Alternative interpretation of the Plücker quadric's ambient space and its application

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Funded by FWF Project Grant No. P24927-N25

Austrian Science Fund

Basics: Line Geometry

It is well known [14] that there exists a bijection between the set $\mathcal{L}$ of lines of the projective 3-space P^3 and all real points of the so-called Plücker quadric

$$\Psi: \quad l_{01}l_{23} + l_{02}l_{31} + l_{03}l_{12} = 0$$

of P^5, where the homogeneous 6-tuple $(l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12})$ are the Plücker coordinates of the lines. The 2-dimensional generator space $L: \ l_{01} = l_{02} = l_{03} = 0$ of Ψ corresponds to the set of ideal lines.

A Line l of the Euclidean 3-space E^3 is represented by a real point of $\Psi \setminus L$ where:

- $\mathbf{l} := (l_{01}, l_{02}, l_{03}) \neq \mathbf{o}$ gives the direction of the line l,
- $\widehat{\mathbf{l}} := (l_{23}, l_{31}, l_{12})$ is the moment-vector computed by $\mathbf{p} \times \mathbf{l}$ with $\mathbf{p} := (p_1, p_2, p_3)$ being the coordinate vector of a point P $\in$ l in the Cartesian frame (O; x_1, x_2, x_3).

Basics: Line Geometry

The bijection $\mathcal{L} \to \Psi$ is also known as *Klein mapping*.

The *extended Klein mapping* identifies each point of P^5 with a linear complex $\mathcal{C} := (c_{01} : c_{02} : c_{03} : c_{23} : c_{31} : c_{12})$ of lines fulfilling the equation:

$$c_{01}l_{23} + c_{02}l_{31} + c_{03}l_{12} + c_{23}l_{01} + c_{31}l_{02} + c_{12}l_{03} = 0.$$

This set of lines equals the set of path-normals of an instantaneous motion different from the instantaneous standstill. For an instantaneous translation/rotation/screw motion the corresponding point $\mathsf{C} \in P^5$ of the linear line complex $\mathcal{C}$ has the property $\mathsf{C} \in L$ resp. $\mathsf{C} \in \Psi \setminus L$ resp. $\mathsf{C} \in P^5 \setminus \Psi$.

We give an alternative interpretation for the points of Ψ's ambient space P^5 and discuss its application.

Overview

1. Introduction: Quaternions $\mathbb{H}$

$$1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k} \ldots \text{quaternionic units}$$

$$\circ \ldots \text{quaternion multiplication}$$

$$\mathfrak{Q} := q_0 + q_1\mathfrak{i} + q_2\mathfrak{j} + q_3\mathfrak{k} \ldots \text{quaternion with } q_0, \ldots, q_3 \in \mathbb{R}$$

$$q_0 \ldots \text{scalar part}$$

$$\mathfrak{q} := q_1\mathfrak{i} + q_2\mathfrak{j} + q_3\mathfrak{k} \ldots \text{pure part}$$

$$\widetilde{\mathfrak{Q}} := q_0 - \mathfrak{q} \ldots \text{conjugated quaternion to } \mathfrak{Q} = q_0 + \mathfrak{q}$$

We embed points P of E^4 with coordinates (p_0, p_1, p_2, p_3) with respect to the Cartesian frame (O; x_0, x_1, x_2, x_3) into the set of quaternions by the mapping:

$$\iota : \mathbb{R}^4 \to \mathbb{H} \quad \text{with} \quad (p_0, p_1, p_2, p_3) \mapsto \mathfrak{P} := p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} = p_0 + \mathfrak{p}.$$

1. Lines in Euclidean 4-space

The so-called *homogenous minimal coordinates* [10] of a line l $\in E^4$ can be written as

$$(\mathfrak{L}, \mathfrak{m})\mathbb{R} \quad \text{with} \quad \mathfrak{m} := \widetilde{\mathfrak{L}} \circ \mathfrak{F} \quad \text{where}$$

$\mathfrak{F}$ ... corresponds to the pedal point F of l with respect to the origin O

$\mathfrak{L}$ ... corresponds to the direction of l $\in E^4$

$\mathfrak{m}$ ... is a pure quaternion

Theorem. There is a bijection between the set of lines of E^4 and the points of P^6, which is sliced along the 2-space $l_0 = l_1 = l_2 = l_3 = 0$; i.e. $\mathfrak{L} = 0$.

Let us identify E^3 with the hyperplane $x_0 = 0$.

2. Alternative interpretation

In the following we are only interested in the subset $\mathcal{M}$ of lines of E^4, which are orthogonal to the x_0-direction. As a consequence $\mathfrak{L}$ has to be a pure quaternion, i.e. the *homogenous minimal coordinates* of a line $\mathsf{l} \in \mathcal{M}$ read as:

$$(\mathfrak{l}, \mathfrak{m})\mathbb{R}.$$

Lines of $\mathcal{M}$ belonging to E^3 (given by $x_0 = 0$) are determined by the fact that $\mathfrak{F}$ is a pure quaternion, which is equivalent with the Plücker condition.

Therefore the following alternative interpretation of $P^5 \setminus L$ can be given:

Theorem. There is a bijection between the set $\mathcal{M}$ and the points of P^5, which is sliced along the 2-space $L: \; l_1 = l_2 = l_3 = 0$; i.e. $\mathfrak{l} = 0$. The lines of $\mathcal{M}$ belonging to E^3 correspond to the points of $\Psi \setminus L$.

2. Projection on the Plücker quadric

Recall that every point $(\mathbf{c}, \widehat{\mathbf{c}})\mathbb{R}$ of $P^5 \setminus L$ corresponds to the path-normals of a instantaneous rotation/screw motion. The Plücker coordinates of the so-called axis of this instantaneous rotation/screw motion are given by $(\mathbf{a}, \widehat{\mathbf{a}})\mathbb{R}$.

Therefore we can consider the mapping:

$$\mu: \; P^5 \setminus L \to \Psi \setminus L \quad \text{with} \quad (\mathbf{c}, \widehat{\mathbf{c}})\mathbb{R} \mapsto (\mathbf{a}, \widehat{\mathbf{a}})\mathbb{R}.$$

What is the geometric meaning of μ in terms of our alternative interpretation?

Answer: μ corresponds to the orthogonal projection of the line $\mathsf{l} \in \mathcal{M}$ onto E^3.
We denote this orthogonal projection $E^4 \to E^3$ by π.

3. Line-elements in Euclidean 3-space

For some applications (e.g. 3D shape recognition and reconstruction [6]) it is superior to study so-called line-elements instead of lines. As these geometric objects consist of a line l and a point P on it, we write them as (l, P).

Moreover we call a ruled surface together with a curve on it a *ruled surface strip*.

According to [14] the Plücker coordinates of lines can be extended for line-elements of E^3 by:

$$(\mathbf{l}, \widehat{\mathbf{l}}, l)\mathbb{R} \quad \text{with} \quad l := \langle \mathbf{p}, \mathbf{l} \rangle.$$

Theorem. There is a bijection between the set of line-elements of E^3 and all real points of P^6 located on a cone Λ over Ψ, which is sliced along the 3-dimensional generator space $G: \; l_{01} = l_{02} = l_{03} = 0$ of Λ.

3. Line-elements in Euclidean 4-space

The set $\mathcal{N}$ of line-elements (l, P) of E^4, where l is orthogonal to the x_0-direction, can be written in terms of *homogenous minimal coordinates* [10] by:

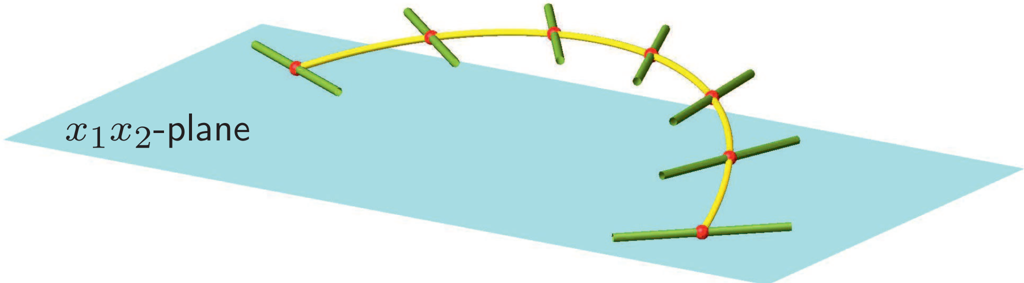
$$(\mathfrak{l}, l + \mathfrak{m})\mathbb{R}.$$

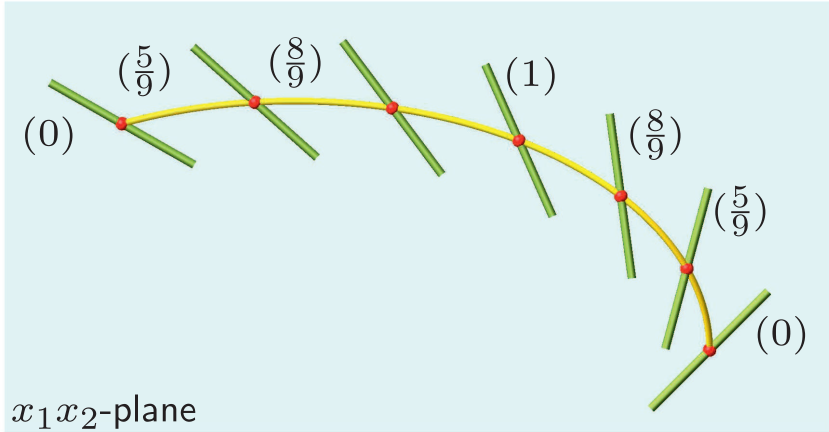
A straight forward extension of results for lines to line-elements yields:

Theorem. There is a bijection between the set $\mathcal{N}$ and the points of P^6, which is sliced along the 3-space $G: \ l_1 = l_2 = l_3 = 0$; i.e. $\mathfrak{l} = 0$. Moreover line-elements of $\mathcal{N}$ belonging to E^3 are located on the cone $\Lambda \setminus G$.

Moreover, the extension of the mapping μ to the set $\mathcal{N}$ corresponds to the orthogonal projection π of the line-element $(\mathsf{l}, \mathsf{P}) \in \mathcal{N}$ onto E^3.

3. Lower-dimensional analogue





Consider the set $\mathcal{Q}$ of line-elements of E^3, those lines are orthogonal to the x_3-direction, and its subset $\mathcal{P}$ of line-elements, which are contained in the x_1x_2-plane.

If we apply an orthogonal projection along the x_3-direction on the x_1x_2-plane (analogue of π) to line-elements of $\mathcal{Q}$, we obtain line-elements of $\mathcal{P}$.

We label the line-elements in the top view by the x_3-coordinate. In German such a map is known as "*kotierte Projektion*".

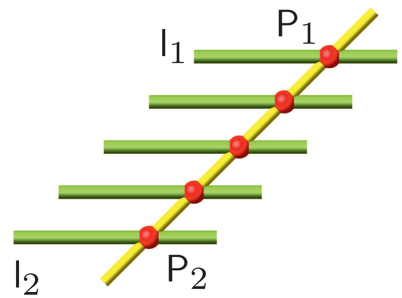
4. Straight lines in $P^6 \setminus G$ resp. $P^5 \setminus L$

Two distinct line-elements $(\mathsf{l}_i, \mathsf{P}_i) \in \mathcal{N}$ $(i = 1, 2)$ span a straight line q in $P^6 \setminus G$. If the underlying lines l_1 and l_2 are:

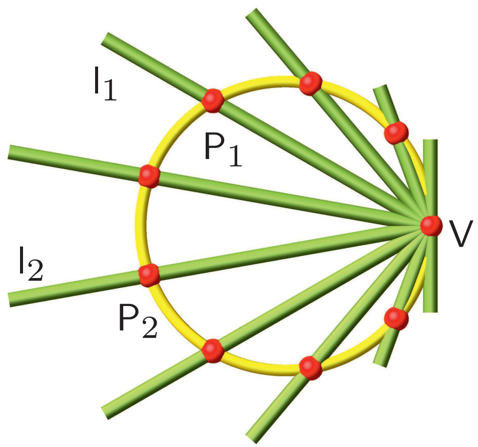
- **coplanar**, then we can distinguish two cases:

⋆ $\mathsf{l}_1 \parallel \mathsf{l}_2$: q corresponds to a ruled surface strip which consists of a parallel line pencil (spanned by l_1 and l_2) with a line on it.

Special case $\mathsf{l}_1 = \mathsf{l}_2$: q corresponds to the set of line-elements which have the same carrier line $\mathsf{l}_1 = \mathsf{l}_2$.



⋆ $\mathsf{l}_1 \nparallel \mathsf{l}_2$: q corresponds to a ruled surface strip consisting of a line pencil, where the vertex V is the intersection point of l_1 and l_2, and a circle on it, which is determined by $\mathsf{V}, \mathsf{P}_1, \mathsf{P}_2$.



4. Properties of (Γ, k)

• **skew**, then q corresponds to a ruled surface strip (Γ, k).

Theorem. Γ is a ruled cubic conoidal 2-surface (with director hyperplane $x_0 = 0$).

Remark: The image of Γ under π is the *Plücker conoid* $(=$ *cylindroid)* [3]. ◇

Theorem. Γ possesses a rational quadratic parametrization and is a LN-surface.

LN-property: For any 3-space there exists a unique parallel tangent plane of Γ. ◇

Theorem. k is a circle, which implies that Γ carries a 2-parametric set of circles.
The striction curve s of Γ is a circle and it is a geodesic curve of Γ.

Remark: Note that $\pi(\mathsf{s})$ coincides with the common normal of $\pi(\mathsf{l}_1)$ and $\pi(\mathsf{l}_2)$.
All other circles k on Γ are mapped to ellipses $\pi(\mathsf{k})$. ◇

5. Kinematic relevance of Γ

Now we want to study the one-parametric motion in E^4, which is generated by reflecting the coordinate frame in the one-parametric set of Γ's rulings. Such a motion is called line-symmetric and Γ is the corresponding *basic surface* (cf. [1]).

In this context the following theorem can be proven:

Theorem. The line-symmetric motion in E^4 with basic surface Γ is a circular Darboux 2-motion, which is neither spherical nor a pure translation, and vice versa.

Circular Darboux 2-motion: All points have circular trajectories. This motion can be interpreted as a straight line in the ambient space of the Study quadric. For more details please see [12]. $\diamond$

6. Application

We perform a projective De Casteljau algorithm in the projective space of dimension:

- 5 for the design of rational ruled surfaces using

$$(l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12})$$

- 6 for the design of rational ruled surface-strips using

$$(l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12} : l)$$

- 7 for the design of rational ruled surface-patches using

$$(l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12} : l_1 : l_2) \quad \text{with} \quad l_i := \langle \mathbf{p}_i, \mathbf{l} \rangle \quad \text{for} \quad i = 1, 2$$

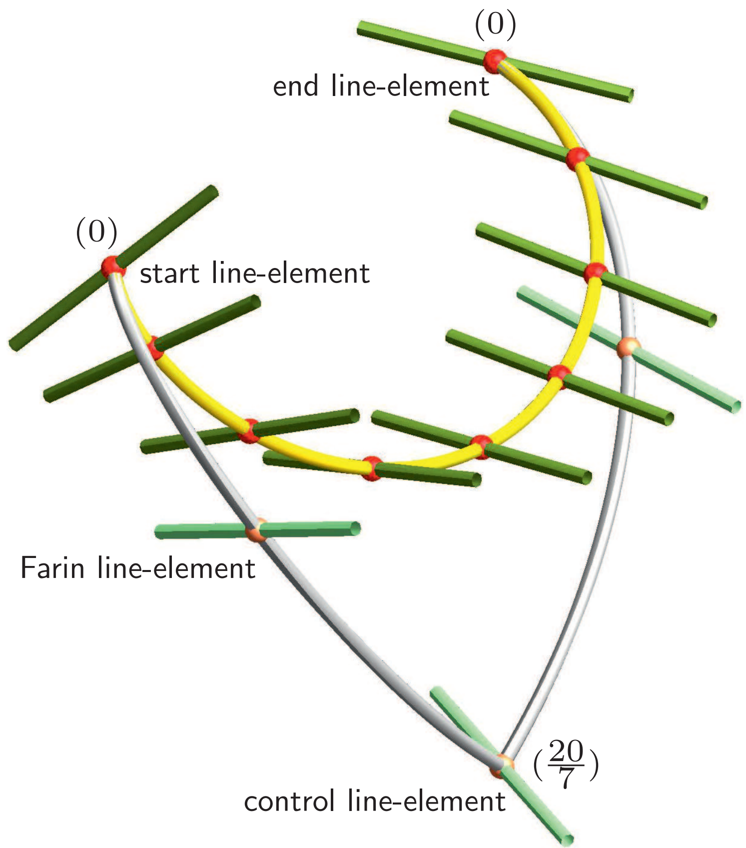
where $\mathbf{p}_i$ are the two boundary points of the patch along the ruling.

6. Application

Projective de Casteljau construction:

The resulting curve $\in P^{5/6/7}$ can be interpreted as a conoidal ruled 2-surface/surface-strip/surface-patch in E^4 with director hyperplane $x_0 = 0$. By applying the orthogonal projection π in x_0-direction we obtain the desired ruled surface/surface-strip/surface-patch in E^3.

By labeling the projected lines/line-elements/line-segments by the x_0-coordinate, the user can modify very intuitively the control structure; i.e. the Farin and control lines/line-elements/line-segments can be changed by *mouse action* and their x_0-heights by the *scroll wheel*.



7. References

Finally we referred to

- [12], where an analogue algorithm for an user-friendly design of rational motions in E^3 is described,
- [13], where the proofs of the presented theorems are given (due to the limitation of pages).

All references refer to the list of publications given in the presented paper:

NAWRATIL, G.: Alternative interpretation of the Plücker quadric's ambient space and its application. In Proc. of ICGG 2018 (L. Cocchiarella, Ed.), pages 918–929, Springer Nature (2019)