Congruent Stewart Gough platforms with non-translational self-motions

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Overview

1. Stewart Gough Platform (SGP)

The geometry of a SGP is given by the six base anchor points M_i and by the six platform points m_i for $i = 1, \ldots, 6$.

M_i and m_i are connected with a S$\underline{P}$S leg.

Theorem 1
A SGP is *singular* (*infinitesimal flexible, shaky*) if and only if the carrier lines of the six S$\underline{P}$S legs belong to a linear line complex.

A SGP is called *architecturally singular* if it is singular in any possible configuration.

1. Self-motions and the Borel Bricard problem

If all P-joints are locked, a SGP is in general rigid. But in some special cases the manipulator can perform an n-parametric motion ($n > 0$), which is called *self-motion*.

All self-motions of SGPs are solutions to the famous Borel Bricard problem [1,2,9].

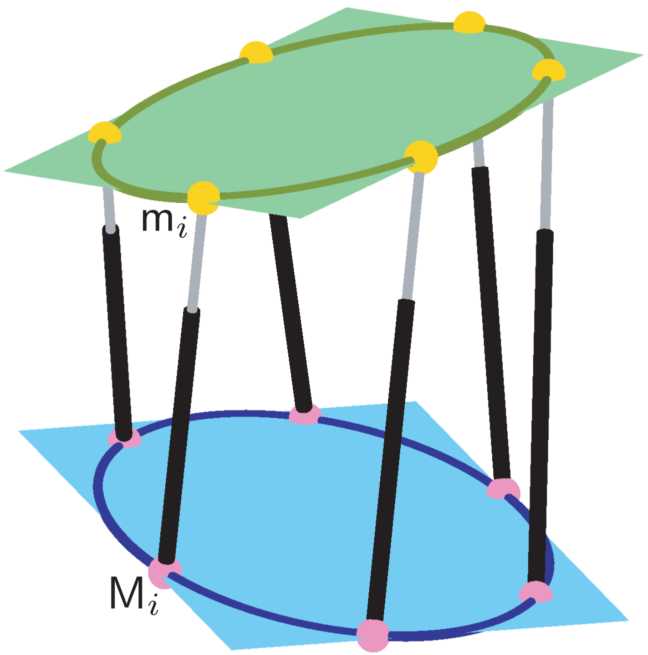
Borel Bricard problem (still unsolved)
Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.

1. Known results on projective SGPs

A SGP is called *planar* if $\mathsf{M}_1, \ldots, \mathsf{M}_6$ are coplanar and $\mathsf{m}_1, \ldots, \mathsf{m}_6$ are coplanar.

A SGP is called *projective* if the platform and base are coupled by a regular projective mapping κ: $\mathsf{M}_i \mapsto \mathsf{m}_i$ for $i = 1, \ldots, 6$.

- Architecturally singular planar projective SGPs ($\Leftrightarrow$ anchor points are located on conic section) possess self-motions in each pose (over $\mathbb{C}$).
- Non-architecturally singular planar projective SGPs can only have self-motions (pure translations) if κ is an affinity.



The non-planar case remains open, in which we focus on so-called *congruent* SGPs (κ is a congruence).

2. Study parameters

We use Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$ for the parametrization of SE(3). Note that $(e_0 : e_1 : e_2 : e_3)$ are the so-called Euler parameters of SO(3).

All real points of the Study parameter space P^7, which are located on the so-called *Study quadric*

$$\Psi : \; e_0 f_0 + e_1 f_1 + e_2 f_2 + e_3 f_3 = 0,$$

correspond to an Euclidean displacement with exception of the 3-dimensional subspace $e_0 = e_1 = e_2 = e_3 = 0$ of Ψ, as its points cannot fulfill the condition $N = 1$ with

$$N = e_0^2 + e_1^2 + e_2^2 + e_3^2.$$

All points of P^7, which cannot fulfill this normalizing condition, are located on the so-called *exceptional quadric* $N = 0$.

2. Direct kinematics of SGPs

The solution of the direct kinematics is based on a quadratic homogeneous equation in Study parameters (cf. Husty [8]) expressing that the point $\mathrm{m}_i := (a_i, b_i, c_i)$ is located on a sphere centered in $\mathrm{M}_i := (A_i, B_i, C_i)$ with radius R_i. This is the so-called *sphere condition* Λ_i with:

$$\begin{aligned}\Lambda_i: \quad & (a_i^2 + b_i^2 + c_i^2 + A_i^2 + B_i^2 + C_i^2 - R_i^2)N + 4(f_0^2 + f_1^2 + f_2^2 + f_3^2) \\ & - 2(a_iA_i + b_iB_i + c_iC_i)e_0^2 + \ldots + 4(c_iB_i - b_iC_i)e_0e_1 + \ldots \\ & + 4(a_i - A_i)(e_0f_1 - e_1f_0) + \ldots = 0.\end{aligned}$$

Now the solution of the direct kinematics over $\mathbb{C}$ can be written as the algebraic variety V of the ideal spanned by $\Psi, \Lambda_1, \ldots, \Lambda_6, N = 1$. In general V consists of a discrete set of points, which correspond to a maximum of 40 configurations.

2. Bonds for SGPs

We assume that a given SGP has a d-dimensional self-motion. As a d-dimensional self-motion corresponds with a d-dimensional solution of the direct kinematics problem, the seven quadrics $\Psi, \Lambda_1, \ldots, \Lambda_6$ have to have a d-dimensional set of points in common, which is called *algebraic motion*.

Now the points of this algebraic motion with $N \neq 0$ equal the kinematic image of the algebraic variety V. But we can also consider the points of the algebraic motion, which belong to the exceptional quadric $N = 0$. These points are the so-called *bonds* of the d-dimensional self-motion.

Theorem 2. Nawratil [18]
The set $\mathcal{B}$ of bonds depends on the geometry of the SGP, but not on $R_1, \ldots, R_6$.

2. Computation of Bonds

1st Step: We project the algebraic motion of the SGP into the Euler parameter space P^3 by the elimination of $f_0, \ldots, f_3$. This projection is denoted by π_f.

2nd Step: We determine those points of the projected point set $\pi_f(V)$, which are located on the quadric $N = 0$. Note that this set of projected bonds cannot be empty for a non-translational self-motion.

The kernel of this projection π_f equals the group of translational motions. Therefore this computational approach is restricted to non-translational self-motions.

But this poses no problem, as all SGPs with pure translational self-motions were already characterized in Nawratil [18]. Based on this result we can easily prove:

3. Basic results

Lemma 1.
Non-planar congruent SGPs can only have a pure translational self-motion if all legs have equal non-zero lengths. The resulting self-motion $\mathcal{T}$ is 2-dimensional.

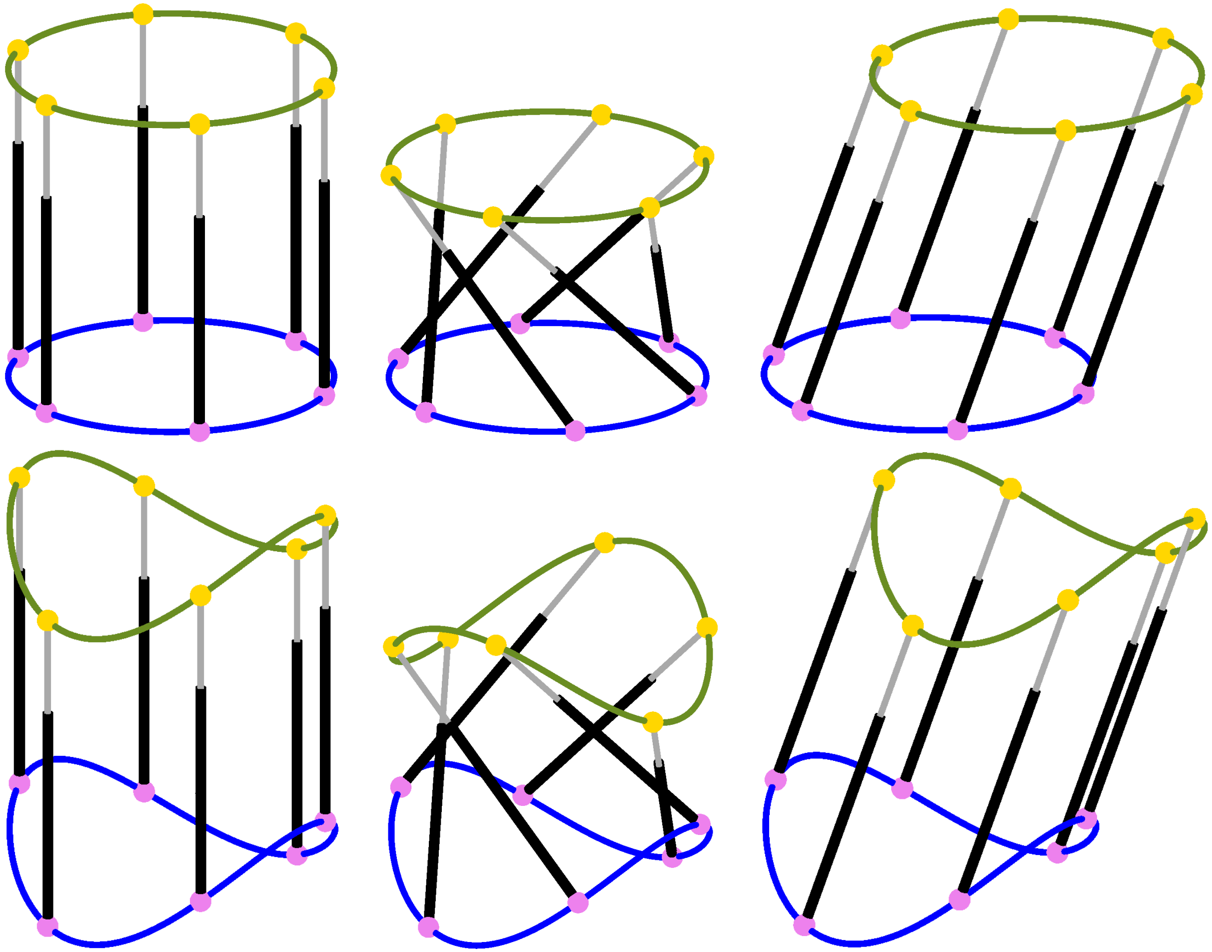
Moreover, from the results of Karger [13] we obtain:

Lemma 2.
A non-planar congruent SGP is architecturally singular if and only if four anchor points are collinear.

The following example demonstrates the existence of non-translational self-motions of non-planar congruent SGPs, which are not architecturally singular.

Remark: This existence is not self-evident, as planar congruent SGPs can only have translational self-motions if they are not architecturally singular. $\diamond$

3. Wren platform and its modification



Branching singularity (left) of the self-motion $\mathcal{T}$ (right) and the 1-dim Schönflies self-motion (center).

Therefore the Wren platform is *kinematotropic* [23].

Remark: The modified Wren platform shows that the property of kinematotropy is not restricted to architecturally singular SGPs. ◇

4. Cylinders of Rotation

A cylinder of revolution Φ equals the set of all points, which have equal distance to its rotation axis s (finite line). Under the assumption that Φ has at least one real point, we can distinguish the following four cases:

1. s is real and Φ is not reducible: Φ is a cylinder of revolution over $\mathbb{R}$.
2. s is real and Φ is reducible: Φ equals a pair of isotropic planes γ_1 and γ_2, which are conjugate complex. Trivially s carries the only real points of Φ.
3. s is imaginary and Φ is not reducible: Φ is a cylinder of revolution over $\mathbb{C}$. The real points are on the 4th order intersection curve of Φ and its conjugate $\overline{\Phi}$.
4. s is imaginary and Φ is reducible: In this case Φ equals a pair of isotropic planes γ_1 and γ_2, which are not conjugate complex. Moreover Φ contains two real lines g_i $(i = 1, 2)$, which are the intersections of γ_i and its isotropic conjugate $\overline{\gamma}_i$.

4. Computation of cylinders of revolution

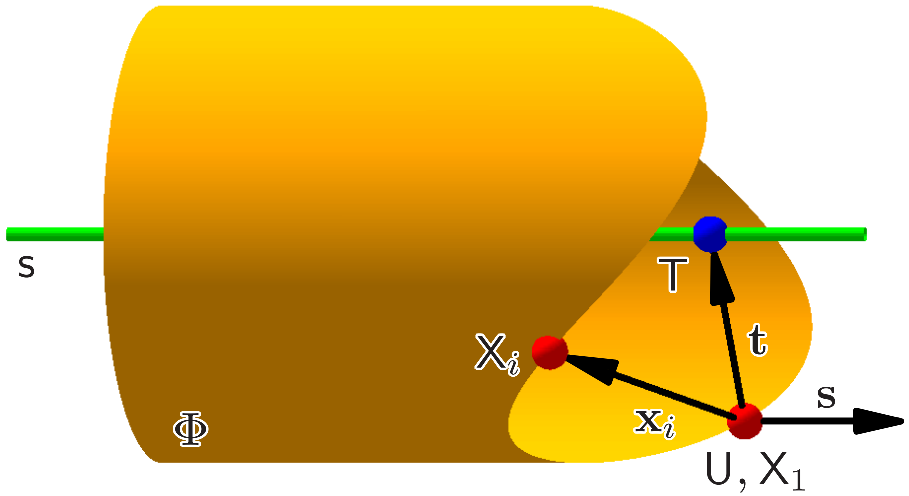
We use the approach of Zsombor-Murray and El Fashny [24] for the computation of all cylinders of revolution through a given set of real points $X_1, \ldots, X_n$. This problem is equivalent with the solution of the following system of equations if X_1 equals the origin U of the reference frame:

$$\mathbf{s}^2 = 1 \ldots \text{normalizing condition}$$

$$\Upsilon: \quad \mathbf{s} \cdot \mathbf{t} = 0,$$

$$\Omega_i: \quad (\mathbf{x}_i \times \mathbf{s})^2 - 2\mathbf{s}^2(\mathbf{x}_i \cdot \mathbf{t}) = 0,$$

for $i = 2, \ldots, n$, where T is the foot-point on s with respect to $U = X_1$.



Remark: For $n = 5$ there exist in general six cylinders of revolution over $\mathbb{C}$ (e.g. [24]). For $n > 5$ no solution exists if $X_1, \ldots, X_n$ are in general configuration. $\diamond$

5. Main Theorem: Necessity

Main Theorem: Necessity
A non-planar congruent SGP can have a real non-translational self-motion only if the six base (resp. platform) anchor points have equal distance to a finite line s, i.e. they are located on a cylinder of revolution of type 1, 3 or 4.

Proof: If a non-translational self-motion exists, then there has to exist a projected bond. The computation of the existence condition is sketched for the general case. For details and the discussion of special cases, please see the presented paper.

We compute $\Delta_{j,i} := \Lambda_j - \Lambda_i$, which is only linear in the Study parameters $f_0, \ldots, f_3$. We can solve the linear system of equations $\Psi, \Delta_{2,1}, \Delta_{3,1}$ for f_1, f_2, f_3. We plug the obtained expressions into $\Delta_{4,1}, \Delta_{5,1}, \Delta_{6,1}$ and consider their numerators, which are homogeneous polynomials P_4, P_5, P_6 of degree three in the Euler parameters.

5. Main Theorem: Necessity

We eliminate e_0 from P_i and $N = 0$ by computing the resultant Q_i of these two expressions for $i = 4, 5, 6$. Q_i factors into $16F_i^2$ with

$$F_i[27] = \sum_{j+k+l=3} g_{jkl} e_1^j e_2^k e_3^l \quad \text{and} \quad j \in \{0, 1, 2\}, \quad k, l \in \{0, \ldots, 3\}.$$

Now the necessary condition for the existence of a bond is that the cubics F_4, F_5 and F_6 in the projective plane spanned by e_1, e_2, e_3 have a point in common.

Due to the number of variables and the degree of the involved equations, the corresponding algebraic conditions for the existence of a common point cannot be computed explicitly (e.g. by applying a resultant based elimination method), and therefore it seems that we cannot prove the theorem.

5. Main Theorem: Necessity

But due to the example of the modified Wren platform, we conjecture that bonds can only exist if the six anchor points are located on a cylinder of revolution. Therefore we consider the system of equations $\Upsilon, \Omega_2, \ldots, \Omega_6$ with respect to the six anchor points.

We can solve $\Upsilon, \Omega_2, \Omega_3$, which are linear in the coordinates of $\mathfrak{t}$, for these unknowns. We plug the obtained expressions into $\Omega_4, \Omega_5, \Omega_6$ and consider their numerators, which are homogeneous polynomials $F_4^\star, F_5^\star, F_6^\star$.

For the substitution $\mathfrak{s} := (s_1, s_2, s_3) \leftrightarrow (e_1, e_2, e_3)$ the polynomial $F_j^\star$ equals F_j for $j = 4, 5, 6$.

Therefore the existence of a cylinder of revolution through the six anchor points implies the existence of a projected bond and vice versa. $\square$

5. Main Theorem: Sufficiency

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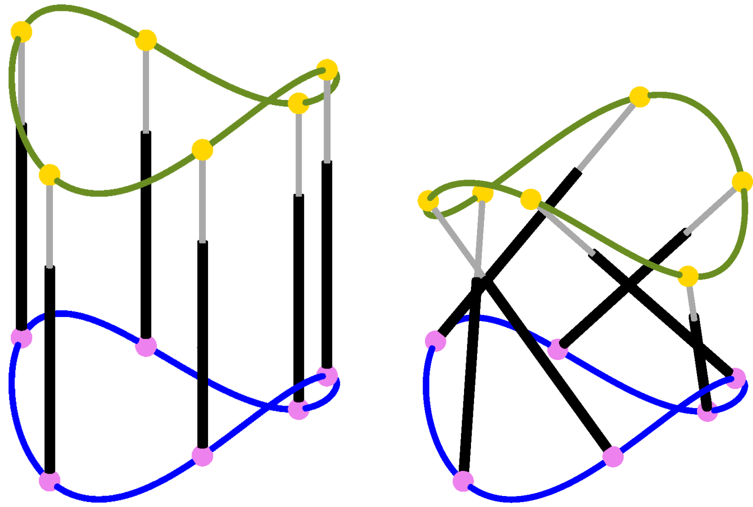
The condition of the main theorem is sufficient for the existence of a self-motion over $\mathbb{C}$.



Proof: For each cylinder of revolution there exists the corresponding Schönflies self-motion over $\mathbb{C}$, which was recognized for the modified Wren platform. This can easily be verified algebraically as follows:

As for this Schönflies self-motion all legs have equal length, we set $R_1 = \ldots = R_6$. Then we do not have to eliminate e_0 by applying the resultant with $N = 0$, but P_i already equals F_i up to a non-zero factor (for $i = 4, 5, 6$).
Therefore e_0 remains free and parametrizes the self-motion. $\square$

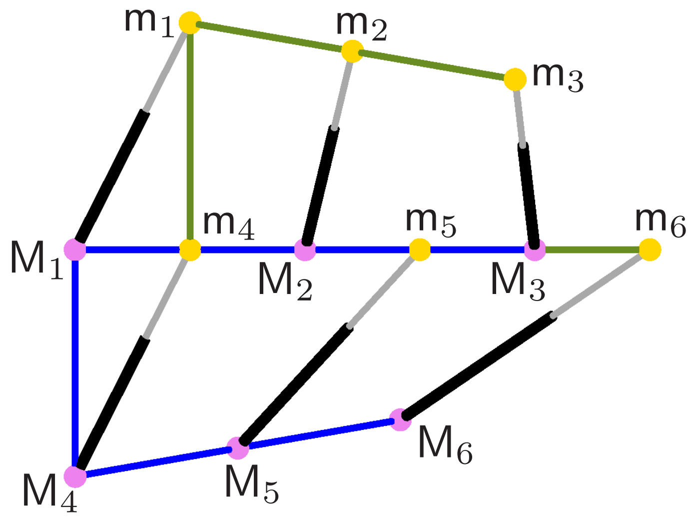
6. Remarks and Examples

The Schönflies self-motion is real if and only if the points are located on a real cylinder of revolution, as the direction $(s_1 : s_2 : s_3)$ of the axis s equals the direction $(e_1 : e_2 : e_3)$ of the rotation axis of the Schönflies self-motion.

But the Schönflies self-motions are not the only non-translational self-motions, which can be performed by SGPs characterized in the Main Theorem.

Counter examples:

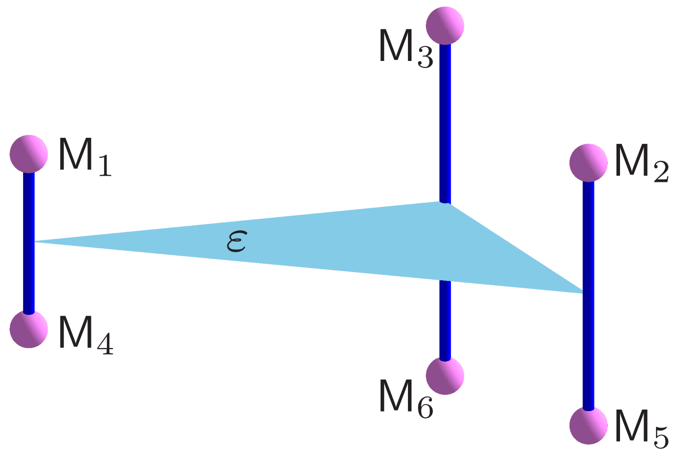
1. Architecturally singular SGPs.
2. Butterfly self-motions (see Figure).
3. Example 3.3.8 of Husty et al. [11].



6. New Self-motions

We study non-planar congruent SGPs, which are plane-symmetric, i.e. the fourth, fifth and sixth anchor point are obtained by reflecting the first, second and third one on a plane ε.

W.l.o.g. we can assume that ε is the xy-plane.



Due to the plane-symmetry there exist four cylinder of revolution through the six anchor points. Therefore the bond-set contains the four bonds of the Schönflies self-motions (up to conjugation of coordinates) implied by these cylinders.

But beside these self-motions there exist the following ones characterized by $e_3 = 0$, which are new to the best knowledge of the author.

6. New Self-motions

The unknowns f_1, f_2, f_3 can be computed from $\Psi, \Delta_{2,1}, \Delta_{4,1}$. If we set

$$R_5^2 = \frac{c_2}{c_1}(R_4^2 - R_1^2) + R_2^2 \quad \text{and} \quad R_6^2 = \frac{c_3}{c_1}(R_4^2 - R_1^2) + R_3^2$$

where c_i denotes the z-coordinate of the i^{th} anchor point, then $\Delta_{5,1}$ is fulfilled identically and $\Delta_{3,1} = \Delta_{6,1}$ holds. The numerator of this condition is a homogeneous polynomial P of degree 3 in e_0, e_1, e_2.

Hence for given design parameters (5+scaling), the cubic P implies a 4-parametric set $\mathcal{S}$ of self-motions, as it depends on the four leg lengths R_1, R_2, R_3, R_4.
For more details (e.g. special subsets of $\mathcal{S}$) please see the presented paper.

Supplementary data: For a discussion of exemplary self-motions (incl. animations) generated by a plane-symmetric congruent SGP see the author's homepage. $\diamond$

7. Open Problems and References

- A complete list of non-translational self-motions of congruent SGPs is missing.
- A restriction of the sufficiency condition with respect to $\mathbb{R}$ remains open as well.

Remark: The necessary condition of the Main Theorem also holds for the more general case that κ is no congruence transformation but a similarity (cf. [21]). $\diamond$

All references refer to the list of publications given in the presented paper:

Nawratil, G.: Congruent Stewart Gough platforms with non-translational self-motions. In Proc. of 16th International Conference on Geometry and Graphics, Innsbruck, Austria, August 4-8, 2014