mathbf{M}_1 = (0,0)$$
$$\mathbf{M}_2 = (1,0)$$
$$\mathbf{M}_3 = (0,1)$$
$$\mathbf{m}_1 = (0,0)$$
$$\mathbf{m}_2 = (1,1)$$
$$\mathbf{m}_3 = (0,1)$$

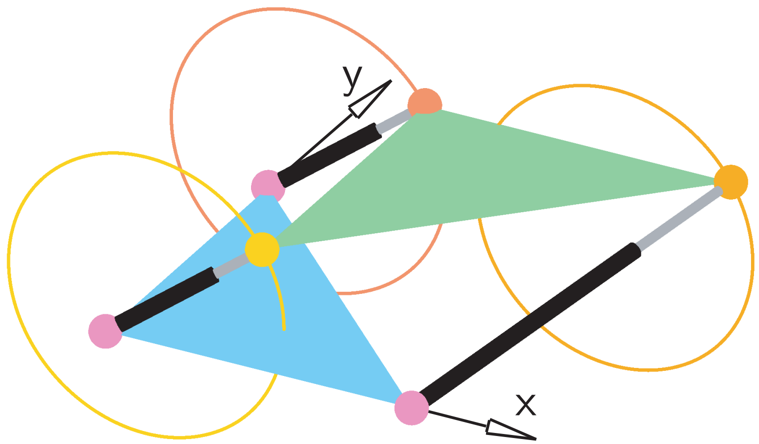
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

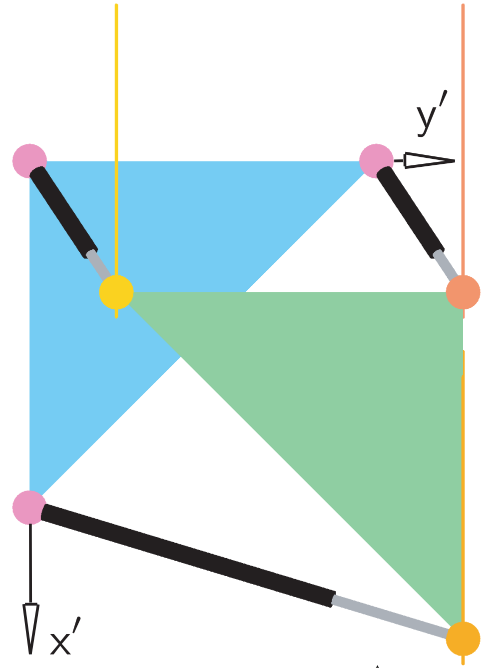
$$s_1 = (\sqrt{5}-1)/2$$

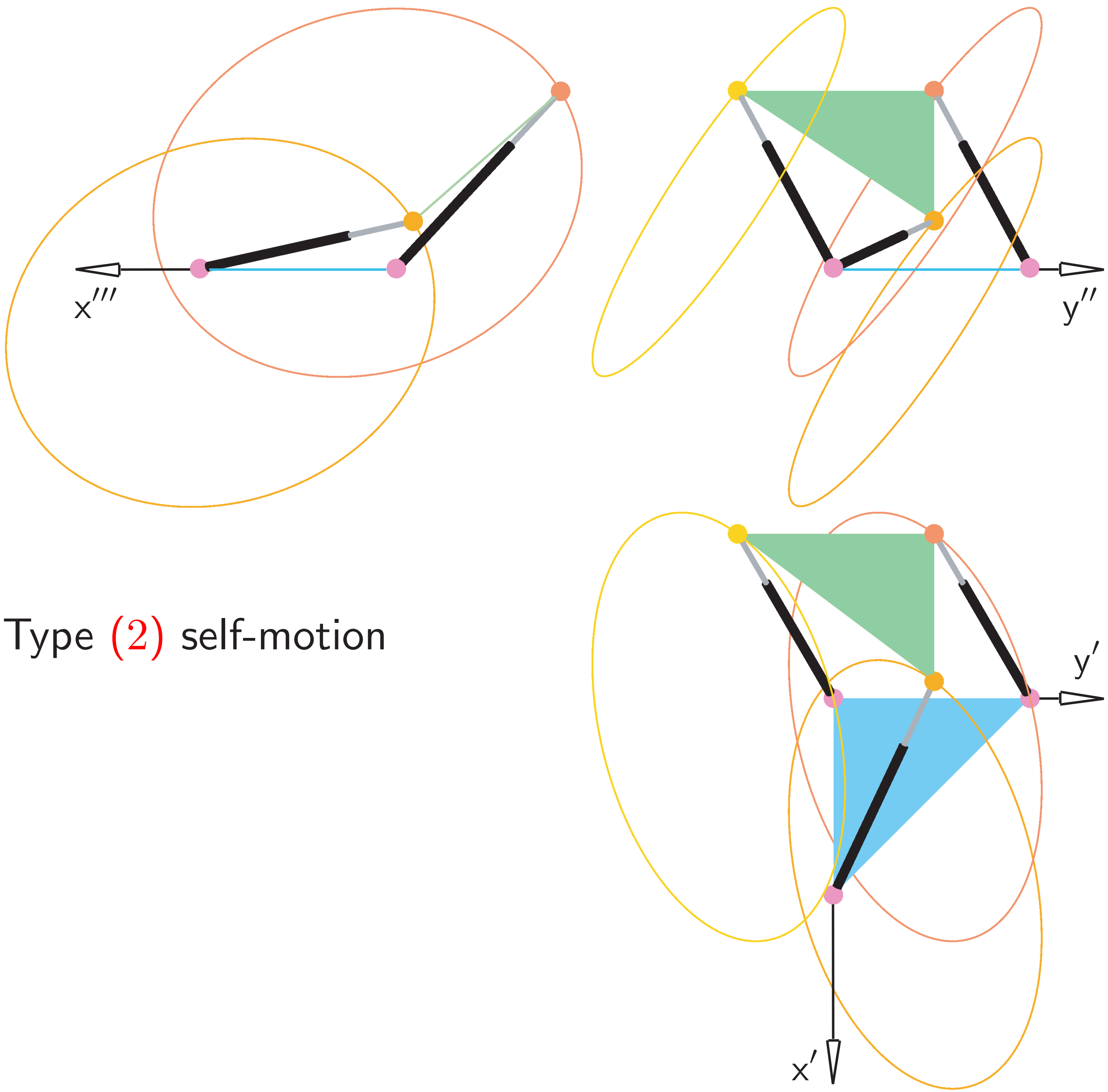
$$s_2 = (\sqrt{5}+1)/2$$



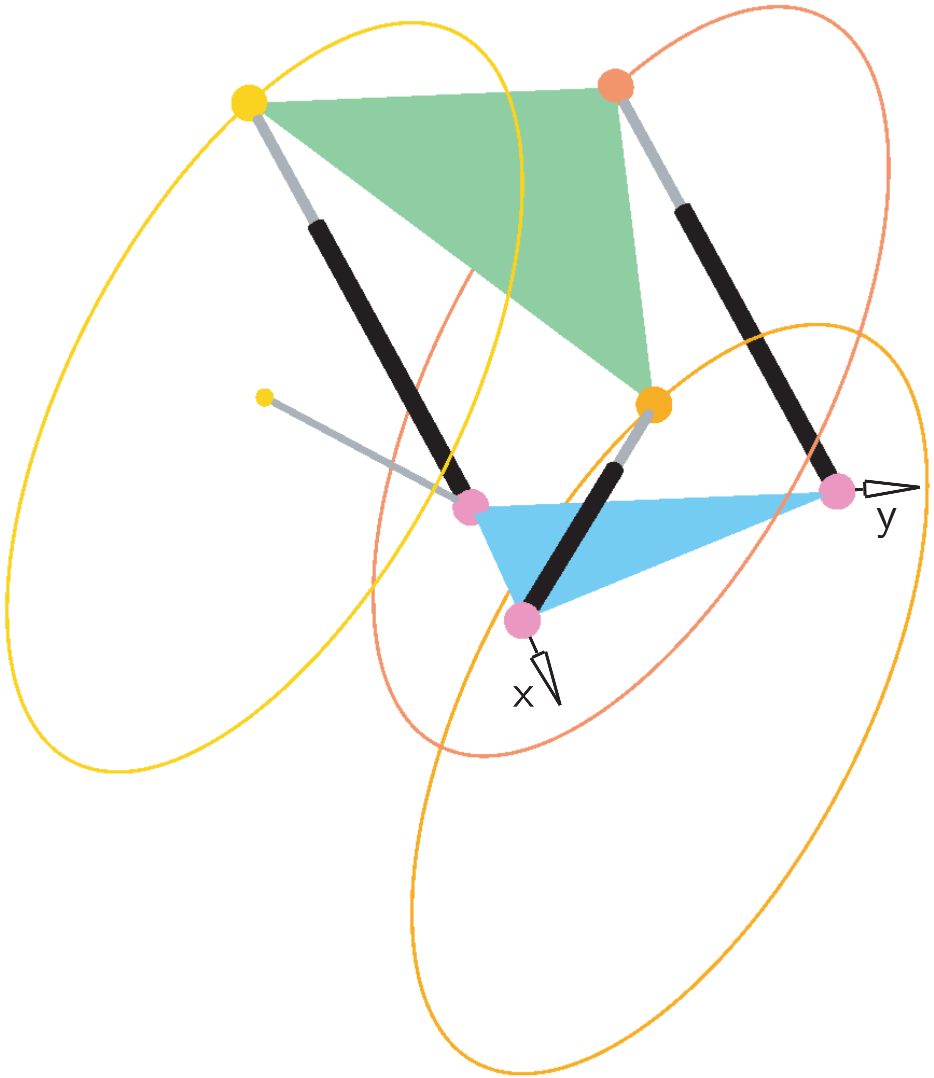
Type (1) self-motion



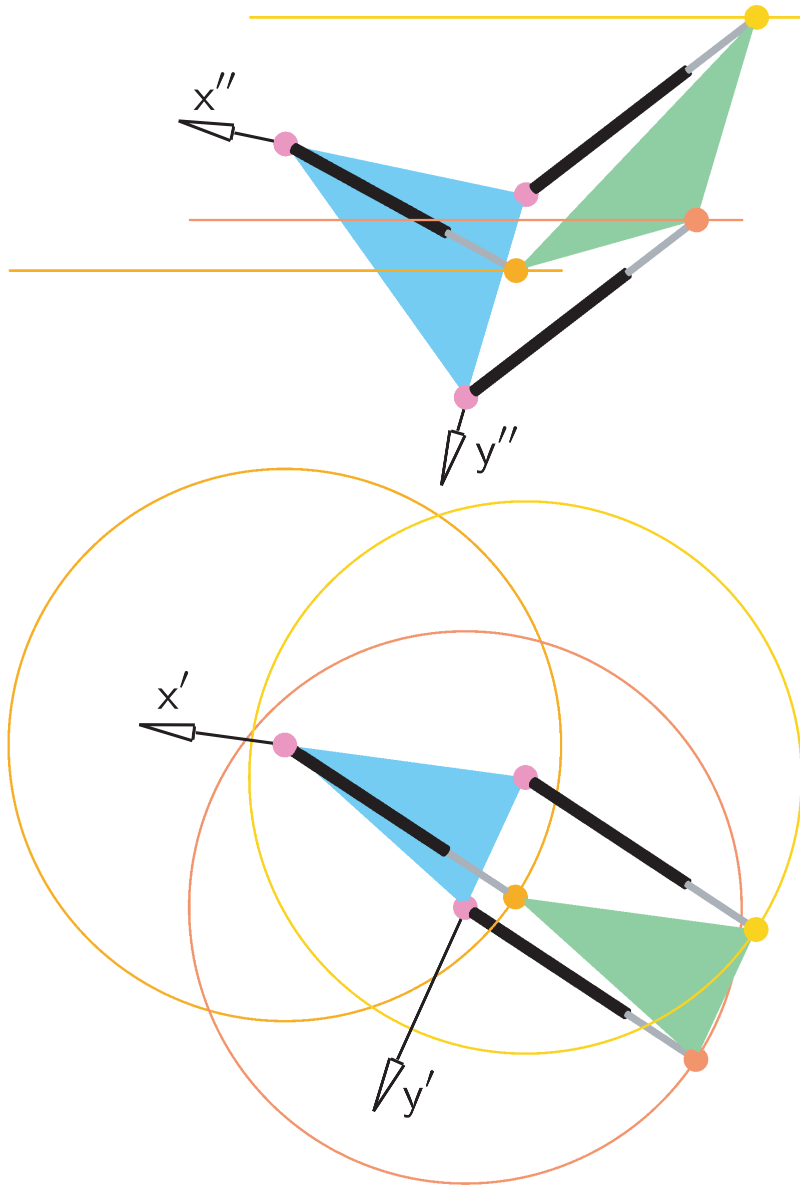




Type (2) self-motion



y
x



x''
y''
x'
y'

4. Remarks on self-motions of planar affine SGPs

- As all self-motions of planar affine SGPs are pure translations (cf. Karger [2]), the trajectories of all platform anchor points are congruent.

- For one-dimensional self-motions the trajectories are circles.
  $\Rightarrow$ All one-parametric self-motions are circular translations.

- The self-motion is two-dimensional, if and only if, the platform and the base are congruent and all legs have equal length.

- Theorem 6 also implies the result of Karger [5], that non-architecturally singular planar equiform SGPs cannot have self-motions, as $s_1 = s_2 \neq 1$ holds.

5. Elliptic self-motions

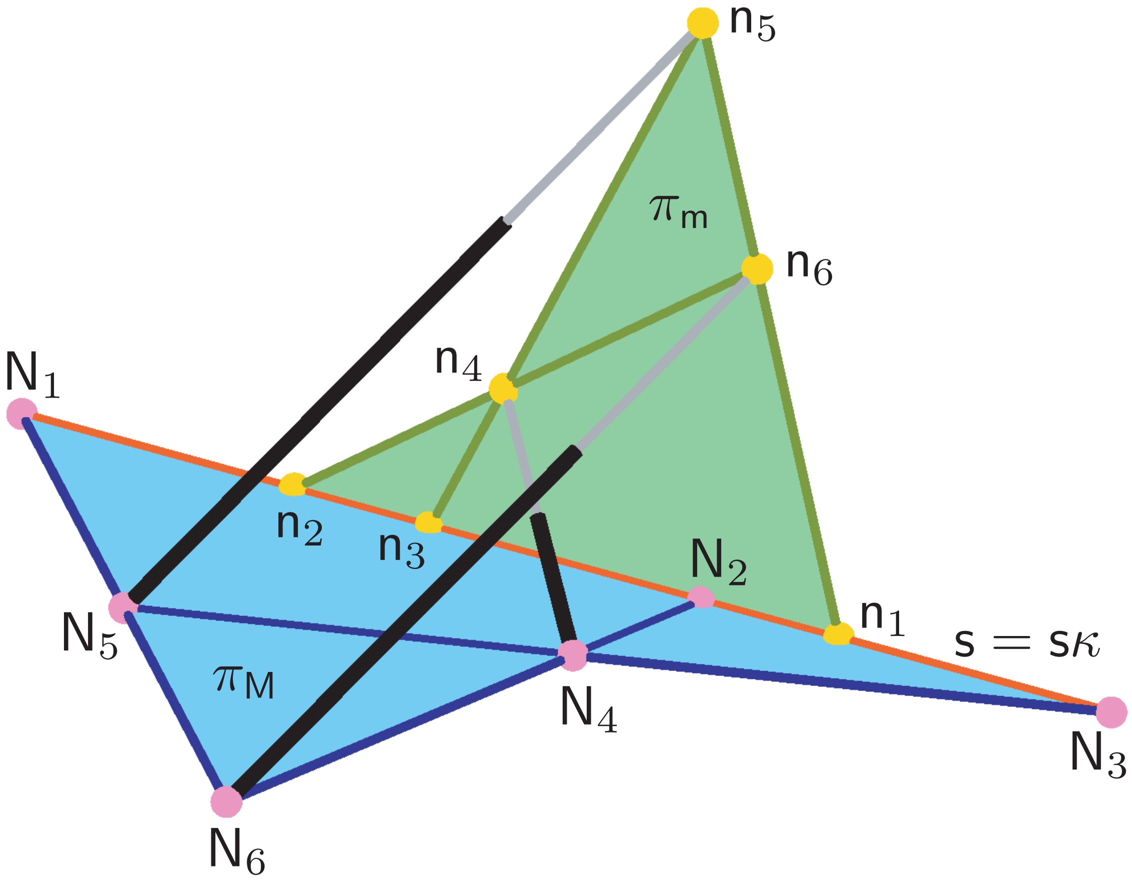
It remains open, whether elliptic self-motions even exist, as no example is known. In the case of existence the following theorem has to hold:

Theorem 7
A non-architecturally singular planar projective SGP possesses in each pose of an elliptic self-motion exactly two instantaneous degrees of freedom.

Proof of Theorem 7
Due to Lemma 1 and the results of Borras et al. [6], we can replace the original six legs $\overline{\mathsf{m}_i\mathsf{M}_i}$ by a new set of six legs $\overline{\mathsf{n}_i\mathsf{N}_i}$ without changing the direct kinematics and singularity surface, if:

- $\mathsf{n}_i\kappa = \mathsf{N}_i$ holds for $i = 1, \ldots, 6$ and
- $\mathsf{n}_1, \ldots, \mathsf{n}_6$ are not located on a conic section.

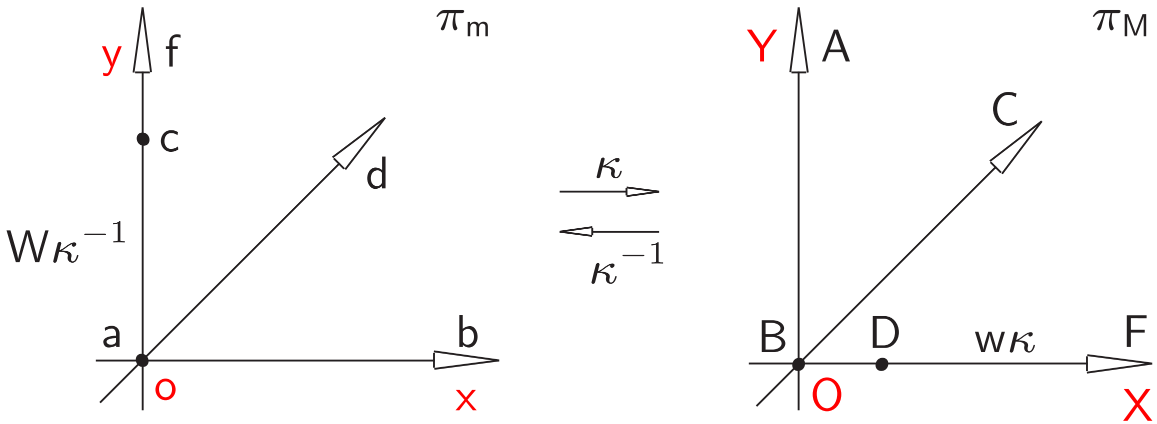
5. Elliptic self-motions



Therefore, $n_1, \ldots, n_6$ can be selected as shown in the figure.

As the carrier lines of the legs $\overline{n_1 N_1}$, $\overline{n_2 N_2}$ and $\overline{n_3 N_3}$ coincide, the lines $[n_i, N_i]$ with $i = 1, \ldots, 6$ can only span a linear congruence of lines. $\square$

5. Basic quadrangles (a, b, c, d) and (A, B, C, D) of κ



$$\begin{aligned}
\mathbf{a} &= (1:0:0) \\
\mathbf{b} &= (0:1:0) \\
\mathbf{c} &= (1:0:\beta) \\
\mathbf{d} &= (0:1:1) \\
\mathbf{f} &= (0:0:1)
\end{aligned}$$

$$\begin{aligned}
\mathbf{A} &= (0:0:1) \\
\mathbf{B} &= (1:0:0) \\
\mathbf{C} &= (0:1:1) \\
\mathbf{D} &= (1:\alpha:0) \\
\mathbf{F} &= (0:1:0)
\end{aligned}$$

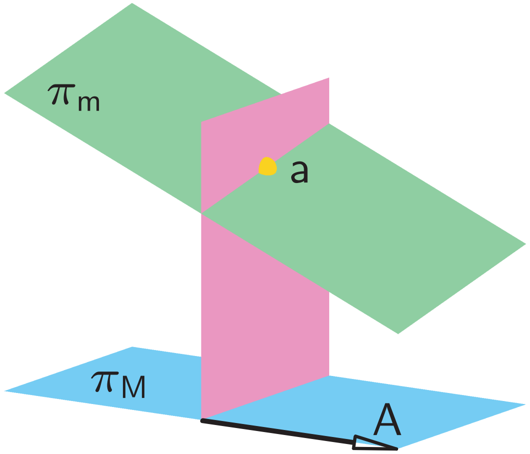
w ... ideal line of π_m
W ... ideal line of π_M

(o, x, y) and (O, X, Y) are Cartesian coordinate systems in π_m resp. π_M.

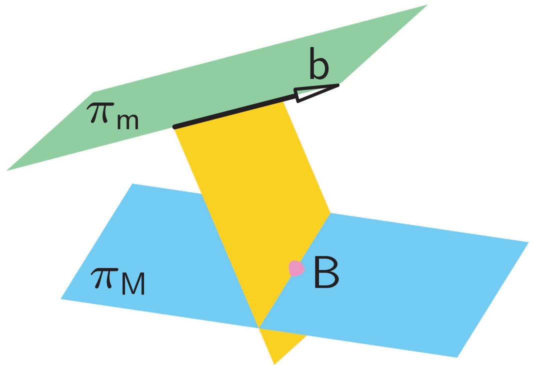
We can eliminate the factor of similarity by setting $\alpha = 1$. Therefore, the matrix $\mathbf{P}$ of κ only depends on $\beta \in \mathbb{R} \setminus \{0\}$.

5. The special legs $\overline{aA}, \overline{bB}, \overline{cC}, \overline{dD}, \overline{fF}$

- The attachment of the special leg $\overline{aA}$ (resp. $\overline{cC}$) corresponds with the so-called Darboux constraint (cf. [7]), that the platform anchor point a (resp. c) moves in a fixed plane orthogonal to A (resp. C).



- The attachment of the special leg $\overline{bB}$ (resp. $\overline{dD}$) corresponds with the so-called Mannheim constraint (cf. [7]), that a plane of the moving system orthogonal to b (resp. d) slides through the point B (resp. D).



- The attachment of the special leg $\overline{fF}$ corresponds with the so-called angle constraint (cf. [B]), that the ideal points f and F enclose a constant angle.

5. Study parameters $e_0 : \ldots : e_3 : f_0 : \ldots : f_3$

They represent an Euclidean displacements, if $\Phi : \sum_{i=0}^{3} e_i f_i = 0$ and $K = 1$ hold with $K := e_0^2 + e_1^2 + e_2^2 + e_3^2$. The rotational matrix is given by:

$$\mathbf{R} := (r_{ij}) = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}.$$

The translation vector $\mathbf{t} = (t_1, t_2, t_3)$ equals:

$$t_1 := 2(e_0f_1 - e_1f_0 + e_2f_3 - e_3f_2),$$

$$t_2 := 2(e_0f_2 - e_1f_3 - e_2f_0 + e_3f_1),$$

$$t_3 := 2(e_0f_3 + e_1f_2 - e_2f_1 - e_3f_0).$$

Moreover, we define the following 3 variables:

$$\bar{t}_1 := 2(e_0f_1 - e_1f_0 - e_2f_3 + e_3f_2),$$

$$\bar{t}_2 := 2(e_0f_2 + e_1f_3 - e_2f_0 - e_3f_1),$$

$$\bar{t}_3 := 2(e_0f_3 - e_1f_2 + e_2f_1 - e_3f_0).$$

5. Algebraic formulation of the constraints

With respect to the special coordinate systems $(\mathsf{o}, \mathsf{x}, \mathsf{y})$ and $(\mathsf{O}, \mathsf{X}, \mathsf{Y})$, introduced in π_{m} and π_{M}, respectively, the constraints can be written as follows (cf. [B]):

Darboux constraint:

$$\Omega_{\mathsf{A}}^{\mathsf{a}}: \; t_2 + L_a K = 0, \qquad L_a \in \mathbb{R},$$

$$\Omega_{\mathsf{C}}^{\mathsf{c}}: \; t_1 + t_2 + L_c K + \beta(r_{12} + r_{22}) = 0, \qquad L_c \in \mathbb{R},$$

Mannheim constraint:

$$\Pi_{\mathsf{B}}^{\mathsf{b}}: \; \bar{t}_1 + g_b K = 0, \qquad g_b \in \mathbb{R},$$

$$\Pi_{\mathsf{D}}^{\mathsf{d}}: \; \bar{t}_1 + \bar{t}_2 + g_d K - (r_{11} + r_{12}) = 0, \qquad g_d \in \mathbb{R},$$

Angle constraint:

$$\sphericalangle_{\mathsf{F}}^{\mathsf{f}}: \; r_{12} - \gamma K = 0 \quad \text{with} \quad \gamma \in]-1, 1[,$$

as $\arccos(\gamma)$ equals the enclosed angle.

5. Orthogonal elliptic self-motions

Based on the six constraints Ω_A^a, Ω_C^d, Π_B^b, Π_D^d, $\sphericalangle_F^f$, Φ we can prove the following:

Theorem 8 (Proof was given in [B])
An elliptic self-motion of a non-architecturally singular planar projective SGP has to be a one-parametric motion.

We introduce a geometric classification of elliptic self-motions as follows:

Definition 3
An elliptic self-motion is called orthogonal, if the angle enclosed by the unique pair of ideal points (f, F) with $f\kappa = F$ equals $\pi/2$ $(\Leftrightarrow \gamma = 0)$.

Theorem 9
There does not exist a non-architecturally singular planar projective SGP with an orthogonal elliptic self-motion.

5. Proof of Theorem 9: classical approach

An elliptic self-motion corresponds with a common curve of the seven hyperquadrics $\Omega_{\mathsf{A}}^{\mathsf{a}}$, $\Omega_{\mathsf{C}}^{\mathsf{c}}$, $\Pi_{\mathsf{B}}^{\mathsf{b}}$, $\Pi_{\mathsf{D}}^{\mathsf{d}}$, $\triangleleft_{\mathsf{F}}^{\mathsf{f}}$, Φ, $\Theta_{\mathsf{M}}^{\mathsf{m}}$ of the 7-dimensional projective Study parameter space.

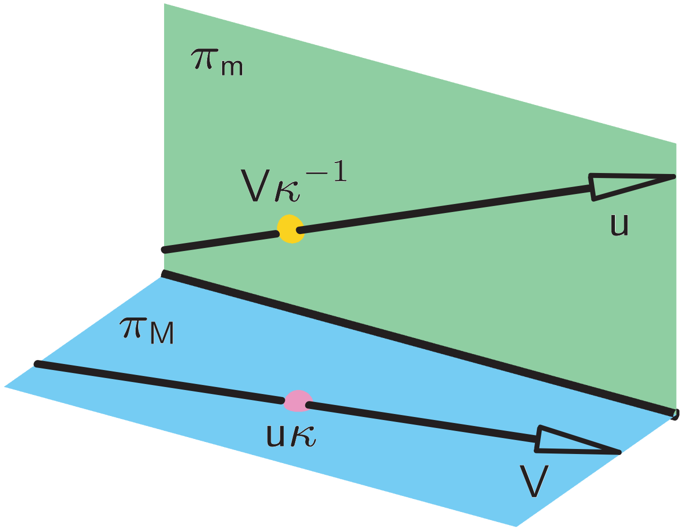
$\Theta_{\mathsf{M}}^{\mathsf{m}}$ is the so-called sphere constraint (cf. Husty [8]), that $\mathsf{m} \in \pi_{\mathsf{m}}$ is located on a sphere with radius R and center $\mathsf{M} := \mathsf{m}\kappa$ in π_{M}. In order to get a very compact expression, we choose $\mathbf{m} = (1 : -\beta : 0)$ and $\mathbf{M} = (1 : 0 : -1)$, which yields:

$$\Theta_{\mathsf{M}}^{\mathsf{m}}: \ (R^2 - \beta^2 - 1)K - 4(f_0^2 + f_1^2 + f_2^2 + f_3^2) - 2t_2 + 2\beta(\bar{t}_1 + r_{21}) = 0.$$

An elimination process yields the polynomial $\Upsilon[24685]$ of degree 16 in e_1 and e_2. For an elliptic self-motion the coefficients of $\Upsilon[24685]$ have to vanish identically. We were not able to solve the resulting system of 17 equations in $L_a, L_c, g_b, g_d, \beta, R$.

5. Proof of Theorem 9: alternative approach

u ... ideal point of $\pi_\mathsf{m} \setminus \{\mathsf{f}\}$ ... $\mathbf{u} = (0 : 1 : u)$
V ... ideal point of $\pi_\mathsf{M} \setminus \{\mathsf{F}\}$ ... $\mathbf{V} = (0 : v : 1)$
u and V are the ideal points of s and sκ iff:



$$\mathsf{u} = \mathsf{V} \quad \Longleftrightarrow \quad u = -\frac{r_{31}}{r_{32}}, \quad v = -\frac{r_{23}}{r_{13}},$$

$$\mathsf{V}\kappa^{-1} \in \pi_\mathsf{M} \quad \Longleftrightarrow \quad \Xi_1 : \ r_{12}t_3 - \beta r_{23}r_{32} = 0,$$

$$\mathsf{u}\kappa \in \pi_\mathsf{m} \quad \Longleftrightarrow \quad \Xi_2 : \ r_{32}\bar{t}_3 + r_{13}r_{31} = 0.$$

Ξ_1, Ξ_2 are quartic equations in the Study parameters, but only linear in $f_0, \ldots, f_3$.

$\Omega^\mathsf{a}_\mathsf{A}$, $\Omega^\mathsf{c}_\mathsf{C}$, $\Pi^\mathsf{b}_\mathsf{B}$, $\Pi^\mathsf{d}_\mathsf{D}$, $\triangleleft^\mathsf{f}_\mathsf{F}$, Φ, $\Xi_i \quad \Longrightarrow \quad \Upsilon_i[1960]$ of degree 12 in e_1 and e_2 for $i = 1, 2$

Coefficients imply a much more simpler system of 26 equations in $L_a, L_c, g_b, g_d, \beta$. This system can be used to prove Theorem 9. For details see [B].

5. Conjecture

Conjecture

Non-architecturally singular planar projective SGPs with an elliptic self-motion do not exist.

Clearly, the first idea to prove this conjecture, is to do it similarly to Theorem 9. There is only one more unknown, namely the variable γ:

The two corresponding polynomials Υ_1 and Υ_2 can be computed with MAPLE on a high capacity computer (78GB RAM). Each of these two expressions has 8259 terms and is again of degree 12 in e_1 and e_2. We tried hard to solve the resulting system of 26 equations, but we failed due to its high degree of non-linearity.

Remark: Note that with the classical approach, we were not even able to compute Υ with MAPLE, as the high capacity computer ran out of memory. $\diamond$

5. Historical results

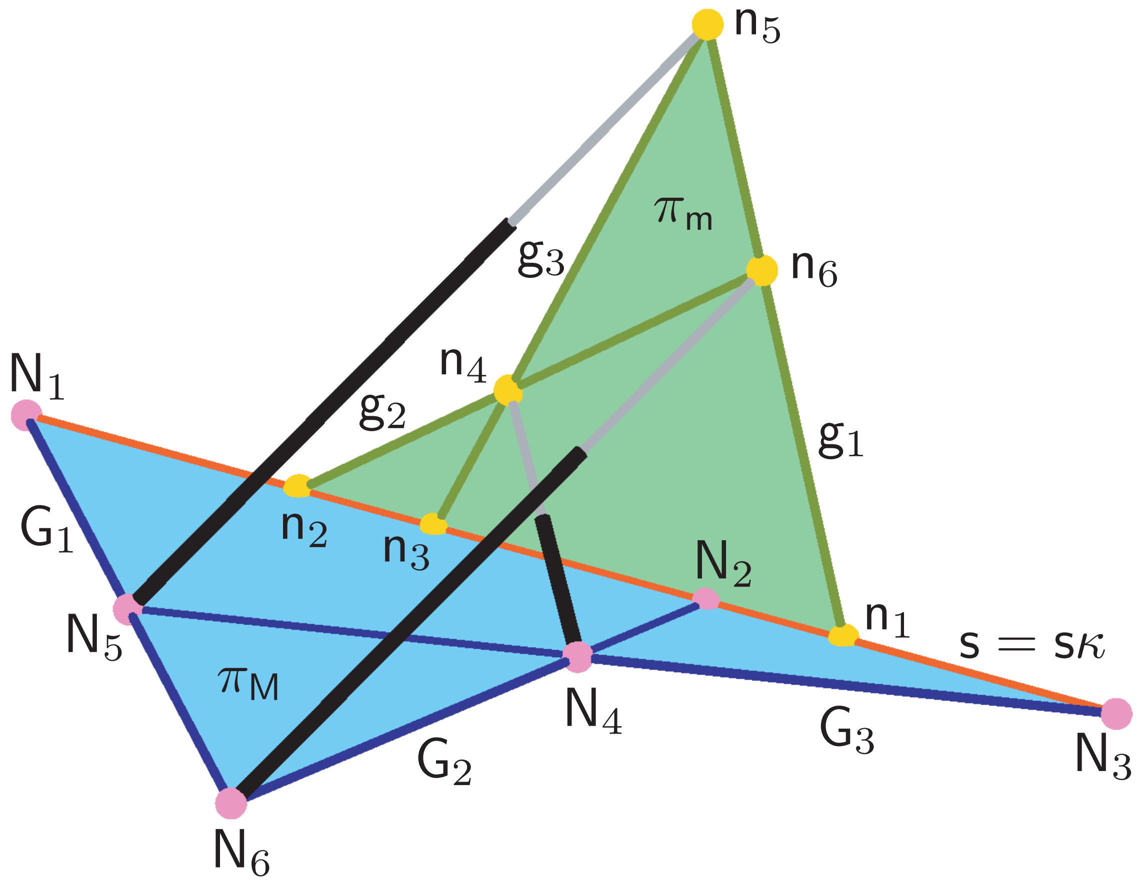
In 1873, the following theorem was given by Henrici [9]:

Theorem 10
If the generators of a hyperboloid Φ of one sheet are constructed of rods, jointed at the points of crossing in a way that at each intersection point one rod is free movable about the other one, then the surface is not rigid, but permits a deformation into a one-parametric set $\mathcal{H}$ of hyperboloids.

In 1899, Schur [10] presented a very elegant proof for Henrici's theorem, which also showed, that this theorem remains valid if the one-sheeted hyperboloid is replaced by a hyperbolic paraboloid.

Based on these results, Wiener [11] made some deformable models of one-sheeted hyperboloids and hyperbolic paraboloids.

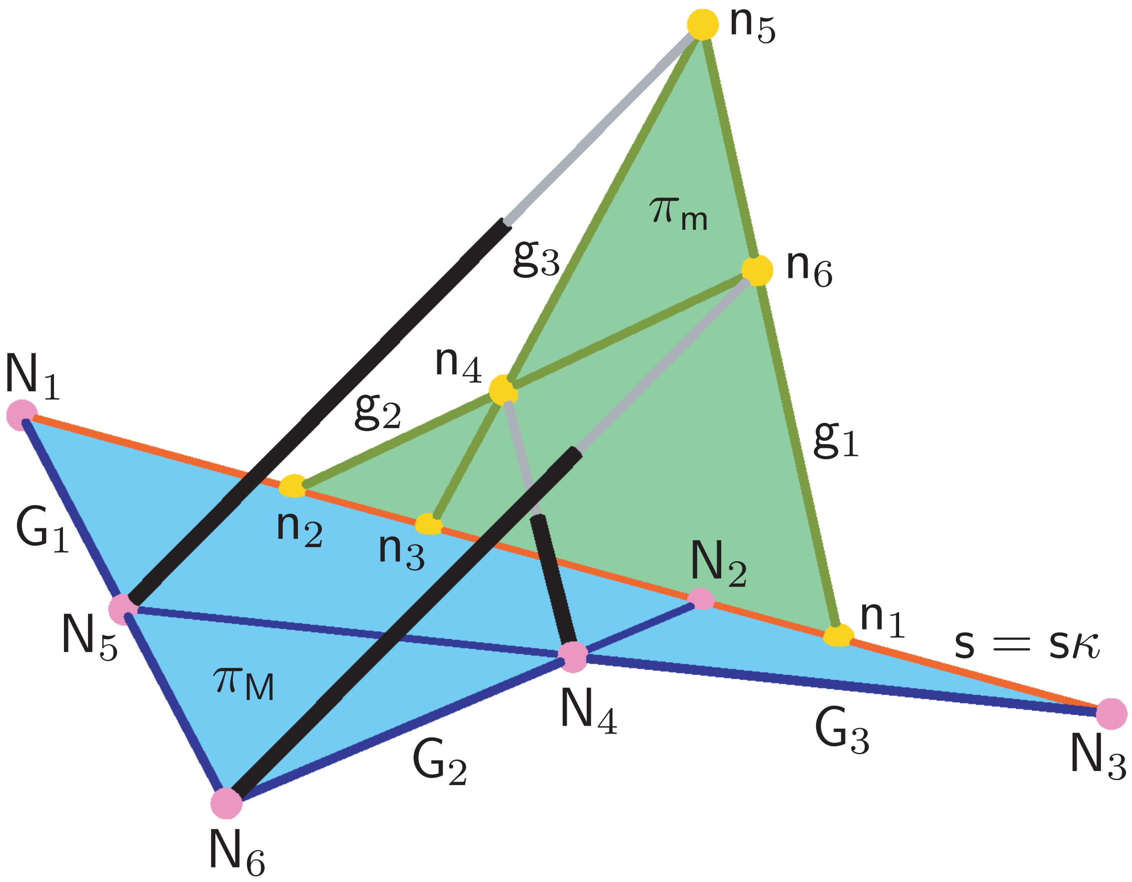
5. Proof of the Conjecture



Due to Lemma 1, we can add the one-parametric set of legs $\overline{nN}$ with $n \in g_1$, $N \in G_1$ and $n\kappa = N$ without disturbing the elliptic self-motion.

The lines g_1 and G_1 are skew ($\Leftrightarrow n_1 \neq N_1$), as the projectivity of s onto itself is elliptic.

5. Proof of the Conjecture



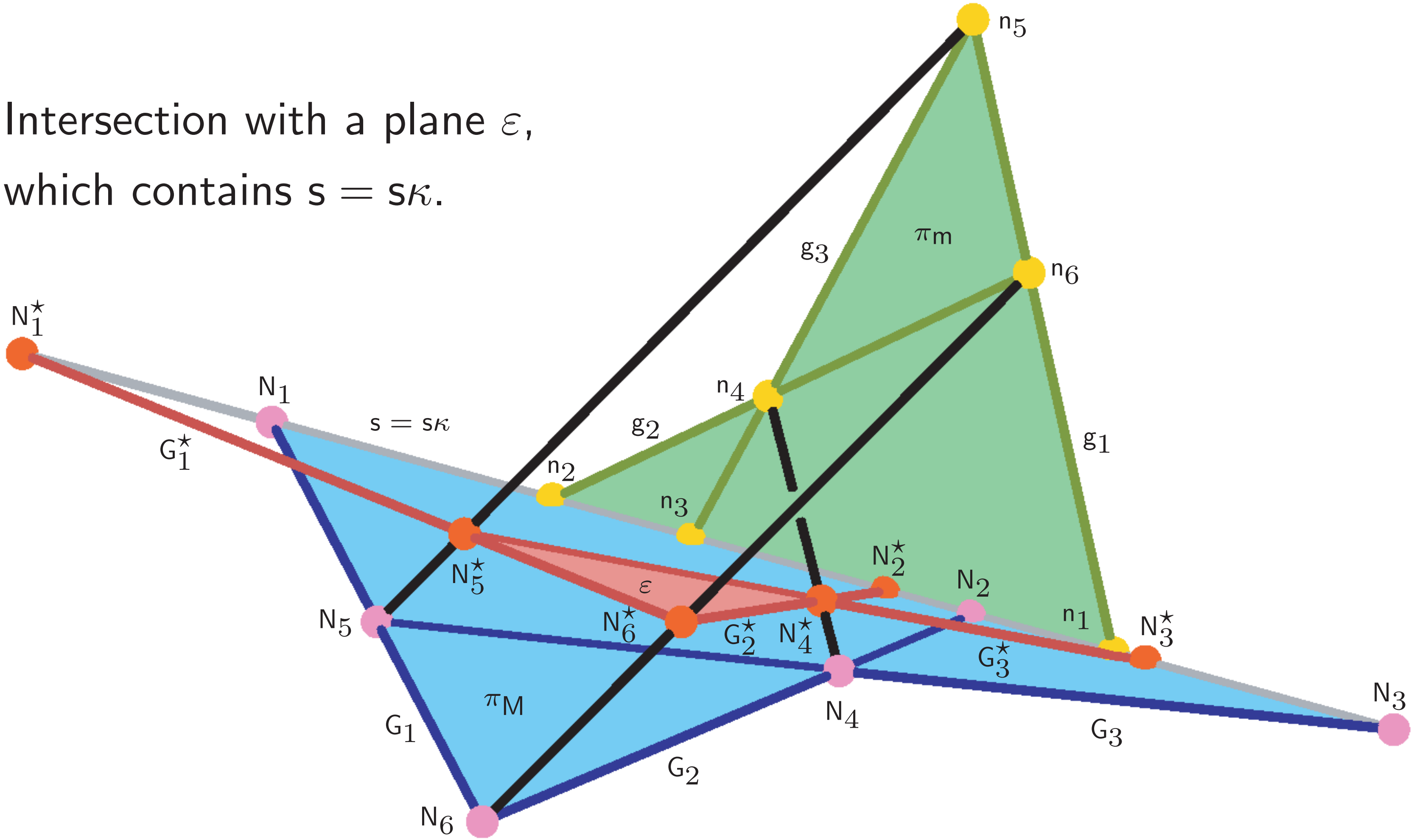
Therefore, the one-parametric set $\mathcal{R}_1$ of lines $[n, N]$ is a regulus of a regular ruled quadric Φ_1.

Due to the results of Henrici and Schur, we can add even arbitrary lines of the associated regulus $\mathcal{R}_1^\times$ without restricting the elliptic self-motion.

5. Proof of the Conjecture

Intersection with a plane ε,

which contains $s = s\kappa$.



5. Proof of the Conjecture

Lemma 2
There exists a non-singular projectivity $\kappa^\star$ with $\mathsf{n}_i\kappa^\star = \mathsf{N}_i^\star$ for $i = 1, \ldots, 6$. Therefore, the manipulator with platform anchor points $\mathsf{n}_1, \ldots, \mathsf{n}_6$ and base anchor points $\mathsf{N}_1^\star, \ldots, \mathsf{N}_6^\star$ is also a planar projective SGP with an elliptic self-motion.

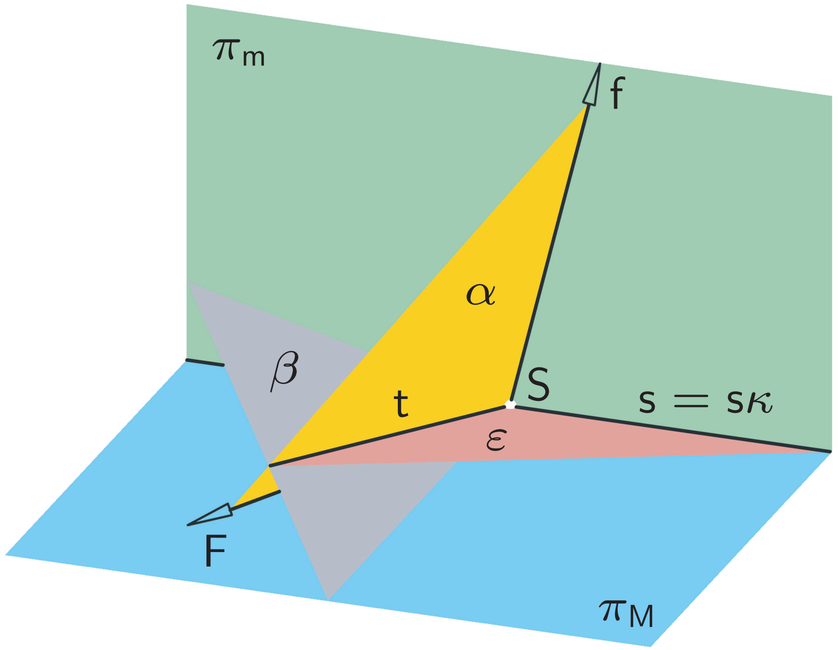
Proof of Lemma 2
It can easily be seen that $\mathsf{N}_1^\star, \mathsf{N}_2^\star, \mathsf{N}_4^\star, \mathsf{N}_5^\star$ always form a quadrangle. Therefore, the mapping $\mathsf{n}_i \mapsto \mathsf{N}_i^\star$ for $i = 1, 2, 4, 5$ uniquely defines a regular projectivity $\kappa^\star$, which also yields $\mathsf{n}_3\kappa^\star = \mathsf{N}_3^\star$ and $\mathsf{n}_6\kappa^\star = \mathsf{N}_6^\star$.

The elliptic self-motion of the manipulator $\mathsf{n}_1, \ldots, \mathsf{N}_6$ is transmitted by the motion of the reguli $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ onto the manipulator $\mathsf{n}_1, \ldots, \mathsf{N}_6^\star$.
This resulting self-motion is elliptic too, as a fixed point of the restriction of $\kappa^\star$ on $\mathsf{s} = \mathsf{s}\kappa^\star$ also has to be a fixed point of the restriction of κ on $\mathsf{s} = \mathsf{s}\kappa$. $\square$

5. Proof of the Conjecture

Construction for a special choice of ε:

- S ... finite point of $s = s\kappa$
- α ... plane spanned by $[S, f, F]$
- β ... plane orthogonal to f through S
- t ... intersection line of α and β
- ε ... plane spanned by t and $s = s\kappa$



$f\kappa^\star$ equals the ideal point of t and therefore, the self-motion of the planar projective SGP $n_1, \ldots, N_6^\star$ is orthogonal. Theorem 9 yields the contradiction. $\square$

6. Conclusion

Theorem 11
A planar projective SGP, which is not architecturally singular, can only have a self-motion if the projectivity is an affinity $\mathbf{M}_i = \mathbf{a} + \mathbf{A}\mathbf{m}_i$, where the singular values s_1 and s_2 of the 2×2 transformation matrix $\mathbf{A}$ with $0 < s_1 \leq s_2$ fulfill the condition $s_1 \leq 1 \leq s_2$.
All one-parametric self-motions are circular translations. Moreover, the self-motion is a two-dimensional translation, if and only if, the platform and the base are congruent and all legs have equal length.

Der Wissenschaftsfonds.

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