Noname manuscript No.
(will be inserted by the editor)

Types of self-motions of planar Stewart Gough platforms

Georg Nawratil

Received: date / Accepted: date

Abstract We show that the self-motions of general planar Stewart Gough platforms can be characterized in the complex extension of the Euclidean 3-space by the movement of three platform points in planes orthogonal to the planar base (3-point Darboux motion) and a simultaneous sliding of three planes orthogonal to the planar platform through points of the base (3-plane Mannheim motion). Based on this consideration, we prove that all one-parametric self-motions of a general planar Stewart Gough platform can be classified into two types (type I DM and type II DM, where DM abbreviates Darboux Mannheim). We also succeed in presenting a set of 24 equations yielding a type II DM self-motion that can be computed explicitly and that is of great simplicity seen in the context of self-motions. These 24 conditions are the key for the complete classification of general planar Stewart Gough platforms with type II DM self-motions, which is an important step in solving the famous Borel Bricard problem.

Keywords Self-motion · Stewart Gough platform · Borel Bricard problem · Darboux motion · Mannheim motion

1 Introduction

A Stewart Gough (SG) platform is a parallel manipulator consisting of a moving platform which is connected via six Spherical-Prismatic-Spherical legs with the base.

G. Nawratil
Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstrasse 8-10/104, Vienna, A-1040, Austria
Tel.: (+43)-(0)1-58801/104362
Fax: (+43)-(0)1-58801/10493
E-mail: nawratil@geometrie.tuwien.ac.at

Therefore, the geometry of a SG platform with planar platform and planar base (which is also known as planar SG platform) is given by the six base anchor points M_i with coordinates $\mathbf{M}_i := (A_i, B_i, 0)^T$ with respect to the fixed system Σ_0 and by the six platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, b_i, 0)^T$ with respect to the moving system Σ.

It is well known [1], that a SG platform is singular if and only if the carrier lines of the prismatic legs belong to a linear line complex, or analytically seen, if $det(\mathbf{J}) = 0$ holds, where the i^{th} row of the 6×6 matrix $\mathbf{J}$ equals the Plücker coordinates $\underline{\mathbf{l}}_i$ of the i^{th} carrier line.

If the geometry of the SG platform is given as well as the six leg lengths R_i, then the manipulator is in general rigid in one of its 40 possible assembly modes. But, under particular conditions, the manipulator can perform an n-parametric motion ($n > 0$), which is called self-motion. Clearly, in each pose of a self-motion the SG platform is singular. Moreover, all self-motions of SG manipulators are solutions to the famous Borel Bricard problem [2–5]. This still unsolved problem was posed 1904 by the French Academy of Science for the Prix Vaillant and reads as follows: *"Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths."*

Beside the historical and theoretical interest in the solution of the Borel Bricard problem, there is also a practical one in the context of SG platforms [4,6]. If one is aware of all SG designs with self-motions, it is an easy task to avoid such geometries in the design process. For examples of designs without self-motions see [7].

1.1 Review

It is already known, that manipulators, which are singular in every possible configuration, possess self-motions in each pose. These manipulators are so-called architecturally singular SG platforms [8] and they are well studied: For the characterization of architecturally singular planar SG platforms, we refer to [9–12]. For the non-planar case, we refer to [13,14]. Therefore, we are only interested in the computation of self-motions of non-architecturally singular SG platforms. Until now, only few self-motions of this type are known, as their computation is a very complicated task.

To the best knowledge of the author, a complete and detailed review on non-architecturally singular SG platforms with self-motions was given in [7]. For the understanding of the following pages, it is enough to notice, that Karger [15,16] presented a method for designing planar SG platforms with self-motions of the type $e_0 = 0$, where e_0 denotes an Euler parameter. Karger's method is based on the study and classification of all self-motions of the original SG platform (cf. [17]).

1.2 Related work and overview

A motion, where n points of a rigid body move in n corresponding fixed planes, is called n-point Darboux motion. In the third note of Darboux to the book of Kœnigs [18] the case $n = 3$ was extensively studied. The inverse motion of the Darboux motion is the Mannheim motion [19]. Therefore, a motion, where n planes of a rigid body slide through n corresponding fixed points, is called n-plane Mannheim motion.

Before we give an overview for the article at hand, we repeat some results of Röschel and Mick [11], which are of importance for our considerations: A planar SG platform (where not all six base anchor points or all six platform anchor points are collinear) is architecturally singular if and only if $rk(\mathbf{T}) < 6$ holds with $\mathbf{T} = (\mathbf{p}_1, \ldots, \mathbf{p}_6)$ and where $\mathbf{p}_i$ equals

$$(w_iW_i, w_iX_i, w_iY_i, x_iW_i, x_iX_i, x_iY_i, y_iW_i, y_iX_i, y_iY_i)^T.$$

Note that $(w_i : x_i : y_i)$ and $(W_i : X_i : Y_i)$ are the homogeneous coordinates of the platform and base anchor points, respectively, i.e.

$$a_i = \frac{x_i}{w_i}, \;\; b_i = \frac{y_i}{w_i}, \;\; A_i = \frac{X_i}{W_i}, \;\; B_i = \frac{Y_i}{W_i}. \qquad (1)$$

Moreover, the criterion $rk(\mathbf{T}) < 6$ is invariant with respect to (even different) non-singular collineations in the platform and the base, respectively.

In Section 2, all the preparatory work is done for identifying each self-motion of a general planar SG platform as a 3-point Darboux motion and a simultaneous 3-plane Mannheim motion in the complex extension of the Euclidean 3-space. For the rest of this article, these motions are called Darboux Mannheim self-motions (DM self-motions). Moreover, in Section 3, we distinguish between different types (type I, type II, ...) of one-parametric self-motions of SG platforms, which are not architecturally singular, and compare the computation of ordinary type II self-motions and type II DM self-motions. In Section 3.3, we give the construction of non-architecturally singular planar SG platforms with a type II DM self-motion. Moreover, in Section 4, we propose a clear classification scheme for all one-parametric self-motions of general planar SG platforms. Finally, in Section 5, we give a geometric interpretation of a large set of already known type II DM self-motions. In addition, all given results are demonstrated on the basis of concrete examples. We close this article with conclusions and an outlook on future research.

2 Preliminary considerations

A major role in the given study and classification of self-motions of planar SG platforms play redundant SG platforms, which were studied by Husty et al. [20] and Mielczarek et al. [21]. It turned out, that if a planar SG platform $\mathsf{m}_1, \ldots, \mathsf{M}_6$ is not architecturally singular, then at least a one-parametric set of legs exists, which can be attached to the given manipulator without changing the forward kinematics. The underlying linear system of equations is given in Eq. (30) of [21]. As the solvability condition of this system is equivalent with the criterion given in Eq. (12) of [22], also the singularity surface of the manipulator does not change by adding legs of this one-parametric set.

Moreover, it was shown that in general[1], the base anchor points M_i as well as the corresponding platform anchor points m_i are located on planar cubic curves C and c. Note that these cubics can also split up.

Assumption 1 *We assume for the rest of this article, that such cubic curves* c *and* C *(which can also be reducible) exist in the Euclidean domain of the platform and the base, respectively.*

Now, we consider the complex projective extension $P^3_{\mathbb{C}}$ of the Euclidean 3-space E^3 (cf. Eq. (1)). Note that ideal points are characterized by $w_i = 0$ and $W_i = 0$, respectively. Therefore, we denote in the remainder of

[1] Until now, a complete list of the special cases is missing. For known non-trivial special cases see [7].

this article the coordinates of anchor points, which are ideal points, by x_i, y_i and X_i, Y_i, respectively. For all other anchor points we use the coordinates a_i, b_i and A_i, B_i, respectively.

The correspondence between the points of C and c in $P^3_{\mathbb{C}}$, which is determined by the geometry of the manipulator $\mathsf{m}_1, \ldots, \mathsf{M}_6$, can be computed according to [20,21] or [22] under consideration of Eq. (1). As this correspondence has not to be a bijection, a point $\in P^3_{\mathbb{C}}$ of c resp. C is in general mapped to a non-empty set of points $\in P^3_{\mathbb{C}}$ of C resp. c. We denote this set by the term *corresponding location* and indicate this fact by the usage of brackets $\{\,\}$.

In $P^3_{\mathbb{C}}$ the planar cubic C has three ideal points $\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3$, where at least one of these points is real. Without loss of generality (w.l.o.g.), we can assume that this point is U_1. The remaining two points U_2 and U_3 are real or conjugate complex points. Then, we compute the corresponding locations $\{\mathsf{u}_1\}, \{\mathsf{u}_2\}, \{\mathsf{u}_3\}$ of c ($\Rightarrow$ $\{\mathsf{u}_1\}$ contains real points). We denote the ideal points of c by $\mathsf{u}_4, \mathsf{u}_5, \mathsf{u}_6$, where again at least one of these points has to be real. W.l.o.g., we can assume that this point is u_4. The remaining two points u_5 and u_6 are again real or conjugate complex points. Then, we compute the corresponding locations $\{\mathsf{U}_4\}, \{\mathsf{U}_5\}, \{\mathsf{U}_6\}$ of C ($\Rightarrow$ $\{\mathsf{U}_4\}$ contains real points).

Assumption 2 *For guaranteeing a general case, we assume that each of the corresponding locations* $\{\mathsf{u}_1\}$, $\{\mathsf{u}_2\}$, $\{\mathsf{u}_3\}$, $\{\mathsf{U}_4\}$, $\{\mathsf{U}_5\}$, $\{\mathsf{U}_6\}$ *consists of a single point. Moreover, we assume that no four collinear platform points* u_i *or base points* U_i *for* $i = 1, \ldots, 6$ *exist.*

2.1 Basic idea

Now the basic idea can simply be expressed by attaching the special "legs" $\overline{\mathsf{u}_i\mathsf{U}_i}$ with $i = 1, \ldots, 6$ to the manipulator $\mathsf{m}_1, \ldots, \mathsf{M}_6$. We have to quote the word legs in this context, as it is impossible to attach physical legs with infinite length to the platform. Moreover, some of the anchor points can even be complex points. Therefore, we have to choose a different point of view; namely the pure algebraic one:

The constraint that m_i is located on the end-point of a leg attached in M_i, corresponds with the condition that m_i moves on a sphere centered in M_i. By applying a limiting process $\mathsf{M}_i \to \mathsf{U}_i$, it can easily be seen that the sphere degenerates into a plane through m_i orthogonal to the direction of the ideal point U_i. Therefore, the attachment of the "leg" $\overline{\mathsf{u}_i\mathsf{U}_i}$ with $i = 1, 2, 3$ corresponds with the so-called Darboux constraint, that the platform anchor point u_i moves in a plane of the fixed system orthogonal to the direction of the ideal point U_i.

Moreover, the condition that m_i moves on a sphere centered in M_i is equivalent with the condition that M_i is located on sphere with center m_i. By applying a limiting process $\mathsf{m}_i \to \mathsf{u}_i$, it can again be seen that the sphere degenerates into a plane through M_i orthogonal to the direction of the ideal point u_i. Therefore, the attachment of the "leg" $\overline{\mathsf{u}_i\mathsf{U}_i}$ with $i = 4, 5, 6$ corresponds with the so-called Mannheim constraint, that a plane of the moving system orthogonal to u_i slides through the point U_i.

Remark 1 Note that due to Assumption 2 not both points m_i and M_i can be ideal points. ⋄

In the remaining part of the paper we show, that these Darboux and Mannheim constraints are very helpful in the study of self-motions of planar SG platforms. The equations of these constraints are computed in the next section.

2.2 Mathematical framework

For the determination of self-motions, it is advantageous to work in the Study parameter space $P^7_{\mathbb{R}}$, which is a 7-dimensional real projective space with homogeneous coordinates $e_0, \ldots, e_3, f_0, \ldots f_3$. By using these so-called Study parameters for the parametrization of Euclidean displacements, the coordinates $\mathbf{m}'_i$ of the platform anchor points with respect to Σ_0 can be written as $K\mathbf{m}'_i = \mathbf{R}\,\mathbf{m}_i + (t_1, t_2, t_3)^T$ with

$$
\begin{aligned}
t_1 &:= 2(e_0f_1 - e_1f_0 + e_2f_3 - e_3f_2),\\
t_2 &:= 2(e_0f_2 - e_2f_0 + e_3f_1 - e_1f_3),\\
t_3 &:= 2(e_0f_3 - e_3f_0 + e_1f_2 - e_2f_1),
\end{aligned}
$$

and the rotational matrix $\mathbf{R} := (r_{ij})$ given by:

$$
\begin{bmatrix}
e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2)\\
2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1)\\
2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2
\end{bmatrix}
$$

with the Euler parameters $e_0, \ldots, e_3$. Now, all points of $P^7_{\mathbb{R}}$, which are located on the so-called Study quadric $\Psi : \sum_{i=0}^{3} e_if_i = 0$, correspond with Euclidean displacements with exception of the subspace $e_0 = \ldots = e_3 = 0$ of Ψ, as these points cannot fulfill the normalizing condition $K = 1$ with $K := e_0^2 + e_1^2 + e_2^2 + e_3^2 \neq 0$.

Sphere constraint: Husty [24] showed that the condition for m_i to be located on a sphere with center M_i

and radius R_i can be expressed by the following homogeneous quadratic equation Λ_i:

$$\begin{aligned}&(A_i^2+B_i^2+a_i^2+b_i^2-R_i^2)K+4(f_0^2+f_1^2+f_2^2+f_3^2)\\&+2(e_3^2-e_0^2)(A_ia_i+B_ib_i)+2(e_2^2-e_1^2)(A_ia_i-B_ib_i)\\&+4[(f_0e_2-e_0f_2)(B_i-b_i)+(e_1f_3-f_1e_3)(B_i+b_i)\\&+(f_2e_3-e_2f_3)(A_i+a_i)+(f_0e_1-e_0f_1)(A_i-a_i)\\&+e_0e_3(A_ib_i-B_ia_i)-e_1e_2(A_ib_i+B_ia_i)]=0.\end{aligned}$$

Darboux constraint: The constraint that the platform anchor point u_i $(i=1,2,3)$ moves in a plane of the fixed system orthogonal to the direction of the ideal point U_i can be written as

$$(X_i,Y_i,0)^T\cdot\left(\mathbf{R}\,(a_i,b_i,0)^T+(t_1,t_2,t_3)^T\right)+L_iK=0,$$

with $X_i,Y_i,a_i,b_i,L_i\in\mathbb{C}$ and "$\cdot$" denoting the dot product. This yields the Darboux constraint Ω_i:

$$\overline{X}_i(a_ir_{11}+b_ir_{12}+t_1)+\overline{Y}_i(a_ir_{21}+b_ir_{22}+t_2)+L_iK=0,$$

which is a homogeneous quadratic equation in the Study parameters. Note that $\overline{X}_i$ and $\overline{Y}_i$ denote the conjugate complex of X_i and Y_i, respectively, as U_i can also be a complex point for $i=2$ or $i=3$.

Mannheim constraint: The constraint that the plane orthogonal to u_i $(i=4,5,6)$ through the platform point $(g_i,h_i,0)^T$ slides through the point $\mathsf{U}_i\in\Sigma_0$ can be written as

$$\begin{aligned}\left[\left(\mathbf{R}\,(x_i,y_i,0)^T\right)\cdot(A_i,B_i,0)^T\right]K-\\\left(\mathbf{R}\,(x_i,y_i,0)^T\right)\cdot\left(\mathbf{R}\,(g_i,h_i,0)^T+(t_1,t_2,t_3)^T\right)=0,\end{aligned}$$

with $x_i,y_i,A_i,B_i,g_i,h_i\in\mathbb{C}$. Then, we can factor out K and the Mannheim constraint Π_i remains:

$$\begin{aligned}&\overline{x}_i[A_ir_{11}+B_ir_{21}-2(e_0f_1-e_1f_0-e_2f_3+e_3f_2)]\\&+\overline{y}_i[A_ir_{12}+B_ir_{22}-2(e_0f_2+e_1f_3-e_2f_0-e_3f_1)]\\&-K(\overline{x}_ig_i+\overline{y}_ih_i)=0,\end{aligned}$$

where $\overline{x}_i$ and $\overline{y}_i$ denote the conjugate complex of x_i and y_i, respectively, as u_i can also be a complex point for $i=5$ or $i=6$. This is again a homogeneous quadratic equation in the Study parameters.

Remark 2 As the plane of the Mannheim motion has to intersect the x-axis or y-axis of the moving frame, we can set one of the variables g_i,h_i equal to zero. Then, the Darboux constraint and the Mannheim constraint have the same number of terms, namely 24. ◇

2.3 Implications of the given two assumptions

Due to the Assumptions 1 and 2, the following theorem can be proven:

Theorem 1 *Given is a planar SG platform* $\mathsf{m}_1,\ldots,\mathsf{M}_6$, *which is not architecturally singular and which fulfills the Assumptions 1 and 2. Then, the resulting manipulator* $\mathsf{u}_1,\ldots,\mathsf{U}_6$ *is redundant and therefore architecturally singular.*

Proof As the points u_i and U_i are located on the cubics c and C, the corresponding Darboux and Mannheim constraints do not change the direct kinematics and singularity surface of the manipulator $\mathsf{m}_1,\ldots,\mathsf{M}_6$. Therefore, Ω_i and Π_j can be written as the following linear combinations:

$$\Omega_i=\sum_{k=1}^{6}\lambda_{i,k}\Lambda_k,\quad \Pi_j=\sum_{k=1}^{6}\lambda_{j,k}\Lambda_k,\qquad(2)$$

for $i=1,2,3$ and $j=4,5,6$, according to [20,21]. As the Darboux and Mannheim constraints are only linear in the Study parameters $f_0,\ldots,f_3$, in contrast to the sphere constraints, which are quadratic in these parameters, the equations can be rewritten as:

$$\Omega_i=\sum_{k=2}^{6}\delta_{i,k}\Delta_k,\quad \Pi_j=\sum_{k=2}^{6}\delta_{j,k}\Delta_k,$$

as $\Delta_k:=\Lambda_1-\Lambda_k$ is also only linear in $f_0,\ldots,f_3$. Now it can be seen, that the set of the six polynomials $\Omega_1,\Omega_2,\Omega_3,\Pi_4,\Pi_5,\Pi_6$ are redundant, because they can be generated as linear combinations of $\Delta_2,\ldots,\Delta_6$. As a consequence the manipulator $\mathsf{u}_1,\ldots,\mathsf{U}_6$ is redundant and therefore architecturally singular. □

Due to Theorem 1 and the assumption that no four collinear platform anchor points u_i or base anchor points U_i exist (cf. Assumption 2), we can apply the corollary of Lemma 2 given by Karger [9] to our manipulator. This implies that all anchor points of the platform $\mathsf{u}_1,\ldots,\mathsf{u}_6$ and as well of the base $\mathsf{U}_1,\ldots,\mathsf{U}_6$ are distinct.

Example 1 For the verification of Theorem 1 on the basis of an example, we take the data of Example 2 given by Karger [15].[2] In this example a planar SG platform with a self-motion of type $e_0=0$ was computed, where a one-parametric set of legs can be attached without

[2] This example is initialized in [15] by the following coordinates of the first four pairs of anchor points: $A_1=B_1=B_2=a_1=b_1=b_2=0$, $a_2=b_3=2$, $A_2=a_3=b_4=3$, $A_3=a_4=1$, $B_3=5$, $A_4=-21155/1872$ and $B_4=165/8$.

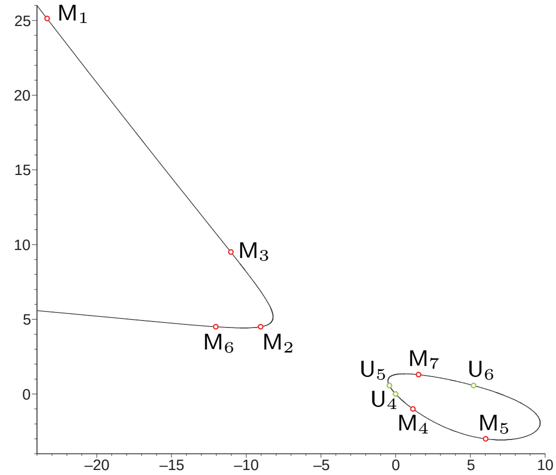

Fig. 1 The base of the manipulator given by Karger [15]. Note that the ideal points $\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3$ are given by the collinearity of the following point triples: $(\mathsf{U}_4, \mathsf{U}_5, \mathsf{U}_3), (\mathsf{U}_5, \mathsf{U}_6, \mathsf{U}_1), (\mathsf{U}_4, \mathsf{U}_6, \mathsf{U}_2)$.

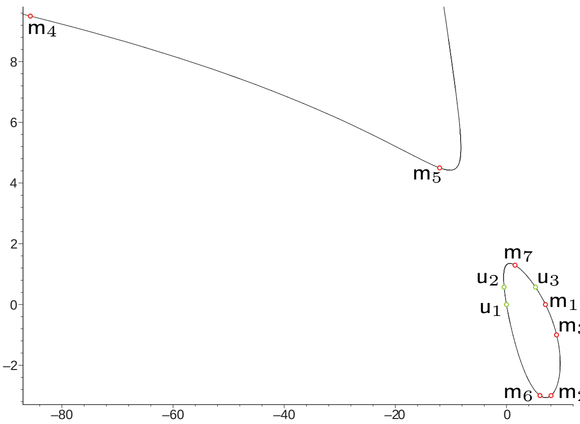

Fig. 2 The platform of the manipulator given by Karger [15]. Note that this figure is scaled in order to give a proper illustration. Again, the ideal points $\mathsf{u}_4, \mathsf{u}_5, \mathsf{u}_6$ are determined by the collinearity of the following points triples: $(\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_6), (\mathsf{u}_1, \mathsf{u}_3, \mathsf{u}_5), (\mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_4)$.

disturbing the self-motion. For this example we get the following coordinates for $\mathsf{u}_1, \ldots, \mathsf{U}_6$:

$$
\begin{aligned}
X_1 &= 1, & Y_1 &= 0,\\
X_2 &= 1677/400 + \sqrt{3895129}/400, & Y_2 &= 1,\\
X_3 &= 1677/400 - \sqrt{3895129}/400, & Y_3 &= 1,\\
A_4 &= 1203/100, & B_4 &= -9/2,\\
A_5 &= 5049/350 - \sqrt{3895129}/700, & B_5 &= -55/14,\\
A_6 &= 5049/350 + \sqrt{3895129}/700, & B_6 &= -55/14,\\
a_1 &= -301/50, & b_1 &= 3,\\
a_2 &= -2537/700 - \sqrt{3895129}/700, & b_2 &= 25/7,\\
a_3 &= -2537/700 + \sqrt{3895129}/700, & b_3 &= 25/7,\\
x_4 &= 1, & y_4 &= 0,\\
x_5 &= 1677/400 + \sqrt{3895129}/400, & y_5 &= 1,\\
x_6 &= 1677/400 - \sqrt{3895129}/400, & y_6 &= 1.
\end{aligned}
$$

By plugging these coordinates into the matrix $\mathbf{T}$ given in Section 1.2, one can easily verify that $rk(\mathbf{T}) = 5$ holds. The points $\mathsf{U}_1, \ldots, \mathsf{U}_6$ are displayed in Fig. 1 after applying the translation $(-A_4, -B_4, 0)^T$ to all base points. The points $\mathsf{u}_1, \ldots, \mathsf{u}_6$ are displayed in Fig. 2 after applying the translation $(-a_1, -b_1, 0)^T$ to all platform points. ◇

We finish the preparatory work for the later classification (cf. Section 4) by proving the following lemma:

Lemma 1 *Architecturally singular SG platforms with no four platform or base anchor points collinear, have the property* $rk(\underline{\mathbf{l}}_1, \ldots, \underline{\mathbf{l}}_6) = 5$.

Proof If $rk(\underline{\mathbf{l}}_1, \ldots, \underline{\mathbf{l}}_6) = 5 - n$ with $n > 0$ holds, then the architecturally singular manipulator has to contain a so-called degenerated manipulator [23]. Such degenerated manipulators can only be item 3,4,5,6,7,8,10 of Theorem 3 given by Karger [13], as for the types 1,2,9 and the two degenerated planar cases 11 and 12 (cf. [10]) all six anchor points have to fulfill certain geometric conditions. As every of these seven manipulators, which are listed in the Appendix, has four collinear anchor points, Lemma 1 is proven. □

3 Types of non-architecturally singular self-motions

Given our assumptions, we can add a one-parametric set of legs to the manipulator without changing the direct kinematics and the singularity surface. However, due to Borras et al. [22] the replacement of the original six legs by other six legs of this one-parametric set, is invariant with respect to the singularity surface if and only if the resulting manipulator is not architecturally singular.

Now, we assume that a one-parametric self-motion $\mathcal{M}$ of a non-architecturally singular planar SG platform $\mathsf{m}_1, \ldots, \mathsf{M}_6$ is given. In our assumed general case, we can add a one-parametric set of legs without disturbing the self-motion. Now, we make a leg-replacement in such a way that we get an architecturally singular manipulator $\mathsf{s}_1, \ldots, \mathsf{S}_6$ with $rk(\underline{\mathbf{l}}_1, \ldots, \underline{\mathbf{l}}_6) = 5$. We introduce the following notation:

Definition 1 $\mathcal{M}$ is a one-parametric self-motion of type n with respect to $\mathsf{s}_1, \ldots, \mathsf{S}_6$ (with $rk(\underline{\mathbf{l}}_1, \ldots, \underline{\mathbf{l}}_6) = 5$) if $\mathsf{s}_1, \ldots, \mathsf{S}_6$ has an n-parametric self-motion $\mathcal{S}$.[3]

[3] Note that the numbering of types is done with Roman numerals.

Note that the self-motion $\mathcal{S}$ includes the self-motion $\mathcal{M}$. This can easily be seen as follows: If we attach the legs $\overline{\mathsf{s}_i\mathsf{S}_i}$ for $i = 1, \ldots, 6$ to $\mathsf{m}_1, \ldots, \mathsf{M}_6$, we do not change the direct kinematics and singularity surface. Therefore, also the self-motion $\mathcal{M}$ remains unchanged. By removing the legs $\overline{\mathsf{m}_i\mathsf{M}_i}$ the self-motion $\mathcal{M}$ can only be enlarged.

In the remainder of the article, we focus on self-motions of type II with respect to $\mathsf{s}_1, \ldots, \mathsf{S}_6$, as we can construct more self-motions than only $\mathcal{M}$ by attaching an arbitrary leg to $\mathsf{s}_1, \ldots, \mathsf{S}_6$. If the anchor points of the arbitrary leg are located in the carrier planes of the platform and the base, we can even add a one-parametric set of legs without disturbing the self-motion (see also Section 3.3). But, if the resulting SG platform is non-planar, we cannot attach further legs without disturbing the self-motion in the general case [21].

3.1 Computation of ordinary type II self-motions

The word ordinary in this context means, that all anchor points of the architecturally singular manipulator are in E^3, thus we denote these points by $\mathsf{m}_1, \ldots, \mathsf{M}_6$. Therefore, we are interested in the computation of two-parametric self-motions of $\mathsf{m}_1, \ldots, \mathsf{M}_6$.

It was shown by the author in [25], that the rank condition $rk(\mathbf{T}) < 6$ implies the existence of a linear combination $\sum_{i=1}^{6} \Lambda_i \rho_i = 0$. This already shows that all planar architecturally singular manipulators $\mathsf{m}_1, \ldots, \mathsf{M}_6$ are redundant and have self-motions[4] (over $\mathbb{C}$). Due to this redundancy, the problem reduces to the computation of two-parametric self-motions of 5-legged planar parallel manipulators.

We consider the variety $\mathcal{V}$ spanned by $\Psi, \Lambda_1, \ldots, \Lambda_5$. In general, the solution variety is one-dimensional but for special geometries and leg lengths $\mathcal{V}$ can be two-dimensional. In the following, we describe how the necessary and sufficient conditions can be computed theoretically:

Under consideration of the polynomials Δ_i, the variety $\mathcal{V}$ is also spanned by $\Psi, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Lambda_1$. Then, one can solve the linear system $\Psi, \Delta_2, \Delta_3, \Delta_4$ for the unknowns $f_0, \ldots, f_3$. Now, we plug the obtained expressions in the remaining two equations Λ_1 and Δ_5, which yield in general homogeneous polynomials in the Euler parameters of degree eight and four, respectively. Up to this stage, everything can be computed explicitly; Δ_5 has 3 552 terms and Λ_1 has 284 248 terms for the special coordinate systems $A_1 = B_1 = B_2 = a_1 = b_1 = b_2 = 0$. Finally, one has to compute the resultant of these two polynomials with respect to one of the Euler parameters, but due to the degree and length of Δ_5 and Λ_1 the general computation fails.

This demonstrates, that the determination of type II self-motions is hopeless with this approach.

[4] Trivially, this sentence also holds for the excluded case, where all platform or base anchor points are collinear.

3.2 Computation of special type II self-motions

In this section, we show that the computation of type II self-motions simplifies considerably, if we take the special manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ instead of any ordinary architecturally singular manipulator $\mathsf{m}_1, \ldots, \mathsf{M}_6$. Hence, we are interested in two-parametric DM self-motions.

W.l.o.g., we can assume that the variety of a two-parametric DM self-motion is spanned by $\Psi, \Omega_1, \Omega_2, \Omega_3, \Pi_4, \Pi_5$ (as otherwise we can consider the inverse motion). Now, the crucial point is that all six involved equations are only linear in the unknowns $f_0, \ldots, f_3$. Before starting the elimination procedure, we can make following simplifications:

Lemma 2 *W.l.o.g., we can choose special coordinate systems in Σ_0 and Σ with $X_2(X_2 - X_3)x_5 \neq 0$ and*

$$a_1 = b_1 = y_4 = A_4 = B_4 = Y_1 = h_4 = g_5 = 0, \qquad X_1 = Y_2 = Y_3 = x_4 = y_5 = 1. \tag{3}$$

Proof W.l.o.g., we can assume that U_4 is located in the origin of Σ_0 and that U_1 is the ideal point of the x-axis. For Σ, we choose u_1 as origin and u_4 as the ideal point of the x-axis. This yields: $a_1 = b_1 = y_4 = A_4 = B_4 = Y_1 = 0$. Moreover, we can set $h_4 = 0$ (cf. Remark 2). As all platform and base anchor points have to be distinct (cf. Section 2.3), we can set $g_5 = 0$ w.l.o.g.. Moreover, we can relabel the point pairs $(\mathsf{u}_2, \mathsf{U}_2)$ and $(\mathsf{u}_3, \mathsf{U}_3)$ in such a way that U_2 does not coincide with the ideal point of the y-axis. If u_5 is the ideal point of the y-axis of Σ, we can interchange the indices of the fifth and sixth point pair w.l.o.g.. Therefore, we can assume $X_1X_2Y_2Y_3(X_2Y_3 - X_3Y_2)x_4x_5y_5 \neq 0$. Moreover, as we have homogeneous coordinates we can set $X_1 = Y_2 = Y_3 = x_4 = y_5 = 1$. □

We solve the linear system of equations $\Psi, \Omega_1, \Omega_2, \Pi_4$ for $f_0, \ldots, f_3$ and plug the obtained expressions in the remaining two equations. This yields in general two homogeneous polynomials $\Omega_3^\star[40]$ and $\Pi_5^\star[96]$ in the Euler parameters of degree two and four, respectively. The number in the square brackets gives the number of terms. Finally, we compute the resultant of $\Omega_3^\star$ and $\Pi_5^\star$ with respect to one of the Euler parameters. W.l.o.g., we assume that this is e_0. In this case, the general computation can be done easily and yields $\Gamma[117\,652]$, which is homogeneous in e_1, e_2, e_3 of degree eight. In the following, we list the coefficients of $e_1^i e_2^j e_3^k$ of Γ, which are denoted by Γ_{ijk}:

$$\Gamma_{080} = F_1[8]F_2[18]^2, \quad \Gamma_{800} = (b_2 - b_3)^2(L_1 - g_4)^2 F_3[8],$$
$$\Gamma_{170} = F_2[18]F_4[283], \; \Gamma_{710} = (b_2 - b_3)(L_1 - g_4)F_5[170].$$

The remaining 20 coefficients do not factor:

$$\begin{array}{llll}
\Gamma_{620}[2054], & \Gamma_{602}[1646], & \Gamma_{260}[6126], & \Gamma_{062}[4916], \\
\Gamma_{026}[5950], & \Gamma_{116}[3066], & \Gamma_{530}[4538], & \Gamma_{512}[4512], \\
\Gamma_{152}[6514], & \Gamma_{440}[7134], & \Gamma_{422}[6314], & \Gamma_{242}[7622], \\
\Gamma_{044}[6356], & \Gamma_{314}[6934], & \Gamma_{224}[7096], & \Gamma_{134}[6656], \\
\Gamma_{206}[5950], & \Gamma_{350}[7166], & \Gamma_{404}[5766], & \Gamma_{332}[6982].
\end{array}$$

This yields 24 equations $\Gamma_{ijk} = 0$ in the 14 unknowns $a_2, b_2, a_3, b_3, A_5, B_5, X_2, X_3, x_5, L_1, L_2, L_3, g_4, h_5$.

Only those solutions of this set of equations, which do not cause a vanishing of the coefficient of the highest power of e_0 in $\Omega_3^\star$ and $\Pi_5^\star$, respectively, correspond to two-parametric DM self-motions. If one of these coefficients vanish, then this case has to be studied separately by recomputation of the resultant.

Moreover, if the common factor of $\Omega_3^\star$ and $\Pi_5^\star$ determining the two-parametric DM self-motion equals $e_0e_2 - e_1e_3 = 0$, one has to be careful, as this factor appears in the denominator of the f_i's. In this case, we recommend to recompute the f_i's as solution of the linear system $\Psi, \Omega_1, \Pi_4, \Pi_5$. Then, the remaining equations Ω_2 and Ω_3 have to be fulfilled identically for $e_0e_2 - e_1e_3 = 0$.

Remark 3 Based on this set of 24 equations, the author [26,27] was already able to prove the necessity of three conditions for obtaining a two-parametric DM self-motion. Then, these three necessary conditions were used by the author [28] to determine all planar SG platforms (fulfilling Assumptions 1 and 2) with a type II DM self-motion (cf. Definition 2). ⋄

Example 2 Continuation of Example 1. We translate the platform along the vector $(-a_1, -b_1, 0)^T$ and the base along the vector $(-A_4, -B_4, 0)^T$, in order to get the special coordinate systems of Lemma 2. Putting the resulting coordinates in the equation $\Gamma_{080} = 0$ yields

$$3895129L_1 - 200\sqrt{3895129}(L_2 - L_3) = 0. \tag{4}$$

By solving this equation for L_1, also the coefficient of the highest power of e_0 in $\Omega_3^\star$ vanishes. Therefore, we recompute the resultant of $\Omega_3^\star$ and $\Pi_5^\star$ with respect to e_0. This yields a homogeneous quadratic expression in e_1, e_2, e_3, which even splits up into two linear terms. Now, the discussion is trivial and we see that a common factor of $\Omega_3^\star$ and $\Pi_5^\star$ exists if and only if $L_2 = -h_5$ and $L_3 = g_4\sqrt{3895129}/200 - h_5$ ($\Rightarrow L_1 = -g_4$) hold. Due to Karger's construction the common factor equals e_0.[5] Moreover, it should be noted, that the design parameters g_4, h_5 can still be chosen freely. We continue this example in the next section. ⋄

3.3 Construction of SG platforms with two-parametric DM self-motions

Assuming we have computed a two-parametric DM self-motion, the question remains open how to construct a SG platform with a one-parametric self-motion from it. Clearly, we can attach an arbitrary sixth leg to the manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_5$. We can choose arbitrarily $\mathsf{M}_6 \in E^3$ in the planar base, $\mathsf{m}_6 \in E^3$ in the planar platform and the leg length R_6. The corresponding algebraic constraint is given by Λ_6. The resulting manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_5, \mathsf{m}_6, \mathsf{M}_6$ is not architecturally singular as $(\mathsf{m}_6, \mathsf{M}_6) \neq (\mathsf{u}_6, \mathsf{U}_6)$ holds.

In the following, we want to determine the one-parametric set of legs, which can be attached to this manipulator without changing the direct kinematics. Analogous considerations as in [20,21] yield that the constraint Λ_7 of the redundant leg has to be a linear combination of the following type:

$$\Upsilon: \; \mu_1\Omega_1 + \mu_2\Omega_2 + \mu_3\Omega_3 + \mu_4\Pi_4 + \mu_5\Pi_5 + \mu_6\Lambda_6 - \Lambda_7 = 0.$$

Now, the homogeneous quadratic equation Υ has to vanish independently of the Study parameters, where Υ has the following coefficients:

$$\begin{array}{llllll}
\Upsilon_{e_0e_3}, & \Upsilon_{e_0f_1}, & \Upsilon_{e_0f_2}, & \Upsilon_{e_3f_1}, & \Upsilon_{e_3f_2}, & \Upsilon_{e_i^2}, \\
\Upsilon_{e_1e_2}, & \Upsilon_{e_1f_0}, & \Upsilon_{e_1f_3}, & \Upsilon_{e_2f_0}, & \Upsilon_{e_2f_3}, & \Upsilon_{f_i^2},
\end{array}$$

with $i = 0, \ldots, 3$. Note that e.g. $\Upsilon_{e_0e_3}$ denotes the coefficient of e_0e_3 of Υ. As the following relations hold:

$$\begin{aligned}
&\Upsilon_{f_0^2} = \Upsilon_{f_1^2} = \Upsilon_{f_2^2} = \Upsilon_{f_3^2}, \\
&\Upsilon_{e_0^2} - \Upsilon_{e_1^2} - \Upsilon_{e_2^2} + \Upsilon_{e_3^2} = 0, \\
&\Upsilon_{e_0f_2} + \Upsilon_{e_3f_1} + \Upsilon_{e_1f_3} + \Upsilon_{e_2f_0} = 0, \\
&\Upsilon_{e_0f_1} + \Upsilon_{e_3f_2} + \Upsilon_{e_2f_3} + \Upsilon_{e_1f_0} = 0, \\
&\Upsilon_{e_0f_2} - \Upsilon_{e_3f_1} - \Upsilon_{e_1f_3} + \Upsilon_{e_2f_0} = 0, \\
&\Upsilon_{e_0f_1} - \Upsilon_{e_3f_2} - \Upsilon_{e_2f_3} + \Upsilon_{e_1f_0} = 0,
\end{aligned}$$

we can restrict to the following 10 coefficients:

$$\begin{array}{lllll}
\Upsilon_{e_0e_3}, & \Upsilon_{e_1e_2}, & \Upsilon_{e_0f_1}, & \Upsilon_{f_0^2}, & \Upsilon_{e_0^2}, \\
\Upsilon_{e_0f_2}, & \Upsilon_{e_1f_3}, & \Upsilon_{e_2f_3}, & \Upsilon_{e_1^2}, & \Upsilon_{e_2^2},
\end{array} \tag{5}$$

in eleven unknowns $(a_7, b_7, A_7, B_7, R_7, \mu_1, \ldots, \mu_6)$.

[5] Note that $e_0 = 0$ is preserved by translations of the reference frames.

From now on, everything can be done analogously to the method described in [20,21] (see also Example 3). Finally, we end up with the corresponding cubics c and C in the platform and the base, respectively.

Example 3 Continuation of Example 2. We choose the sixth pair of platform and base anchor points as follows: $\mathbf{m}_6 = (-a_1, -b_1, 0)^T$ and $\mathbf{M}_6 = (-A_4, -B_4, 0)^T$, which equals the translated first pair of anchor points in Karger's example (cf. footnote 2). Then, we compute the ten equations implied by the coefficients of Eq. (5). After expressing R_7 from $\Upsilon_{e_2^2} = 0$, we solve the linear system

$$\Upsilon_{f_0^2} = \Upsilon_{e_0 f_1} = \Upsilon_{e_0 f_2} = \Upsilon_{e_0 e_3} = \Upsilon_{e_1 e_2} = \Upsilon_{e_2 f_3} = 0, \quad (6)$$

for $\mu_1, \ldots, \mu_6$. Now we compute a_7 and b_7 from $\Upsilon_{e_0^2} = 0$ and $\Upsilon_{e_1^2} = 0$, and plug the obtained expressions into the remaining equation $\Upsilon_{e_1 f_3} = 0$, which yields:

$$3527550A_7 + 3275637B_7 - 1959750A_7B_7 - 2331696B_7^2 - 502500A_7^2 - 135350B_7^3 - 167700A_7B_7^2 + 20000A_7^2B_7 = 0.$$

If we express A_7 and B_7 from $\Upsilon_{e_0^2} = 0$ and $\Upsilon_{e_1^2} = 0$, then $\Upsilon_{e_2 f_3} = 0$ yields exactly the same cubic in the moving system (see Figs. 1 and 2). Clearly, this cubic is also the same as the one given by Karger, if one takes the different coordinate systems under consideration. We choose the following anchor points:

$$\mathbf{M}_1 = \left(-\frac{1091879}{46800}, \frac{201}{8}, 0\right)^T, \quad \mathbf{m}_1 = \left(\frac{351}{50}, 0, 0\right)^T,$$
$$\mathbf{M}_2 = \left(-\frac{903}{100}, \frac{9}{2}, 0\right)^T, \quad \mathbf{m}_2 = \left(\frac{401}{50}, -3, 0\right)^T,$$
$$\mathbf{M}_3 = \left(-\frac{1103}{100}, \frac{19}{2}, 0\right)^T, \quad \mathbf{m}_3 = \left(\frac{451}{50}, -1, 0\right)^T,$$
$$\mathbf{M}_4 = \left(\frac{1103}{950}, -1, 0\right)^T, \quad \mathbf{m}_4 = \left(-\frac{8569}{100}, \frac{19}{2}, 0\right)^T,$$
$$\mathbf{M}_5 = \left(\frac{301}{50}, -3, 0\right)^T, \quad \mathbf{m}_5 = \left(-\frac{1203}{100}, \frac{9}{2}, 0\right)^T,$$

where the pairs $(\mathsf{M}_i, \mathsf{m}_i)$ for $i = 1, 2, 3$ equal the translated fourth, second and third pair of Karger's example (cf. footnote 2). The manipulator $\mathsf{m}_1, \ldots, \mathsf{M}_6$ is not architecturally singular as $rk(\mathbf{T}) = 6$ holds. If we replace $(\mathsf{M}_1, \mathsf{m}_1)$ by $(\mathsf{M}_7, \mathsf{m}_7)$ with

$$\mathbf{M}_7 = \left(\frac{64565047}{41985100}, \frac{1087419}{839702}, 0\right)^T,$$
$$\mathbf{m}_7 = \left(\frac{50829}{68050}, \frac{640323}{-34025}, 0\right)^T,$$

we get an architecturally singular manipulator. It can easily be checked, that the SG platform $\mathsf{m}_2, \ldots, \mathsf{M}_7$ has only a one-parametric self-motion, where the self-motion $e_0 = 0$ is only one branch. The second branch of the self-motion is characterized by the vanishing of a homogeneous polynomial of degree three in the Euler parameters. Therefore, the self-motion $e_0 = 0$ is a type II self-motion with respect to $\mathsf{u}_1, \ldots, \mathsf{U}_6$ and a type I self-motion with respect to $\mathsf{m}_2, \ldots, \mathsf{M}_7$. ◇

4 Classification of non-architecturally singular self-motions

At the end of Example 3, we have seen, that there exist self-motions which are of different types with respect to different referencing manipulators. As this could yield to confusions, we introduce a clear classification scheme:

As the notation of types (cf. Section 3) depends on the choice of an architecturally singular manipulator with $rk(\underline{\mathbf{l}}_1, \ldots, \underline{\mathbf{l}}_6) = 5$, we suggest to take the uniquely determined manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ as reference manipulator. As this can always be done due to Lemma 1, the following definition is valid:

Definition 2 $\mathcal{M}$ denotes a one-parametric self-motion of a planar SG platform $\mathsf{m}_1, \ldots, \mathsf{M}_6$, which is not architecturally singular. Then, $\mathcal{M}$ is of type n DM if the corresponding manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ has a n-parametric self-motion $\mathcal{U}$, where $\mathcal{U}$ includes $\mathcal{M}$.

Theorem 2 *All one-parametric self-motions of non-architecturally singular planar SG platforms fulfilling Assumptions 1 and 2 are type I or type II DM self-motions.*

Proof Due to Theorem 1, given in Section 2.3, each one-parametric self-motion of a general planar SG platform $\mathsf{m}_1, \ldots, \mathsf{M}_6$ is a type n DM self-motion. Therefore, we only have to show that $n > 2$ yields a contradiction:

W.l.o.g., we can assume that $\mathsf{m}_1, \ldots, \mathsf{M}_6$ is given by the six constraints $\Lambda_1, \ldots, \Lambda_6$. Moreover, the polynomial $\Omega_3^\star$ can also be computed as

$$(\overline{X}_2 - \overline{X}_3)\Omega_1 - \Omega_2 + \Omega_3. \quad (7)$$

As the Darboux constraints Ω_i $(i = 1, 2, 3)$ can be written as linear combination of sphere constraints (cf. Eq. (2)), also the polynomial $\Omega_3^\star$ is a certain linear combination of $\Lambda_1, \ldots, \Lambda_6$.

For a type n DM self-motion with $n > 2$ the polynomial $\Omega_3^\star$ has to be fulfilled identically, but this already yields a contradiction as then $\mathsf{m}_1, \ldots, \mathsf{M}_6$ is architecturally singular. □

We want to close this section by giving some comments:

- Karger [15] showed that for non-architecturally singular planar manipulators, there exists a linear combination Φ of Λ_i for $i = 1, \ldots, 6$, which is only a quadratic homogeneous polynomial in the Euler parameters. Due to our considerations (cf. Eq. (7)), Φ has to correspond to $\Omega_3^\star$ and therefore, we also found the missing kinematic interpretation of Φ.

- According to the elimination procedure proposed by Husty [24], the ideal spanned by $\Psi, \Delta_2, \ldots, \Delta_6$ leads to a polynomial Θ of degree ten for non-planar SG platforms and of degree eight in the planar case. Until now, a kinematic interpretation of this polynomial was also missing. For the planar case, we have such a kinematic meaning, as Θ equals Γ. Therefore, Θ corresponds to the one-parametric motion of the associated manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$.
  On the other hand, we cannot compute Θ from the polynomials $\Psi, \Delta_2, \ldots, \Delta_6$ in its general form, for the following reason: It can easily be seen, that Karger's polynomial Φ is also within the ideal spanned by $\Psi, \Delta_2, \ldots, \Delta_6$ and that Φ has 2208 terms with respect to the special coordinate systems $A_1 = B_1 = B_2 = a_1 = b_1 = b_2 = 0$. Moreover, we can solve the linear system $\Psi, \Delta_2, \Delta_3, \Delta_4$ for the f_i's and plug the resulting expressions into Δ_j $(j = 5, 6)$, which is of degree four and has 3552 terms. In order to compute Θ, one has to build the resultant of Φ and Δ_j with respect to any Euler parameter, but this fails due to the large number of terms of the involved expressions. This again points out the simplicity of the achieved equation $\Gamma[117\,652]$ in the context of self-motions.

Remark 4 Following interesting question arises: What is the kinematic meaning of Θ in the non-planar case? An answer to this question seems to be important for the determination of the corresponding self-motions of non-planar SG platforms. ⋄

5 Known examples of type II DM self-motions

Due to the last two paragraphs of Section 4 it is clear, that all self-motions $\mathcal{K}$ computed by Karger [15,16] are type II DM self-motions.[6]

In the last remark of [15], Karger wrote that the general condition for the geometry of the manipulator yielding a self-motion characterized by $e_0 = 0$ is a very complicated algebraic condition (approx. 1000 terms). Moreover, he noted that it would be interesting to find further special cases beside the original SG platform [17] and the homological configuration [2,3], for which the condition has a geometric interpretation.

Based on our approach, we can give easily a geometric interpretation for a subset of $\mathcal{K}$ as follows:

If we set $e_0 = 0$ the equations $\Omega_3^\star$ and $\Pi_5^\star$ have to vanish identically. By doing this, we only cover a subset $\mathcal{L}$ of $\mathcal{K}$ as for the general case U_1 must not be located on the x-axis of the fixed frame. It should be noted, that the whole set $\mathcal{K}$ is discussed within the general approach (see Remark 3 and [26–28]).

In the following, we denote the coefficient of $e_1^i e_2^j e_3^k$ of $\Omega_3^\star$ and $\Pi_5^\star$ by $\Omega_3^\star(ijk)$ and $\Pi_5^\star(ijk)$, respectively. Moreover, note that due to Assumption 2 no four anchor points are allowed to be collinear.

$\Omega_3^\star(020) - \Omega_3^\star(002) = 0$ can only vanish without contradiction for $b_2 = b_3$. Due to Lemma 2, we can compute a_2 from $\Omega_3^\star(200) - \Omega_3^\star(002) = 0$ w.l.o.g.. Then, $\Omega_3^\star(110) = 0$ implies an expression for a_3. Moreover, we can express L_3 from $\Omega_3^\star(200) = 0$ w.l.o.g.. Then, $\Omega_3^\star$ is fulfilled identically.

Now, e_3 factors out from $\Pi_5^\star$, which is therefore only of degree three in the Euler parameters e_1, e_2, e_3. As $e_3 = 0$ cannot yield a two-parametric DM self-motion, we proceed as follows: $\Pi_5^\star(012) = 0$ can only vanish without contradiction for $L_1 = -g_4$. Moreover, from $\Pi_5^\star(300) = 0$ we can compute L_2 w.l.o.g.. Then, the condition $\Pi_5^\star(102) = 0$ implies an expression for A_5. Now $\Pi_5^\star(120) = 0$ can only vanish without contradiction for $b_2 = B_5$. Then, the condition $\Pi_5^\star(210) = 0$ remains, which can only vanish without contradiction for $X_i - x_5 = 0$ with $i = 2$ or $i = 3$. W.l.o.g., we can set $i = 2$ and compute[7] $x_5 = X_2$. Finally, we get the following result for the not explicitly given expressions of this discussion:

$$A_5 = \overline{X}_3 B_5, \quad a_2 = \overline{X}_3 B_5, \quad a_3 = \overline{X}_2 B_5,$$
$$L_2 = -h_5, \quad L_3 = (\overline{X}_2 - \overline{X}_3) g_4 - h_5.$$

Moreover, f_0 is identically zero.

Theorem 3 *The self-motions $\mathcal{L}$ fulfilling Assumptions 1 and 2 are line-symmetric motions.*

Proof It can easily be seen, that Eq. (7.3) of [19], which gives the parametrization of a line-symmetric motion, equals the given Study parametrization of Euclidean motions (cf. Section 2.2) for $e_0 = f_0 = 0$. This already proves the statement.[8] □

As a consequence of this result, we can apply Theorem 6 of Krames [30]. Therefore, the cubics c and C in the platform and the base have to be congruent (cf. Example 3).

Moreover, we can easily plug the expressions for the f_i's into the equation Λ_6 of a general leg (cf. Section 3.3). This yields a homogeneous polynomial of degree six in e_1, e_2, e_3 with only 260 terms in its general form. The simplicity of the obtained expression again shows the advantage of our approach compared with the one

[6] After plugging the solutions for the f_i's of the linear system $\Psi, \Delta_2, \Delta_3, \Delta_4$ into Δ_5 and Δ_6, these two equations are fulfilled identically under consideration of $e_0 = 0$.

[7] The solution $i = 3$ yields the same result, just for another indexing.

[8] A recent publication of Selig and Husty [29] can also be used for reasoning the correctness of Theorem 3.

used in [15,16]. Moreover, the unknown e_3 only appears with even powers and therefore, this equation can always be solved explicitly for e_3. This already implies the next theorem:

Theorem 4 *The self-motions $\mathcal{L}$ fulfilling Assumptions 1 and 2 can be parametrized with respect to the homogeneous parameter $e_1 : e_2$.*

In the next step, we calculate the uniquely determined sixth point pair $(\mathsf{u}_6, \mathsf{U}_6)$ such that $\mathsf{u}_1, \ldots, \mathsf{U}_6$ is architecturally singular. A straight forward computation (using the conditions implied by $rk(\mathbf{T}) < 6$) yields the solution: $B_6 = B_5$ and

$$x_6 = y_6 \left[\overline{X}_2(X_3 - \overline{X}_3) + \overline{X}_3(\overline{X}_3 - X_2)\right] / (X_3 - X_2),$$
$$A_6 = B_6 \left[X_3(X_2 - \overline{X}_2) + X_2(\overline{X}_3 - X_2)\right] / (\overline{X}_3 - \overline{X}_2). \quad (8)$$

Before we distinguish two cases, we want to introduce the following notation:

Definition 3 A DM self-motion is called octahedral if the following triples of points are collinear:

$$(\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_6), \quad (\mathsf{u}_1, \mathsf{u}_3, \mathsf{u}_5), \quad (\mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_4), \quad (9)$$
$$(\mathsf{U}_4, \mathsf{U}_5, \mathsf{U}_3), \quad (\mathsf{U}_5, \mathsf{U}_6, \mathsf{U}_1), \quad (\mathsf{U}_4, \mathsf{U}_6, \mathsf{U}_2). \quad (10)$$

The reason for this nomenclature is, that all octahedra (cf. Fig. 3) have such a point-configuration, as the cubics in the platform and the plane split up into three lines intersecting each other in the platform and base anchor points.

5.1 U_2 and U_3 are real points

If the points U_2 and U_3 are real ($\Rightarrow \overline{X}_2 = X_2$ and $\overline{X}_3 = X_3$), then the expressions of Eq. (8) simplify to $x_6 = y_6 X_3$ and $A_6 = B_6 X_2$. It can easily be seen, that the type II DM self-motion is octahedral and that

$$\mathsf{u}_1 = \mathsf{U}_4, \quad \mathsf{u}_2 = \mathsf{U}_5, \quad \mathsf{u}_3 = \mathsf{U}_6, \quad (11)$$
$$\mathsf{u}_4 = \mathsf{U}_1, \quad \mathsf{u}_5 = \mathsf{U}_2, \quad \mathsf{u}_6 = \mathsf{U}_3, \quad (12)$$

hold if the moving frame and the fixed frame coincide.[9]

Let us assume, that we attach the leg $\overline{\mathsf{U}_4\mathsf{u}_2}$ to the manipulator (see Fig. 3). As now u_2 moves on a circle about the line $[\mathsf{U}_4, \mathsf{U}_6]$ we can add a further leg $\overline{\mathsf{U}_6\mathsf{u}_2}$ without disturbing the self-motion. Consideration of the inverse motion yields that U_4 moves on a circle about the line $[\mathsf{u}_2, \mathsf{u}_3]$ and therefore, we can also add the leg $\overline{\mathsf{U}_4\mathsf{u}_3}$ without disturbing the self-motion. Now

[9] This is the reason why the congruent cubics C and c are even given by the same equation.

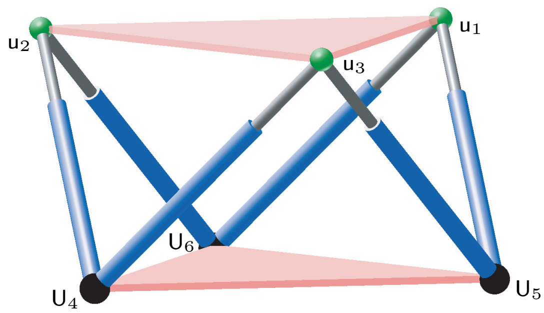

Fig. 3 Sketch of an octahedral manipulator of SG type.

u_3 has to move on a circle about the line $[\mathsf{U}_4, \mathsf{U}_5]$, thus we can attach the leg $\overline{\mathsf{U}_5\mathsf{u}_3}$ without disturbing the self-motion. Finally, we consider again the inverse motion and see that U_6 moves on a circle about $[\mathsf{u}_1, \mathsf{u}_2]$ and that U_5 moves on a circle about $[\mathsf{u}_1, \mathsf{u}_3]$. Therefore, we can attach the legs $\overline{\mathsf{U}_6\mathsf{u}_1}$ and $\overline{\mathsf{U}_5\mathsf{u}_1}$ without disturbing the self-motion.

This already shows that we get a flexible octahedron with a line-symmetric self-motion. Therefore, this can only be a Bricard octahedron [31] of type 1 ($\Leftrightarrow$ all three pairs of opposite vertices are symmetric with respect to a common line). Moreover, this proves the following theorem:

Theorem 5 *Assume that a self-motion of $\mathcal{L}$ is given which fulfills Assumptions 1 and 2. If all points of the corresponding manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ are real, then it is always possible to attach a leg to $\mathsf{u}_1, \ldots, \mathsf{U}_6$, that we get a self-motion of a type 1 Bricard octahedron.*

Example 4 Continuation of Example 3. From the captions of Figs. 1 and 2, we already know that Eqs. (9) and (10) hold. Moreover, it can easily be checked, that also the triangles $\triangle(\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3)$ and $\triangle(\mathsf{U}_4, \mathsf{U}_5, \mathsf{U}_6)$ are congruent. If we choose in Example 3 $(\mathsf{m}_6, \mathsf{M}_6) := (\mathsf{u}_2, \mathsf{U}_4)$ instead of the points $\mathbf{m}_6 = (-a_1, -b_1, 0)^T$ and $\mathbf{M}_6 = (-A_4, -B_4, 0)^T$, then C and c split up into three lines: $\mathsf{C} = [\mathsf{U}_4, \mathsf{U}_5] \cup [\mathsf{U}_4, \mathsf{U}_6] \cup [\mathsf{U}_5, \mathsf{U}_6]$, $\mathsf{c} = [\mathsf{u}_1, \mathsf{u}_2] \cup [\mathsf{u}_1, \mathsf{u}_3] \cup [\mathsf{u}_2, \mathsf{u}_3]$, and we get a self-motion of a type 1 Bricard octahedron. The computation is straight forward. $\diamond$

As a consequence of Theorems 3 and 5, we can immediately formulate the following statement:

Corollary 1 *All flexible octahedra of type 1 have a type II DM self-motion.*

Example 5 We demonstrate Corollary 1 on the basis of an example given by Blaschke [32]:

$$\mathbf{M}_1 = \mathbf{M}_2 = (0, 0, 0)^T, \quad \mathbf{m}_3 = \mathbf{m}_5 = (0, 0, 0)^T, \quad (13)$$
$$\mathbf{M}_3 = \mathbf{M}_4 = (1, 0, 0)^T, \quad \mathbf{m}_1 = \mathbf{m}_4 = (1, 0, 0)^T, \quad (14)$$
$$\mathbf{M}_5 = \mathbf{M}_6 = (0, 1, 0)^T, \quad \mathbf{m}_2 = \mathbf{m}_6 = (0, 1, 0)^T, \quad (15)$$

with leg length $R_1 = R_2 = R_3 = R_5 = \sqrt{2}$, $R_4 = R_6 = 1$. Then, the manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ is given by: $\mathsf{U}_4 = \mathsf{M}_1$, $\mathsf{U}_6 = \mathsf{M}_3$, $\mathsf{U}_5 = \mathsf{M}_5$, $\mathsf{u}_1 = \mathsf{m}_3$, $\mathsf{u}_2 = \mathsf{m}_1$, $\mathsf{u}_3 = \mathsf{m}_2$ and the ideal points $\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3, \mathsf{u}_4, \mathsf{u}_5, \mathsf{u}_6$ are determined by the relations given in Eqs. (9) and (10), respectively. Moreover, we get the unknowns L_i ($i = 1, 2, 3$) by solving the system of equations which arise from the fact that the linear combination

$$\mu_1 \Lambda_1 + \mu_2 \Lambda_2 + \mu_3 \Lambda_3 + \mu_4 \Lambda_4 + \mu_5 \Lambda_5 + \mu_6 \Lambda_6 - \Omega_i = 0$$

has to vanish independently of the Study parameters. This can be done analogously to Section 3.3 and yields $L_1 = 0$ and $L_2 = L_3 = -1$, respectively. Similar considerations for the Mannheim constraints Π_j ($j = 4, 5, 6$) with respect to the linear combination

$$\mu_1 \Lambda_1 + \mu_2 \Lambda_2 + \mu_3 \Lambda_3 + \mu_4 \Lambda_4 + \mu_5 \Lambda_5 + \mu_6 \Lambda_6 - \Pi_j = 0$$

yield under consideration of Remark 2 the following conditions: $g_4 = h_4 = g_5 = h_6 = 0$ and $g_6 = h_5 = 1$. It can easily be checked, that for these values Γ vanishes identically and that the common factor of $\Omega_3^\star$ and $\Pi_5^\star$ is given by $e_1 + e_2 = 0$. Note that we do not get e_0 as common factor due to the chosen coordinate systems in the platform and the base. ◇

5.2 U_2 and U_3 are conjugate complex points

In this case, we have $X_2 = \overline{X}_3$ and the expressions of Eq. (8) simplify to $x_6 = y_6 X_3$ and $A_6 = B_6 X_3$. Now it can easily be checked, that this also yields an octahedral type II DM self-motion, where again the conditions given in Eqs. (11) and (12) hold. Therefore, this case is the complex analogue of the one given in Section 5.1.

Example 6 We initiate this example by setting $B_5 = -17$ and $X_2 = 2 + 3i$. Therefore, $\Omega_1, \Omega_2, \Omega_3, \Pi_4, \Pi_5$ are determined up to the parameters g_4 and h_5. Λ_6 is determined by $a_6 = 5$, $b_6 = -6$, $A_6 = 11$, $B_6 = 13$ and R_6 remains also free. Now, everything can be done as described in Section 3.3. Finally, we end up with

$$A_7 = \frac{1069a_7 - 221b_7^2}{a_7^2 - 4a_7b_7 + 13b_7^2}, \quad B_7 = \frac{311a_7 - 1069b_7}{a_7^2 - 4a_7b_7 + 13b_7^2},$$

and the equation of the cubic c of possible platform points:

$$17a_7^2b_7 - 68a_7b_7^2 + 221b_7^3 - 41a_7^2 - 1602b_7^2 + 475a_7b_7 + 5287a_7 - 18173b_7 = 0,$$

which is displayed in Fig. 4. ◇

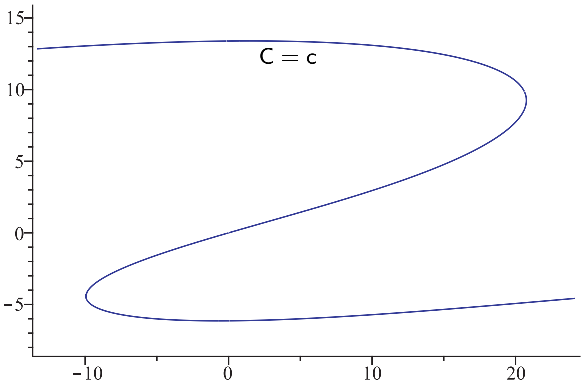

Fig. 4 Both cubics $\mathsf{c} = \mathsf{C}$ of Example 6 are displayed, as they are given by the same equation with respect to Σ and Σ_0.

6 Conclusion and future research

We showed, that one-parametric self-motions of general planar SG platform have to be type I DM or type II DM self-motions (cf. Theorems 1 and 2). We also succeeded in presenting a way on how the set of equations yielding a type II DM self-motion can be computed explicitly (cf. Section 3.2). These 24 conditions are the key for the complete classification of general planar SG platforms with type II DM self-motions (see Remark 3 and [26–28]), which is an important step in solving the famous Borel Bricard problem. In this context, it should also be said, that all planar SG platforms with a type I DM self-motion, which are known to the author, were listed and classified in [7].

In addition, we demonstrated the power of our approach by presenting a geometric interpretation of a large set of known type II DM self-motions (cf. Section 5), which also simplifies their computation considerably.

In our future research, we want to study the problem formulated in Remark 4. Moreover, we want to answer the question if the motions of Bricard octahedra of type 2 and type 3 (cf. [31]) are also type II self-motions with respect to an architecturally singular manipulator $\mathsf{s}_1, \ldots, \mathsf{S}_6$ ($\neq \mathsf{u}_1, \ldots, \mathsf{U}_6$) of $P^3_{\mathbb{C}}$ with $rk(\underline{\mathbf{l}}_1, \ldots, \underline{\mathbf{l}}_6) = 5$? If this is the case, one could immediately compute new classes of non-architecturally singular planar or even non-planar SG platforms with self-motions according to Section 3.

Acknowledgements This research is supported by Grant No. I 408-N13 of the Austrian Science Fund FWF within the project "Flexible polyhedra and frameworks in different spaces", an international cooperation between FWF and RFBR, the Russian Foundation for Basic Research.

Moreover, the author would like to thank the reviewers for their useful comments and suggestions which have helped to improve the quality of the paper.

References

1. Merlet J-P (1992) Singular Configurations of Parallel Manipulators and Grassmann Geometry. Int J Robot Res 8(5):45-56
2. Borel E (1908) Mémoire sur les déplacements à trajectoires sphériques. Mém présenteés par divers savants, Paris(2), 33:1–128
3. Bricard R (1906) Mémoire sur les déplacements à trajectoires sphériques. J École Polyt(2), 11:1–96
4. Husty M (2000) E. Borel's and R. Bricard's Papers on Displacements with Spherical Paths and their Relevance to Self-Motions of Parallel Manipulators. In: Ceccarelli M (ed) International Symposium on History of Machines and Mechanisms, Kluwer, pp 163–172
5. Vogler H (2008) Bemerkungen zu einem Satz von W. Blaschke und zur Methode von Borel-Bricard. Grazer Math Ber 352:1–16
6. Karger A (2010) Parallel manipulators and Borel-Bricard's problem. Comput Aided Geom Des 27(8):669–680
7. Nawratil G (2012) Review and recent results on Stewart Gough platforms with self-motions. Applied Mech Mater 162:151–160
8. Ma O, Angeles J (1992) Architecture Singularities of Parallel Manipulators. Int J Robot Automat 7(1):23-29
9. Karger A (2003) Architecture singular planar parallel manipulators. Mech Mach Theory 38(11):1149–1164
10. Nawratil G (2008) On the degenerated cases of architecturally singular planar parallel manipulators. J Geom Graphics 12(2):141–149
11. Röschel O, Mick S (1998) Characterisation of architecturally shaky platforms. In: Lenarcic J, Husty ML (eds) Advances in Robot Kinematics: Analysis and Control, Kluwer, pp 465–474
12. Wohlhart K (2010) From higher degrees of shakiness to mobility. Mech Mach Theory 45(3):467–476
13. Karger A (2008) Architecturally singular non-planar parallel manipulators. Mech Mach Theory 43(3):335–346
14. Nawratil G (2009) A new approach to the classification of architecturally singular parallel manipulators. In: Kecskemethy A, Müller A (eds) Computational Kinematics, Springer, pp 349–358
15. Karger A (2008) New Self-Motions of Parallel Manipulators. In: Lenarcic J, Wenger P (eds) Advances in Robot Kinematics: Analysis and Design, Springer, pp 275–282
16. Karger A (2008) Self-motions of Stewart-Gough platforms. Comput Aided Geom Des 25(9):775–783
17. Karger A, Husty M (1998) Classification of all self-motions of the original Stewart-Gough platform. Comput Aided Des 30(3):205–215
18. Kœnigs G (1897) Leçons de Cinématique. Librairie Scientifique A Hermann, Paris
19. Bottema O, Roth B (1990) Theoretical kinematics. Dover
20. Husty M, Mielczarek S, Hiller M (2002) A redundant spatial Stewart-Gough platform with a maximal forward kinematics solution set. In: Lenarcic J, Thomas F (eds) Advances in Robot Kinematics: Theory and Applications, Springer, pp 147–154
21. Mielczarek S, Husty ML, Hiller M (2002) Designing a redundant Stewart-Gough platform with a maximal forward kinematics solution set. In: Proceedings of the International Symposium of Multibody Simulation and Mechatronics, Mexico City, Mexico (2002).
22. Borras J, Thomas F, Torras C (2010) Singularity-invariant leg rearrangements in doubly-planar Stewart-Gough platforms. In: Proceedings of Robotics Science and Systems, Zaragoza, Spain (2010).
23. Karger A (1998) Architecture singular parallel manipulators. In: Lenarcic J, Husty ML (eds) Advances in Robot Kinematics: Analysis and Control, Kluwer, pp 445–454
24. Husty ML (1996) An algorithm for solving the direct kinematics of general Stewart-Gough platforms. Mech Mach Theory 31(4):365–380
25. Nawratil G (2004) Zur Geometrie von Stewart Gough Plattformen. Master thesis, TU Vienna
26. Nawratil G (2011) Basic result on type II DM self-motions of planar Stewart Gough platforms. In: Lovasz ECh, Corves B (eds) Mechanisms, Transmissions and Application, Springer, pp 235–244
27. Nawratil G (2011) Necessary conditions for type II DM self-motions of planar Stewart Gough platforms. Technical Report No 219, Geometry Preprint Series, TU Vienna
28. Nawratil G (2011) Planar Stewart Gough platforms with a type II DM self-motion. J Geom 102(1):149–169
29. Selig JM, Husty M (2011) Half-turns and line symmetric motions. Mech Mach Theory 46(2):156–167
30. Krames J (1937) Zur Bricardschen Bewegung, deren sämtliche Bahnkurven auf Kugeln liegen (Über symmetrische Schrotungen II). Mh Math Phys 45:407–417
31. Bricard R (1897) Mémoire sur la théorie de l'octaèdre articulé. J de Mathématiques pures et appliquées, Liouville 3:113–148
32. Blaschke W (1920) Wackelige Achtflache. Math Z 6:85–93

Appendix

In the following, we list the items 3,4,5,6,7,8,10 of Theorem 3 given by Karger [13]:

3. $\mathsf{m}_1,\ldots,\mathsf{m}_5$ and $\mathsf{M}_1,\ldots,\mathsf{M}_5$ are collinear,
4. $\mathsf{m}_1=\mathsf{m}_2=\mathsf{m}_3$, $\mathsf{m}_1,\ldots,\mathsf{m}_5$ are collinear, $\mathsf{M}_4=\mathsf{M}_5$,
5. $\mathsf{m}_1=\mathsf{m}_2=\mathsf{m}_3=\mathsf{m}_4$,
6. $\mathsf{m}_1=\mathsf{m}_2=\mathsf{m}_3$, $\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3$ are collinear,
7. $\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3$ and $\mathsf{M}_3,\mathsf{M}_4,\mathsf{M}_5$ are collinear, $\mathsf{m}_1=\mathsf{m}_2$, $\mathsf{m}_4=\mathsf{m}_5$, $\mathsf{m}_1,\ldots,\mathsf{m}_5$ are collinear,
8. $\mathsf{m}_1,\ldots,\mathsf{m}_4$ and $\mathsf{M}_1,\ldots,\mathsf{M}_4$ are collinear and
$$\begin{aligned} a_4A_2A_3(a_3-a_2)+a_3A_2A_4(a_4-a_2)\\ +a_2A_3A_4(a_4-a_3)=0, \end{aligned} \tag{16}$$
10. Points $\mathsf{m}_1,\ldots,\mathsf{m}_5$ are collinear and pairwise distinct, points $\mathsf{M}_1,\ldots,\mathsf{M}_5$ are coplanar (no three of $\mathsf{M}_1,\ldots,\mathsf{M}_5$ are collinear) and two equations remain,
$$\begin{aligned} &B_3B_4(a_4-a_3)(a_5A_2-A_5a_2)\\ &+B_3B_5(a_3-a_5)(a_4A_2-A_4a_2)\\ &+B_4B_5(a_5-a_4)(a_3A_2-A_3a_2)=0,\\ &B_3B_4A_5(a_4-a_3)(a_5-a_2)\\ &+B_3B_5A_4(a_4-a_2)(a_3-a_5)\\ &+B_4B_5A_3(a_3-a_2)(a_5-a_4)=0. \end{aligned} \tag{17}$$
If $\mathsf{m}_1=\mathsf{m}_2$, points $\mathsf{M}_3,\mathsf{M}_4,\mathsf{M}_5$ must be collinear and (17) yield one equation, a degenerated case.

Note that the formulas (16) and (17) are given with respect to the coordinate systems with $A_1=B_1=B_2=a_1=b_1=b_2=0$.