Review and Recent Results on Stewart Gough Platforms with Self-motions

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What is a self-motion of a SGP?

The geometry of a SGP is given by the six base anchor points $M_i \in \Sigma_0$ and by the six platform points $m_i \in \Sigma$.

A SGP is called planar, if $M_1, \ldots, M_6$ are coplanar and $m_1, \ldots, m_6$ are coplanar.

$M_i$ and $m_i$ are connected with a SPS leg.

If all $P$-joints are locked, a SGP is in general rigid. But under particular conditions, the manipulator can perform an $n$-parametric motion ($n > 0$), which is called self-motion.
Historical background

All self-motions of SGPs are solutions to the still unsolved Borel Bricard (BB) problem, posed by the French Academy of Science for the Prix Vaillant 1904:

Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.

The papers of Borel [2] and Bricard [3] were awarded prizes, but both authors only presented partial solutions (see [4,5]). Known results before the year 1904 are e.g.:

(a) Chasles [6]: If points of two conics are in projective correspondence, then there exists a spatial motion, which keeps the corresponding points at fixed distance.

(b) Bricard [7]: The only non-trivial motion, where all points have spherical paths.

(c) Bricard [8]: All three types of flexible octahedra in the Euclidean 3-space.
It is known, that architecturally singular SGPs possess self-motions (over $\mathbb{C}$) in each pose. Their designs are well studied (see [18–21] for the planar case and [22,23] for the non-planar one), but less is known about their self-motions [24–26].

Therefore, we are only interested in self-motions of non-architecturally singular SGPs. Until now, only a few examples of this type are known:

- **Husty and Zsombor-Murray** [28] reported SGPs with a Schönflies self-motion of item (b). Planar SGPs of this type are also called *polygon platforms* (cf. [29]).

- **Zsombor-Murray et al.** [30] presented SGPs with a line-symmetric self-motion, which was already known to Borel [2], Bricard [3] and Krames [15].

- **Husty and Karger** [31] proved that the list of Schönflies Borel Bricard motions given by Borel [2] is complete.
Review on SGPs with self-motions

- Karger [32] studied all self-motions of planar SGPs, where the platform and the base are affinely equivalent (for the special case of equiform and congruent platform and base see [33] and [34], respectively).

- Nawratil [35,36] gave a complete list of TSSM designs (planar 6–3 SGPs) with self-motions as by-product of the determination of all flexible octahedra in the projective extension of the Euclidean 3-space [37].

- Karger and Husty [39] classified all self-motions of the original SGP.

- Karger [40,41] presented a method for designing planar SGPs with self-motions of the type $e_0 = 0$, where $e_0$ denotes an Euler parameter.

- Geiß and Schreyer [42] gave a pure algebraic method for the computation of further SGPs, where the self-motion has a planar spherical image.
Mielczarek et al. [47] showed that the set $\Lambda$ of additional legs, which can be attached to a given planar SG platform $m_1, \ldots, M_6$ without restricting the forward kinematics, is determined by a linear system of equations (Eq. (30) of [47]).

As the solvability condition of this system is equivalent with the criterion given in Eq. (12) of [48] also the singularity surface of the SGP does not change by adding legs of $\Lambda$.

Moreover, it was shown in [47] that in the general case $\Lambda$ is 1-parametric and that the base anchor points as well as the corresponding platform anchor points are located on planar cubic curves $C$ and $c$, respectively.

**Assumption 1**

We assume, that there exist such cubic curves $c$ and $C$ (which can also be reducible) in the Euclidean domain of the platform and the base, respectively.
**Example: Octahedral SGP**

**Notation:** $U_1, U_2, U_3$ are the ideal points of $C$. $u_4, u_5, u_6$ are the ideal points of $c$.

Moreover the yellow, red and green legs are additional legs belonging to the set $\Lambda$. 
Redundant planar SGPs

As the correspondence between c and C has not to be a bijection (see e.g. octahedral SGP), a point $\in P_C^3$ of c resp. C is in general mapped to a non-empty set of points $\in P_C^3$ of C resp. c. We denote this set by the term corresponding location and indicate this fact by the usage of brackets $\{\}$. 

**Assumption 2**
For guaranteeing a general case, we assume that each of the corresponding locations $\{u_1\}, \{u_2\}, \{u_3\}, \{U_4\}, \{U_5\}, \{U_6\}$ consists of a single point. Moreover, we assume that no four collinear platform points $u_i$ or base points $U_i$ for $i = 1, \ldots, 6$ exist.

Due to Assumption 2, the six pairs of anchor points $(u_i, U_i)$ with $i = 1, \ldots, 6$ are uniquely determined.

**Basic idea:** Attach the special “legs” $u_i U_i$ with $i = 1, \ldots, 6$ to SGP $m_1, \ldots, M_6$. 
Darboux and Mannheim motion

The attachment of the “legs” \( \overline{u_i U_i} \) with \( i = 1, 2, 3 \) corresponds with the so-called **Darboux constraint**, that \( u_i \) moves in a plane of the fixed system orthogonal to the direction of \( U_i \).

The attachment of the “leg” \( \overline{u_j U_j} \) with \( j = 4, 5, 6 \) corresponds with the so-called **Mannheim constraint**, that a plane of the moving system orthogonal to \( u_j \) slides through the point \( U_j \).
Types of self-motions

By removing the originally six legs $m_i \overline{M_i}$ we remain with the manipulator $u_1, \ldots, U_6$.

**Theorem 1** (Proof is given in Nawratil [43])

Given is a planar SGP $m_1, \ldots, M_6$, which is not architecturally singular and which fulfills Assumption 1 and 2. Then, the resulting manipulator $u_1, \ldots, U_6$ is redundant and therefore architecturally singular.

Moreover, it was also proven in [43] that there only exist type I and type II Darboux Mannheim (DM) self-motions, where the definition of types reads as follows:

**Definition 1**

Assume $\mathcal{M}$ is a 1-parametric self-motion of a non-architecturally singular SGP $m_1, \ldots, M_6$. Then $\mathcal{M}$ is of type $n$ DM (Darboux Mannheim) if the corresponding architecturally singular manipulator $u_1, \ldots, U_6$ has an $n$-parametric self-motion $\mathcal{U}$. 
Planar SGPs with type II DM self-motions

Nawratil [44,45] proved the necessity of three conditions for obtaining a type II DM self-motion. Based on these conditions, all planar SGPs fulfilling Assumption 1 and 2 with a type II DM self-motion were determined in [46]. They are either

i. generalizations of line-symmetric Bricard octahedra (12-dim. solution set) or

ii. special polygon platforms (11-dim. solution set).

Moreover, it was shown in [46] that the type II DM self-motions of all these SGPs are line-symmetric and octahedral, where the latter property is defined as follows:

**Definition 2**

A DM self-motion is called octahedral if following triples of points are collinear for $i \neq j \neq k \neq i$ and $i, j, k \in \{1, 2, 3\}$:

- $(u_i, u_j, u_6)$,
- $(u_i, u_k, u_5)$,
- $(u_j, u_k, u_4)$,
- $(U_4, U_5, U_k)$,
- $(U_5, U_6, U_i)$,
- $(U_4, U_6, U_j)$.
The geometric interpretation of the three necessary conditions identifies a new property of line-symmetric Bricard octahedra $\mathcal{O}$ with vertices $1_a, 1_b, 2_a, 2_b, 3_a, 3_b$, where $v_a$ and $v_b$ are symmetric with respect to the line $l$ for $v \in \{1, 2, 3\}$. The following planes have a common line $T_{ijk}$:

- plane orthogonal to $[1_i, 2_j]$ though $3_k'$
- plane orthogonal to $[2_j, 3_k]$ though $1_i'$
- plane orthogonal to $[3_k, 1_i]$ though $2_j'$

with $i \neq i', j \neq j', k \neq k' \in \{a, b\}$.

**New property:** $\mathcal{O}$ possesses the 8 lines $T_{ijk}$.
New results in context of Assumption 1 and 2

For the determination of all planar SGPs with a type II DM self-motion, where only Assumption 1 holds, we can assume w.l.o.g. that four platform or base anchor points are collinear, due to the following theorem:

**Theorem 2**
Given is a planar SGP fulfilling Assumption 1. If one cannot choose anchor points within \( \{u_1\}, \{u_2\}, \{u_3\}, \{U_4\}, \{U_5\}, \{U_6\} \), such that four platform anchor points \( u_i \) or base anchor points \( U_i \) are collinear, then each of these corresponding locations has to consist of a single point.

**Proof:** For the proof, please see page 155 of the presented paper.

**Remark:** A detailed study of these SGPs is dedicated to future research.
New results in context of Assumption 1 and 2

Γ denotes the set of planar architecturally singular SGPs with no four points collinear, which do not belong to item (a) (cf. Chasles [6]).

**Theorem 3**
To any planar SGP \( m_1, \ldots, M_6 \) with exception of the set \( \Gamma \), at least a 1-parametric set of additional legs \( \Lambda \) can be attached.

**Proof:** For the proof, please see pages 155–156 of the presented paper. □

Beside Theorem 3, only one non-trivial exceptional case of Assumption 1 is known:

For a planar SGP, where the platform and base are related by a projectivity \( \kappa \) (so-called planar projective SGP), the set \( \Lambda \) is 2-parametric, whereas the correspondence of anchor points of \( \Lambda \) is given by the projectivity \( \kappa \) itself (cf. Nawratil [52]).

**Remark:** A complete list of special cases is dedicated to future research. ◇
[6] Planar projective SGPs with self-motions

s denotes the lines of intersection of the planar platform and the planar base in the projective extension of the Euclidean 3-space.

**Definition 3**
A self-motion of a non-architecturally singular planar projective SGP is called elliptic, if in each pose of this motion s exists with \( s = s_\kappa \) and the projectivity from s onto itself is elliptic.

Under consideration of this definition, the following result can be proven:

**Theorem 4** (Proof is given in Nawratil [52])
Non-architecturally singular planar projective SGPs can only have either elliptic self-motions or pure translational ones. In the latter case \( \kappa \) has to be an affinity \( t + \mathbf{T}x \), where the singular values \( s_1, s_2 \) of \( \mathbf{T} \) with \( 0 < s_1 \leq s_2 \) fulfill \( s_1 \leq 1 \leq s_2 \).

**Remark:** The study of elliptic self-motions is dedicated to future research.  

\[ \diamond \]
Definition 4
Assume a planar SGP is given, where one can add the set $\Lambda$ of legs. A self-motion of this manipulator is called degenerated if one can add further legs beside $\Lambda$ to the planar manipulator without restricting the self-motion.

Butterfly motion

Spherical 4-bar motion
Planar SGPs with type I DM self-motions

⋆ Degenerated rotational self-motions: The butterfly motion and the spherical 4-bar motion of the octahedral SGP \([53]\) and the corresponding motions of the original SGP \([39]\).
Every SGP with four collinear anchor points possesses a butterfly motion.

⋆ Degenerated Schönflies self-motions: General polygon platforms.

⋆ Non-degenerated self-motions: Bricard octahedra of type 2 and 3.

In view of all known SGPs with self-motions, we have good reasons for the following central conjecture:

**Conjecture 1**
All non-degenerated 1-parametric self-motions of non-architecturally singular planar SGPs, fulfilling Assumption 1 and 2, are octahedral.
References and acknowledgements

All references refer to the list of publications given in the presented paper:


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