Stewart Gough platforms with linear singularity surface

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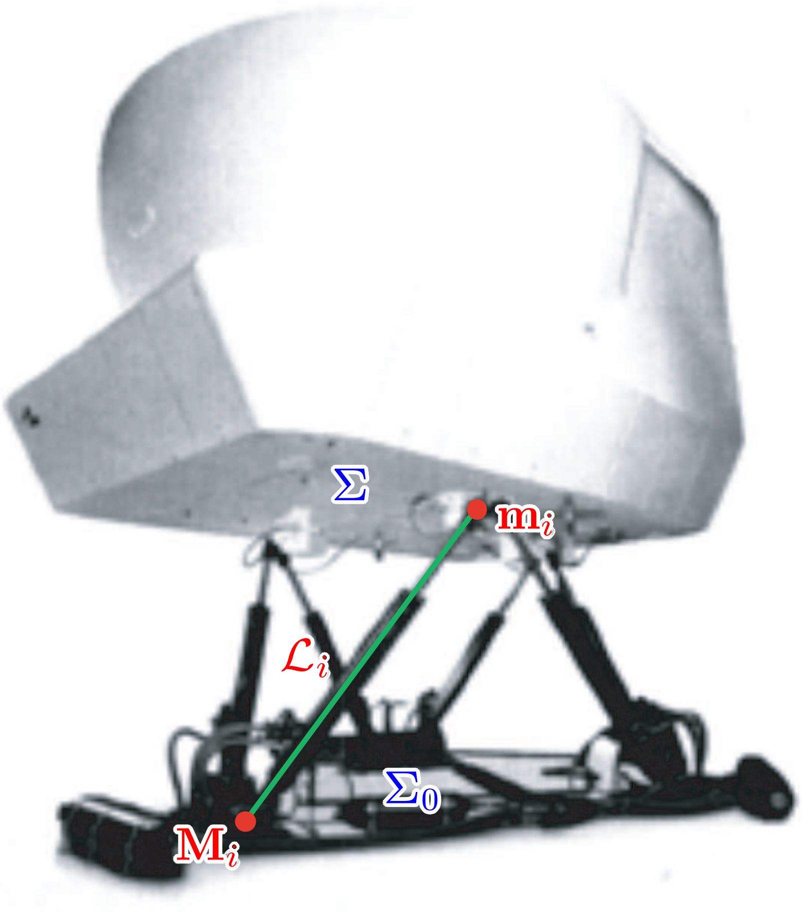
[1] Singular configurations of SGPs

The geometry of a Stewart Gough Platform (SGP) is given by the six base anchor points

$\mathbf{M}_i := (A_i, B_i, C_i)^T$ in the fixed space Σ_0

and by the six platform anchor points

$\mathbf{m}_i := (a_i, b_i, c_i)^T$ in the moving space Σ.



Theorem Merlet [1992]
A SGP is singular iff the carrier lines $\mathcal{L}_i$ of the six legs belong to a linear line complex.

[1] Analytical condition

Plücker coordinates of $\mathcal{L}_i$ can be written as $(\mathbf{l}_i, \widehat{\mathbf{l}}_i) := (\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t} - H\mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$

with $\quad \mathbf{R} := (r_{ij}) = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 + e_0e_3) & 2(e_1e_3 - e_0e_2) \\ 2(e_1e_2 - e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 + e_0e_1) \\ 2(e_1e_3 + e_0e_2) & 2(e_2e_3 - e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix},$

$\mathbf{t} := (t_1, t_2, t_3)^T$ and the homogenizing factor $H := e_0^2 + e_1^2 + e_2^2 + e_3^2$.

$\mathcal{L}_i$ belong to a linear line complex $\iff$ $Q := det(\mathbf{Q}) = 0$ with $\mathbf{Q} := \begin{pmatrix} \mathbf{l}_1 & \widehat{\mathbf{l}}_1 \\ \dots & \dots \\ \mathbf{l}_6 & \widehat{\mathbf{l}}_6 \end{pmatrix}$

Notation: We denote the coefficients of $t_1^i t_2^j t_3^k$ of Q by Q^{ijk}.

[1] Motivation

For designer, it is desirable to have a graphical representation of the singularity set of SGPs, because it simplifies the identification of the singular loci within the given workspace.

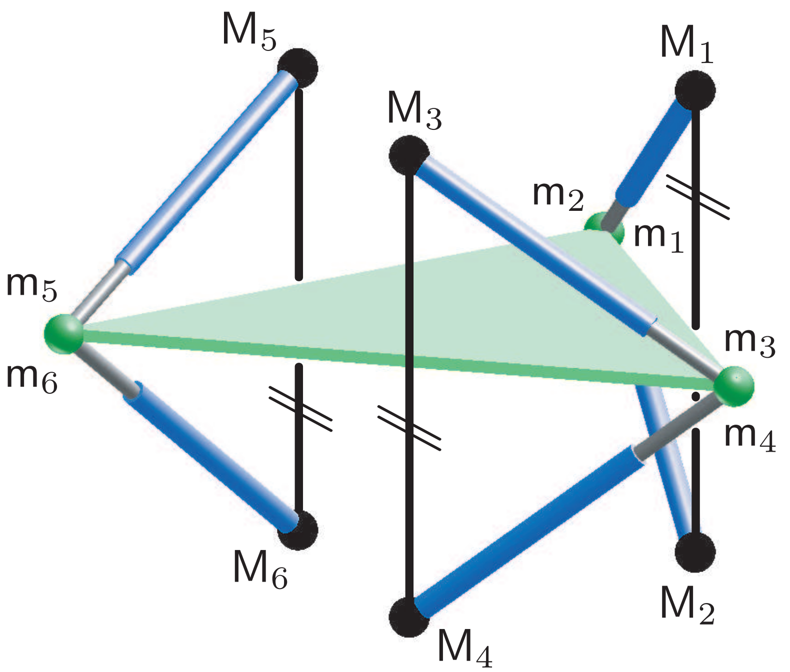
But only the visualization of the singularity surface in 3-dim subspaces make sense.
Usually one fixes the orientational part $\Longrightarrow$ Singularity surface is cubic in t_1, t_2, t_3.

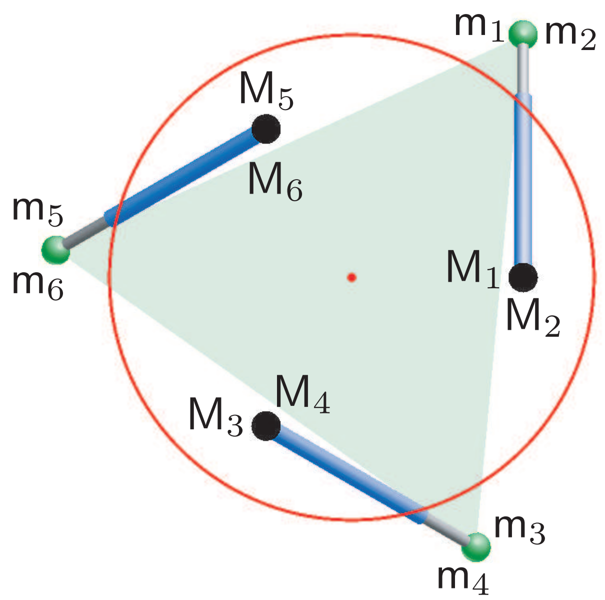
The drawback of cubic surfaces is that they can have very complicated shapes (see figure).

$\Longrightarrow$ We look for SGPs with a simple singularity surface for any orientation of the platform.

[2] Review and related work

The best, one can think of, are **SGPs with a cylindrical singularity surface:**
The manipulators singularity set is for any orientation of the platform a cylindrical surface with rulings parallel to a given fixed direction p in the space of translations.





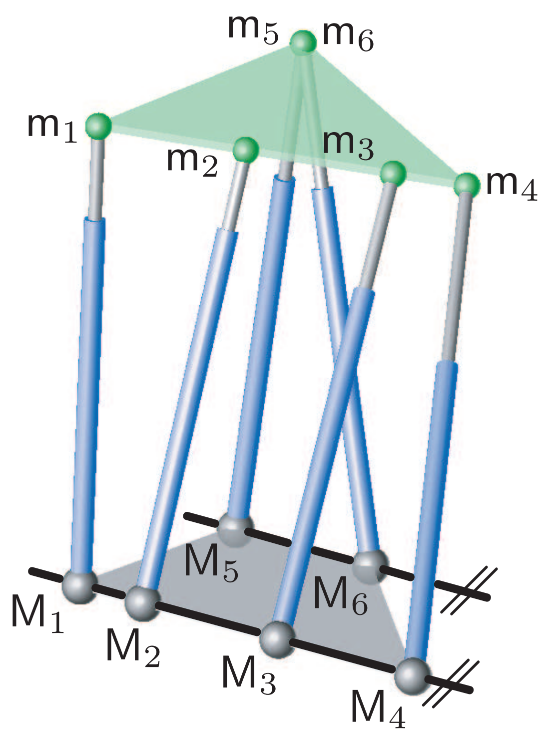
Projection direction is p.

[2] Review and related work

Unfortunately, the set of SGPs with a cylindrical singularity surface has only a very limited variety as there are only two such manipulator designs (cf.**Nawratil [2009]**).

Karger [2006] suggested SGPs with a quadratic singularity surface. They have following advantages:

- all types of quadrics have well known and rather simple shapes,
- it is easier to obtain closed form information about singular positions,
- the computations of singularity free zones for a fixed orientation reduces to the minimization of a quadratic function under a quadratic constraint.



[2] Review and related work

Theorem Nawratil [2010]
A non-architecturally singular SGP possesses a non-cubic (i.e. linear or quadratic) singularity surface if and only if $rk(\mathbf{M}) < 5$ holds with

$$\mathbf{M} = \begin{pmatrix} 1 & a_1 & b_1 & c_1 & A_1 & B_1 & C_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_6 & b_6 & c_6 & A_6 & B_6 & C_6 \end{pmatrix}.$$

This theorem has the following geometric interpretation:

Theorem Nawratil [2010]
A non-architecturally singular SGP possesses a non-cubic singularity surface iff

- there exists a affine correspondence between the platform and the base or
- the SGP has a planar platform and base with $rk(\mathbf{M}) = 4$.

[3] Planar SG platforms

Theorem
A non-architecturally singular SG platform with planar platform and base possesses a linear singularity surface if and only if there exists a affinity between corresponding anchor points.

Karger [2006] demonstrated that SGPs with affine equivalent platform and base possess this property. The proof that no other exists, splits up into two cases:

Proof: No four anchor points are collinear
W.l.o.g. we can choose coordinate systems such that $A_1 = B_1 = B_2 = a_1 = b_1 = b_2 = 0$ and $c_i = C_i = 0$ for $i = 1, \ldots, 6$ hold. Moreover due to **Karger [2003]**

$$a_2 A_2 B_3 B_4 B_5 (a_3 - a_4)(b_3 - b_4) coll(\mathsf{m}_3, \mathsf{m}_4, \mathsf{m}_5) \neq 0$$

holds, where $coll(\mathsf{x}, \mathsf{y}, \mathsf{z}) = 0$ denotes the collinearity condition of the points $\mathsf{x}, \mathsf{y}, \mathsf{z}$.

[3] Planar SG platforms

We can apply the elementary matrix manipulations of **Karger [2003]** to $\mathbf{Q}$ $\Longrightarrow$

$(\mathbf{l}_6, \widehat{\mathbf{l}}_6) = (v_1, v_2, v_3, 0, -w_3, w_2)$ with $v_i = r_{i1}K_1 + r_{i2}K_2, \;\; w_j = r_{j1}K_3 + r_{j2}K_4,$

$$\left.\begin{array}{l} K_1 := |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{a}| \\ K_2 := |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{b}| \\ K_3 := |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Aa}| \\ K_4 := |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Ab}| \end{array}\right\} \text{ and } \quad \mathbf{X} = \begin{bmatrix} X_2 \\ X_3 \\ \vdots \\ X_6 \end{bmatrix}, \; \mathbf{y} = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_6 \end{bmatrix}, \; \mathbf{Xy} = \begin{bmatrix} X_2 y_2 \\ X_3 y_3 \\ \vdots \\ X_6 y_6 \end{bmatrix}.$$

Eqs. (12) and (13) of **Nawratil [2008]** imply $K_1 = K_2 = 0$. Then we consider:

$$Q^{200} = a_2 A_2 r_{13}(r_{31}K_3 + r_{32}K_4)F[24], \quad Q^{020} = A_2 r_{23}(r_{31}K_3 + r_{32}K_4)G[48].$$

$K_3 = K_4 = 0$: architecture singularity, $F = G = 0$: SGP is affinely related. $\square$

[3] Planar SG platforms

Proof: Four anchor points are collinear

W.l.o.g. we can set $A_1 = B_1 = B_2 = B_3 = B_4 = a_1 = b_1 = b_2 = 0$. We compute all non-architecturally singular SGPs with 4 collinear anchor points and $rk(\mathbf{M}) = 4$:

a) $b_3 = b_4 = 0$ and $b_5 = b_6 B_5 / B_6$,

b) $A_4 = \frac{b_4(a_2A_3 - a_3A_2) + a_4 b_3 A_2}{a_2 b_3}$ and

$$A_5 = \frac{a_2b_3A_6B_5 + A_2B_6(a_5b_3 - a_3b_5) - A_2B_5(a_6b_3 - a_3b_6) + a_2A_3(b_5B_6 - b_6B_5)}{a_2b_3B_6}.$$

In both cases the Q^{ijk}'s with $i + j + k = 2$ cannot vanish w.c. (cf. paper). $\square$

Remark 1. In this case we only end up with contradictions as the legs through the 4 collinear anchor points always belong to a regulus if the SGP is affinely related. $\diamond$

[4] Non-planar SG platforms

Lemma 1.
A non-planar SGP with a linear singularity surface has to have 3 collinear points.

Proof: **Karger [2003]** proved in Theorem 1 that a non-planar architecturally singular SGP has to have 3 collinear points. He only used coefficients of Q^{ijk} with $i+j+k>1$ for the proof of this theorem and therefore Lemma 1 is valid. $\square$

Lemma 2.
A non-planar SGP with a linear singularity surface has to have 4 collinear points.

Proof: **Karger [2003]** proved in Theorem 2 that a non-planar architecturally singular SGP has to have 4 collinear points. In his proof he only used in one case the coefficients of $Q^{100}=0$ otherwise coefficients of Q^{ijk} with $i+j+k>1$.
In the presented paper it is shown that also in this exceptional case the contradiction can be concluded from coefficients of Q^{ijk} with $i+j+k>1$. $\square$

[4] Non-planar SG platforms

Theorem
There do not exist non-planar SGPs with a linear singularity surface.

W.l.o.g. can choose coordinate systems in the platform and base such that

$$A_1 = B_1 = B_2 = a_1 = b_1 = b_2 = C_1 = C_2 = C_3 = c_1 = c_2 = c_3 = 0 \quad \text{hold.}$$

W.l.o.g. we can assume that $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3, \mathsf{M}_4$ are not coplanar, i.e. $A_2B_3C_4 \neq 0$. Moreover we know that there exists a affinity κ between corresponding anchor points. κ is given by $\mathsf{M}_i \mapsto \mathsf{m}_i$ for $i = 1, \ldots, 4$.

Due to Lemma 2 there also exist 4 collinear anchor points. If these 4 points are base anchor points then SGP is architecturally singular (cf. remark 1).

Therefore κ has to be a singular affinity with 4 collinear platform anchor points.

[4] Non-planar SG platforms

W.l.o.g. we can assume that the planar platform lies in the xy-plane; i.e. $c_4 = 0$. Moreover we can assume that 3 anchor points from $\mathsf{m}_1, \ldots, \mathsf{m}_4$ are not collinear. W.l.o.g. we can assume that these anchor points are $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_4$, i.e. $a_2 b_4 \neq 0$.

Now we are left with the following 3 possibilities:

$$\left.\begin{array}{ll} \text{a.} & \mathsf{m}_i, \mathsf{m}_j, \mathsf{m}_5, \mathsf{m}_6 \text{ collinear} \\ \text{b.} & \mathsf{m}_i, \mathsf{m}_j, \mathsf{m}_3, \mathsf{m}_k \text{ collinear} \\ \text{c.} & \mathsf{m}_i, \mathsf{m}_3, \mathsf{m}_5, \mathsf{m}_6 \text{ collinear} \end{array}\right\} \quad \text{with } i, j \in \{1, 2, 4\},\ i \neq j \text{ and } k \in \{5, 6\}.$$

A detailed case study (see presented paper) shows that these 3 cases only yield contradictions. Therefore the theorem is proven. □

[5] Main theorem

Main theorem on SGPs with a linear singularity surface
A SGP has a linear singularity surface iff the platform and base are planar and affine equivalent and neither the base nor platform anchor points lie on a conic section.

Proof: There exists a regular affinity κ between corresponding anchor points.

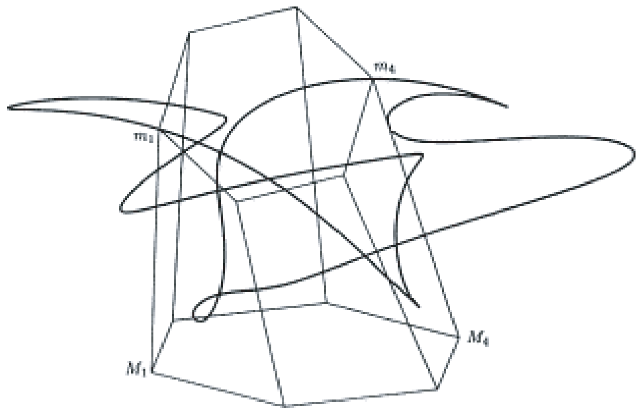
W.l.o.g. we can choose coordinate systems in the platform and the base such that $A_1 = B_1 = B_2 = a_1 = b_1 = b_2 = 0$ and $c_i = C_i = 0$ for $i = 1, \ldots, 6$ hold. Moreover we can assume that the anchor points $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are not collinear.

Therefore κ is determined by $\mathsf{M}_i \mapsto \mathsf{m}_i$ for $i = 1, 2, 3 \Longrightarrow Q = a_2 b_3 H W_1 W_2 W_3$:

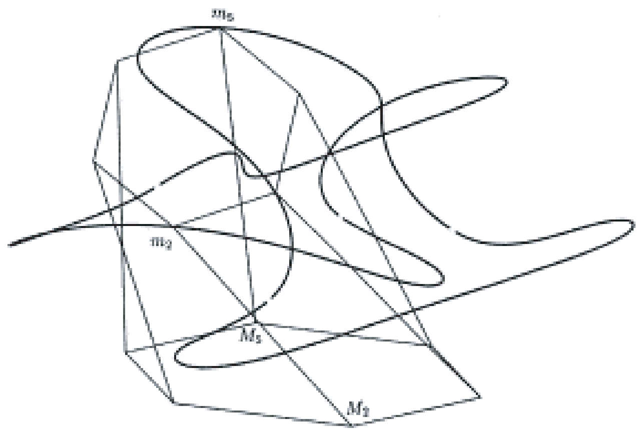
- $W_1[48] = 0 \iff$ Six base points lie on a conic (architecture singularity).
- $W_2[12]$ is independent of t_i and cannot vanish for all poses of the platform.
- $W_3[32]$ is linear in t_i and cannot vanish for all poses of the platform. $\square$

[5] Remarks

Remark 2.
Karger [2002] showed that planar SGPs with affinely equivalent platform and base have self-motions only if they are architecturally singular. Such SGPs only have translatory self-motions (cf. figures from **Karger [2001]**). ◇



Remark 3.
The radius of the largest singularity free sphere in the space of translations (for a fixed orientation) can be given explicitly (via the Hessian normal form of the plane $W_3 = 0$) for all SGPs with a linear singularity surface. ◇



[6] Conclusion

Based on the set of SGPs with a non-cubic singularity surface (cf. **Nawratil [2010]**) we determined all SGPs with a linear singularity surface.

Main theorem on SGPs with a linear singularity surface
A SGP has a linear singularity surface iff the platform and base are planar and affine equivalent and neither the base nor platform anchor points lie on a conic section.

As a side product of this main theorem and the results of **Nawratil [2010]** we can also characterize all SGPs with a quadratic singularity surface:

Corollary
A non-architecturally singular SGP has a quadratic singularity surface if and only if the manipulator is planar and $rk(\mathbf{M}) = 4$ holds or if the manipulator is non-planar and there is a (regular or singular) affinity between corresponding anchor points.

[7] References

- **Karger A. [2001]** Singularities and self-motions of equiform platforms, Mechanism and Machine Theory **36** (7) 801–815.
- **Karger A. [2002]** Singularities and self-motions of a special type of platforms, Advances in Robot Kinematics: Theory and Applications (J. Lenarcic, F. Thomas eds.), 155–164.
- **Karger A. [2003]** Architecture singular planar parallel manipulators, Mechanism and Machine Theory **38** (11) 1149–1164.
- **Karger A. [2006]** Stewart-Gough platforms with simple singularity surface, Advances in Robot Kinematics: Mechanisms and Motion (J. Lenarcic, B. Roth eds.), 247–254.
- **Merlet J.-P. [1992]** Singular Configurations of Parallel Manipulators and Grassmann Geometry, International Journal of Robotics Research **8** (5) 45–56.
- **Nawratil G. [2008]** Results on planar parallel manipulators with cylindrical singularity surface, Advances in Robot Kinematics: Analysis and Design (J. Lenarcic, P. Wenger eds.), 321–330.
- **Nawratil G. [2009]** All planar parallel manipulators with cylindrical singularity surface, Mechanism and Machine Theory **44** (12) 2179–2186.
- **Nawratil G. [2010]** Stewart Gough platforms with non-cubic singularity surface, Technical Report No. 204, Geometry Preprint Series, Vienna University of Technology.