Flexible Oktaeder unter Anbetracht von Fernelementen

Georg Nawratil

Institut für Diskrete Mathematik und Geometrie

Technische Universität Wien, Österreich

Bricard octahedra

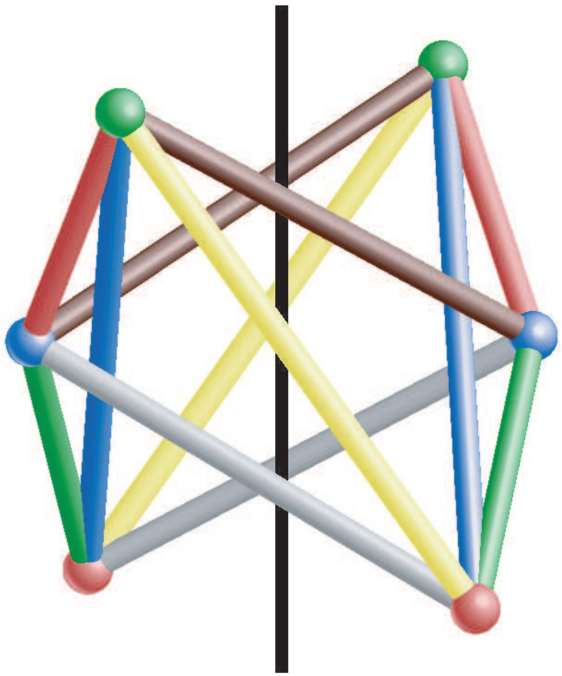
An octahedron is called *flexible*, if its spatial shape can be changed continuously due to changes of its dihedral angles only, i.e. every face remains congruent to itself during the flex.

All flexible octahedra in E^3, where no two faces coincide permanently during the flex, were firstly determined by BRICARD [1].

There are 3 types of these so-called Bricard octahedra:

Bricard octahedra of type I

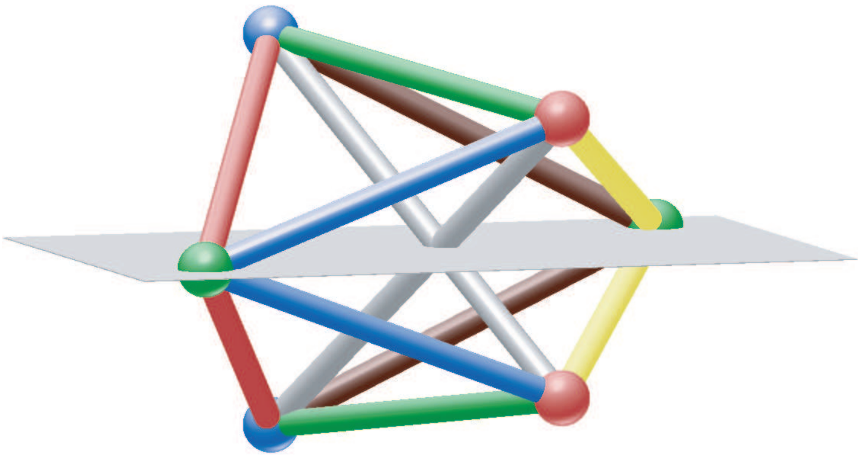
All three pairs of opposite vertices are symmetric with respect to a line.



Bricard octahedra

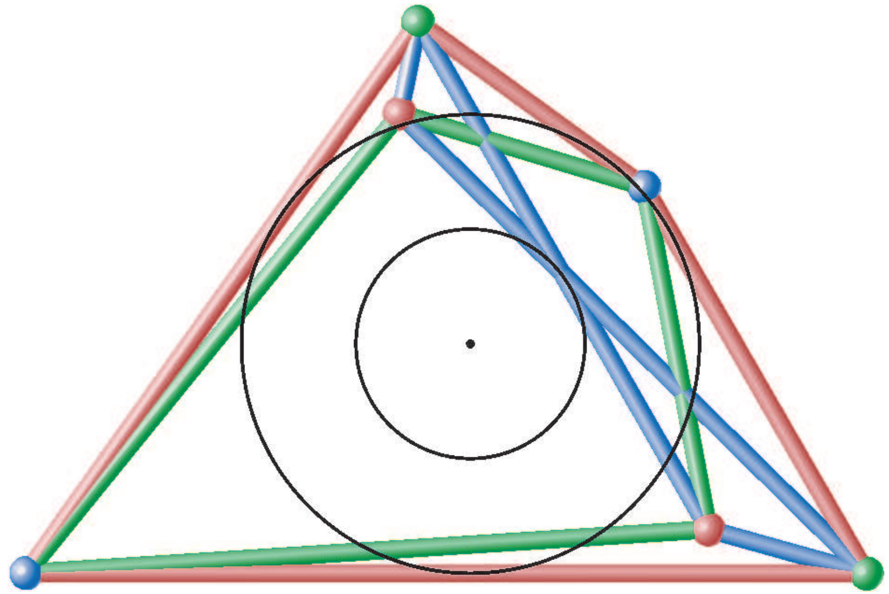
Bricard octahedra of type II

Two pairs of opposite vertices are symmetric with respect to a plane through the remaining two vertices.



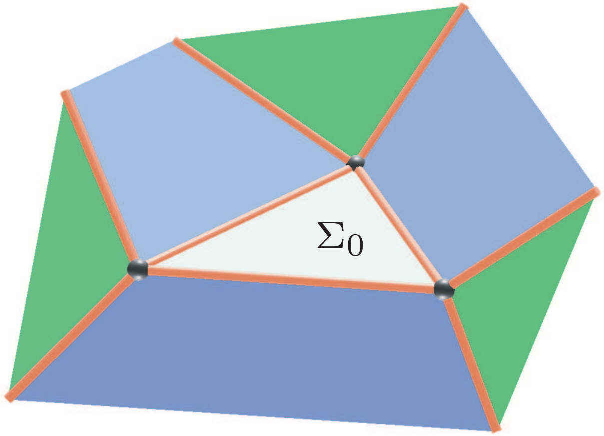
Bricard octahedra of type III

These octahedra possess two flat poses and can be constructed as follows:



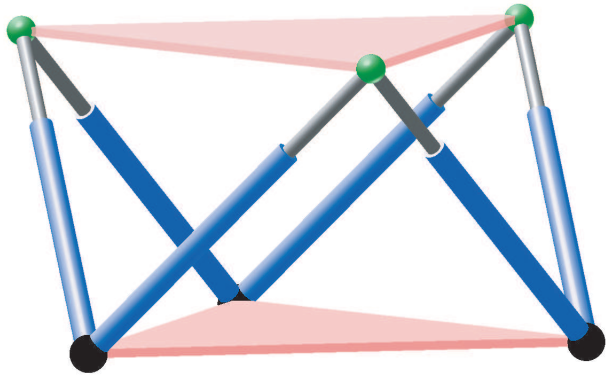
Different points of view

Kokotsakis mesh



A Kokotsakis mesh is a polyhedral structure, consisting of a n-sided central polygon Σ_0, surrounded by a belt of polygons (cf. KOKOTSAKIS [5]).

Stewart Gough platform



A SGP is a parallel manipulator, where the platform is connected via three Spherical-Prismatic-Spherical (S$\underline{\text{P}}$S) legs with the base.

Table of contents

Flexible octahedra in the projective extension of E^3

Presented results are published in:

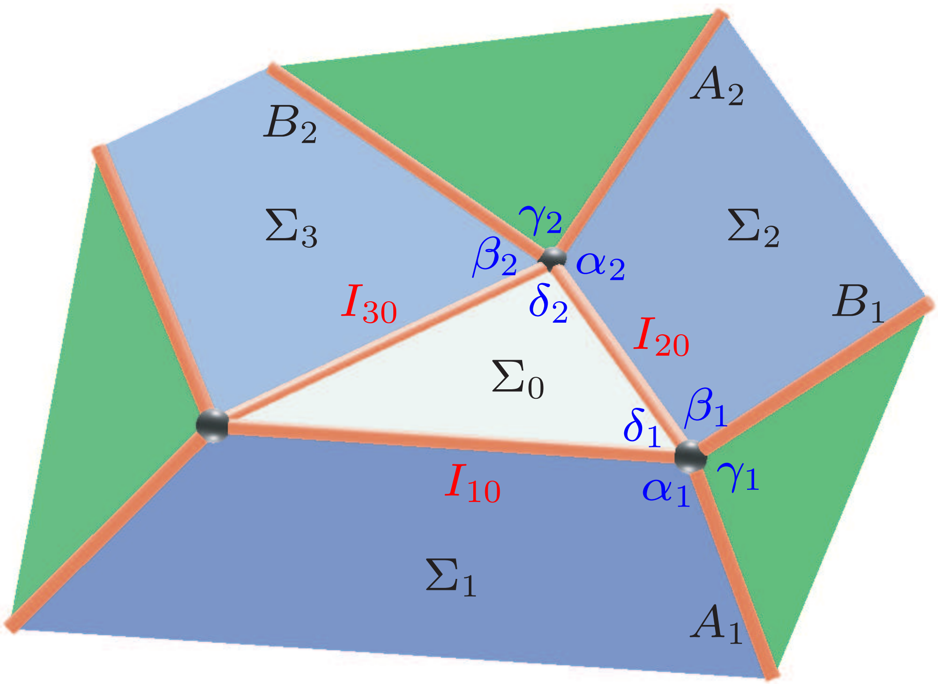
[A] Reducible compositions of spherical four-bar linkages with a spherical coupler component. Mechanism and Machine Theory 46 (5) 725-742 (2011)

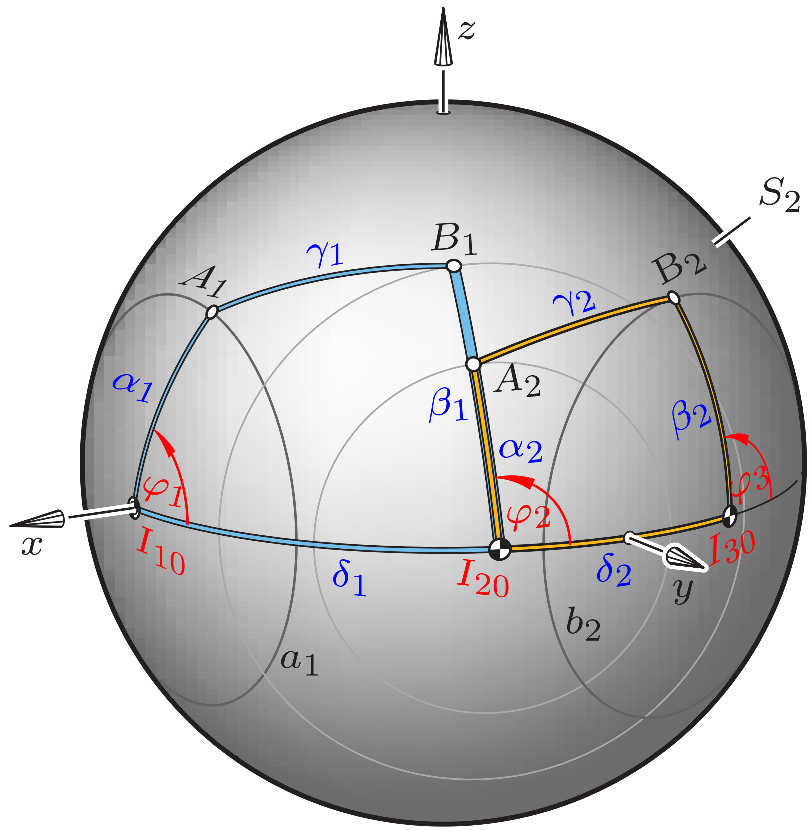
[B] Self-motions of TSSM manipulators with two parallel rotary axes. ASME Journal of Mechanisms and Robotics 3 (3) 031007 (2011)

[C] Flexible octahedra in the projective extension of the Euclidean 3-space. Journal for Geometry and Graphics 14 (2) 147-169 (2010).

Acknowledgements
The research reported in [A,B,C] was supported by Grant No. I 408-N13 of the Austrian Science Fund FWF within the project "Flexible polyhedra and frameworks in different spaces", an international cooperation between FWF and RFBR, the Russian Foundation for Basic Research.

1. Basic idea for the proof of the given results



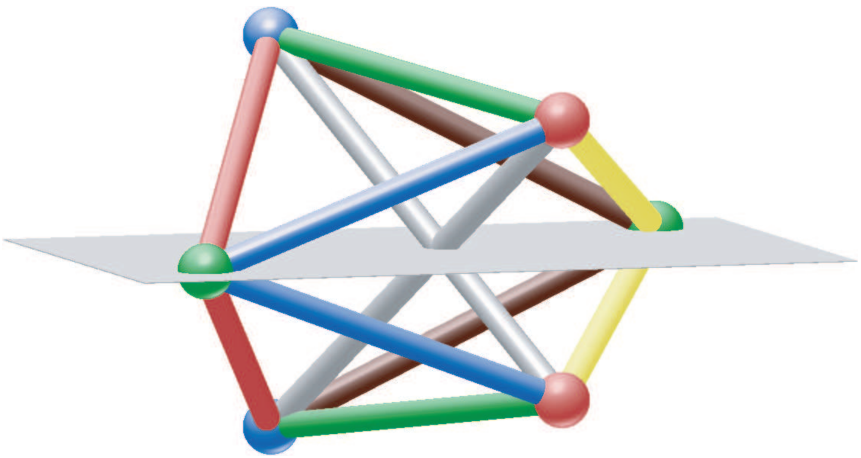


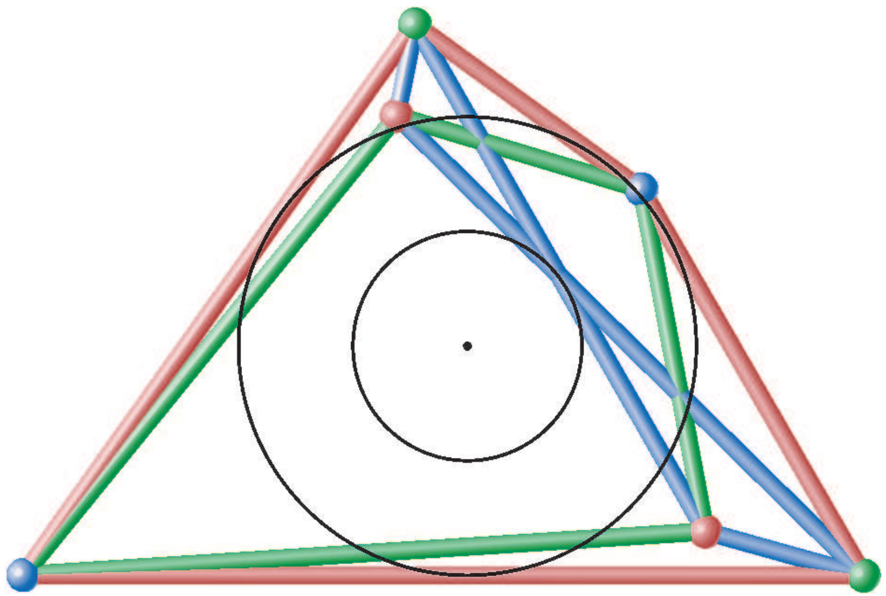
The relative motions Σ_{i+1}/Σ_i is a spherical 4-bar motion (cf. STACHEL [12]).

The Kokotsakis mesh is flexible, if and only if, the transmission of the composition of two spherical 4-bar linkages can be replaced by a single spherical 4-bar linkage.

1a. One vertex is an ideal point

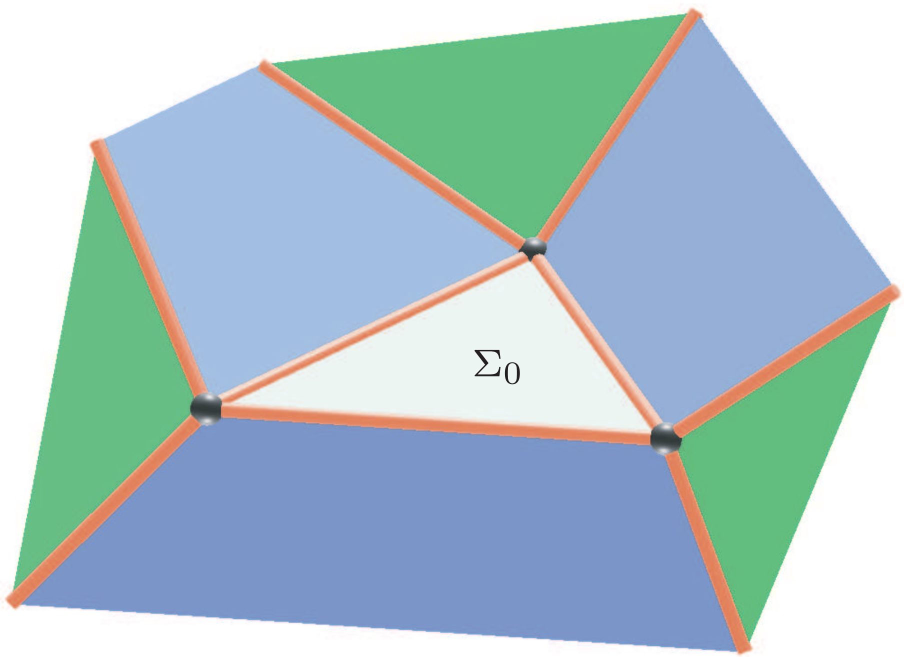
Theorem 1 NAWRATIL [B]
There exist two flexible octahedra, where one vertex is an ideal point: Bricard octahedron II, where one vertex located in the plane of symmetry is an ideal point. Bricard octahedron III, where one vertex is an ideal point (see also STACHEL [11]).





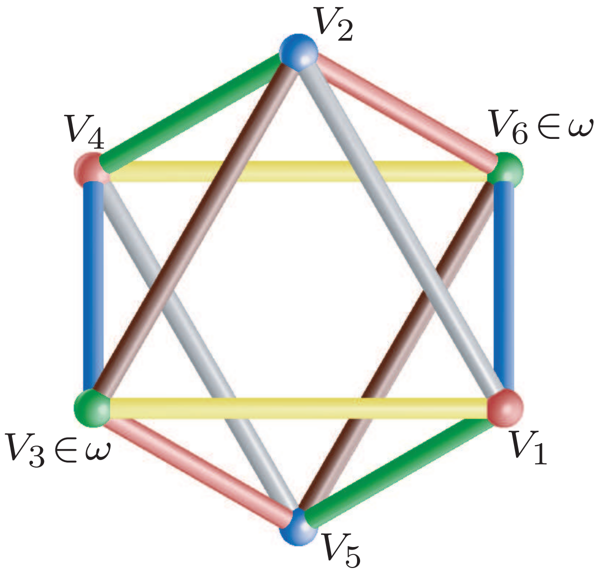
1a. Flexible octahedra with edge or face at infinity

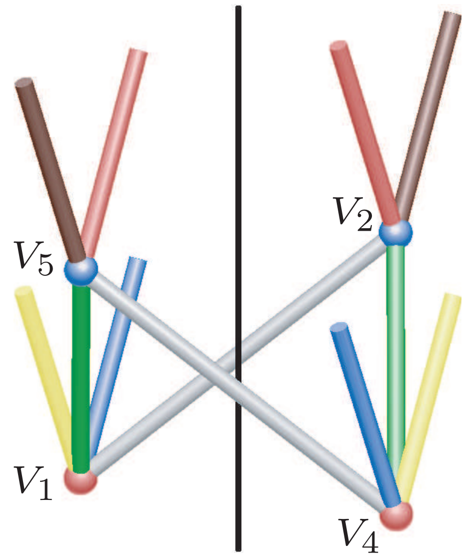
Theorem 2 NAWRATIL [C]
In the projective extension of E^3 there do not exist flexible octahedra with a finite face Σ_0 and one edge or face at infinity.



1b. One pair of opposite vertices are ideal points

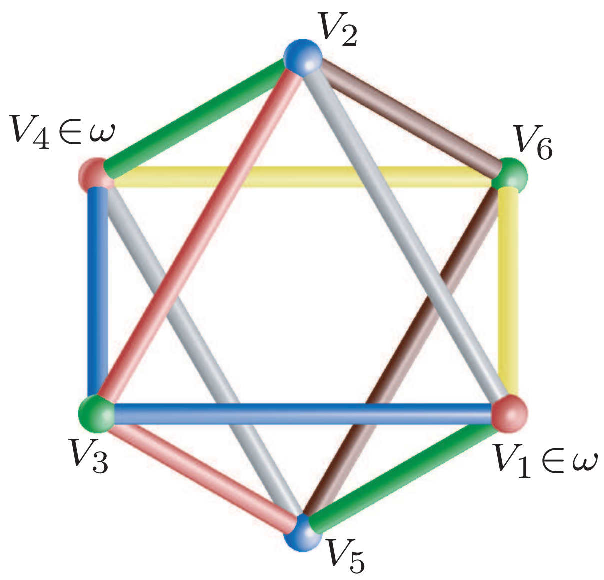
Theorem 3 NAWRATIL [C]
The two pairs of opposite vertices (V_1, V_4) and (V_2, V_5) are symmetric with respect to a common line as well as the edges of the prisms through the ideal points V_3 and V_6, respectively.

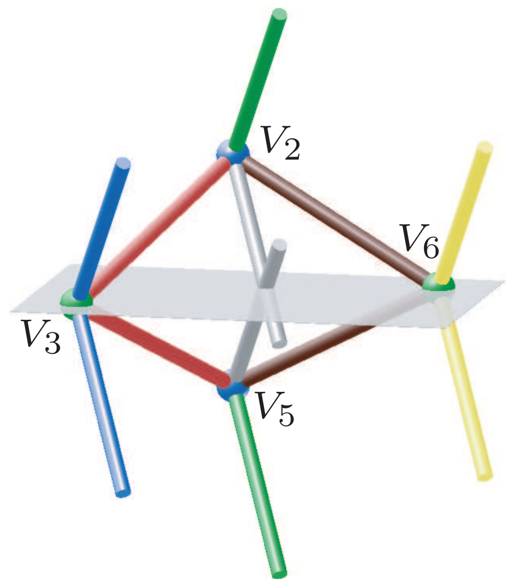




1b. One pair of opposite vertices are ideal points

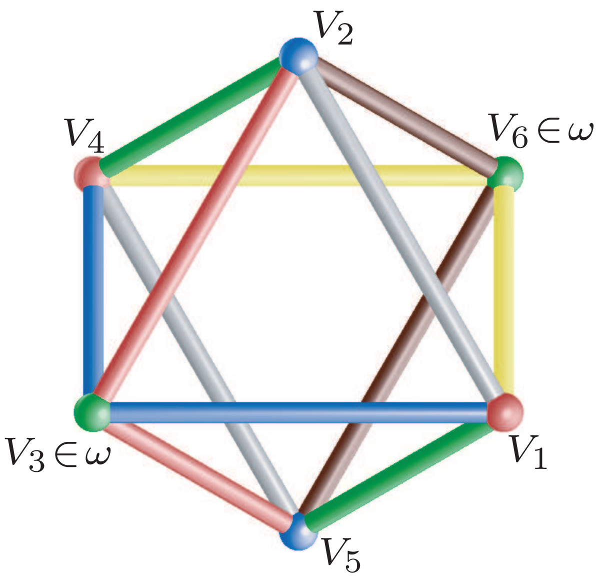
Theorem 4 NAWRATIL [C]
One pair of opposite vertices (V_2, V_5) is symmetric with respect to a plane, which contains the vertices (V_3, V_6). Moreover, also the edges of the prisms through the ideal points V_1 and V_4 are symmetric with respect to this plane.

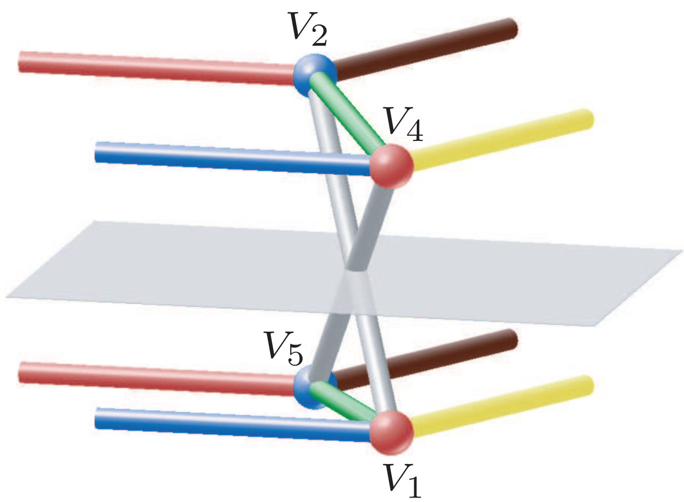




1b. One pair of opposite vertices are ideal points

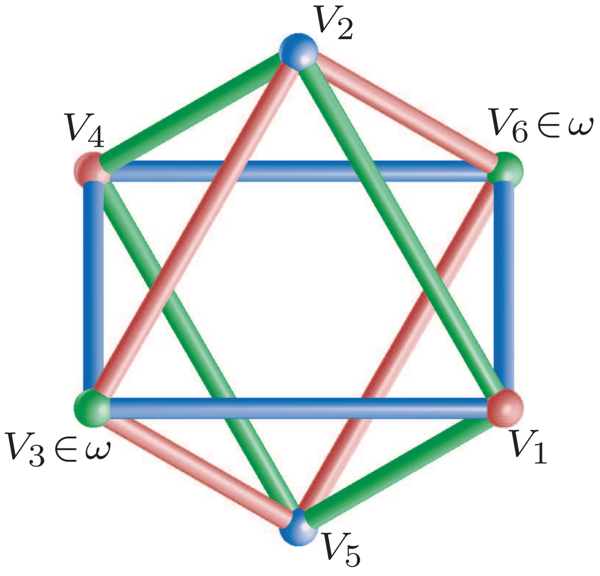
Theorem 5 NAWRATIL [C]
The vertices V_1, V_2, V_4, V_5 are coplanar, form an antiparallelogram and its plane of symmetry is parallel to the edges of the prisms through the ideal points V_3 and V_6, respectively.

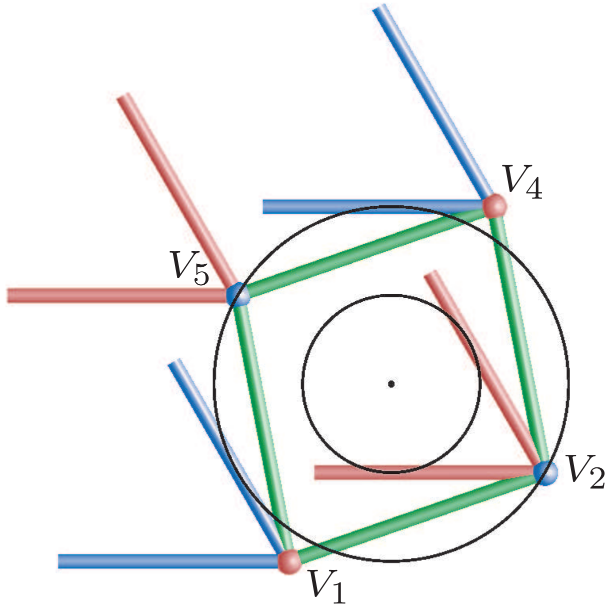




1b. One pair of opposite vertices are ideal points

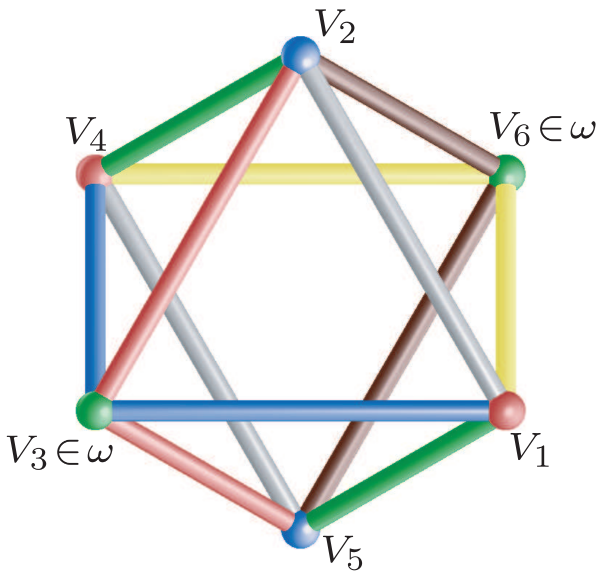
Theorem 6 NAWRATIL [C]
This type is characterized by the existence of two flat poses and consists of two prisms through the ideal points V_3 and V_6, where the orthogonal cross sections are congruent antiparallelograms. The construction can be done as follows:

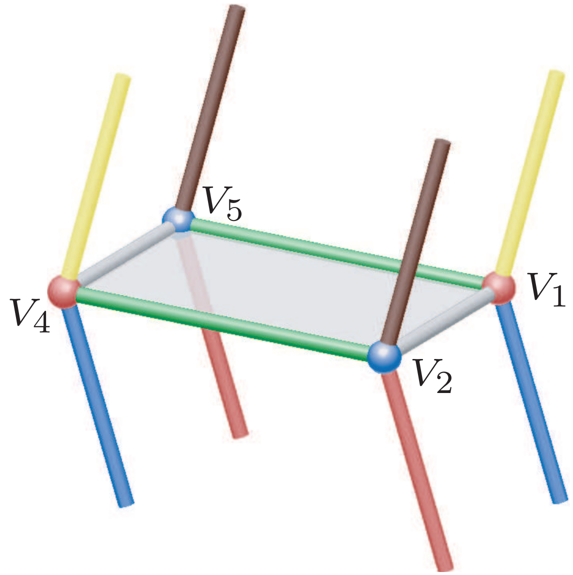




1b. One pair of opposite vertices are ideal points

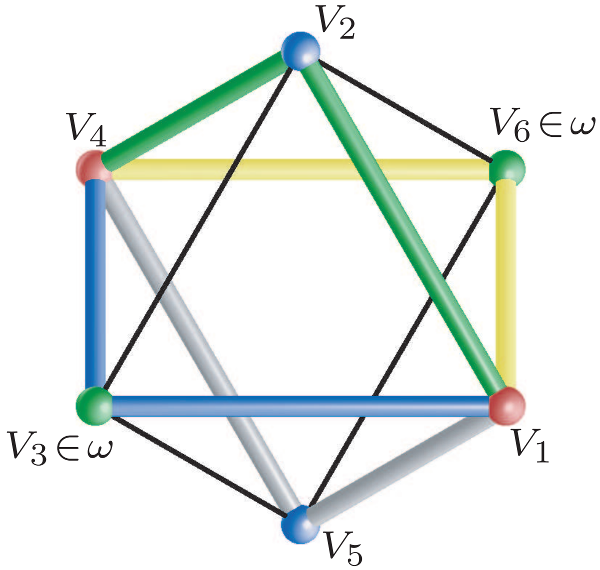
Theorem 7 NAWRATIL [C]
The vertices V_1, V_2, V_4, V_5 are coplanar and form a parallelogram. The ideal points V_3 and V_6 can be chosen arbitrarily.

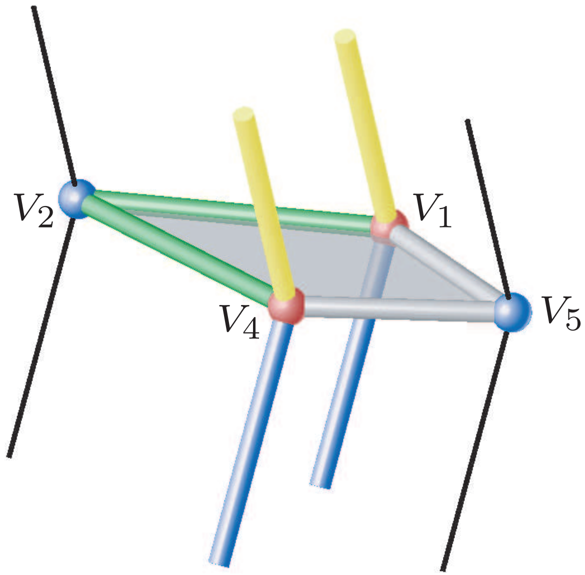




1b. One pair of opposite vertices are ideal points

Theorem 8 NAWRATIL [C]
The vertices V_1, V_2, V_4, V_5 are coplanar, form a deltoid and the edges of the prisms through the ideal points V_3 and V_6 are orthogonal to the deltoid's line of symmetry.

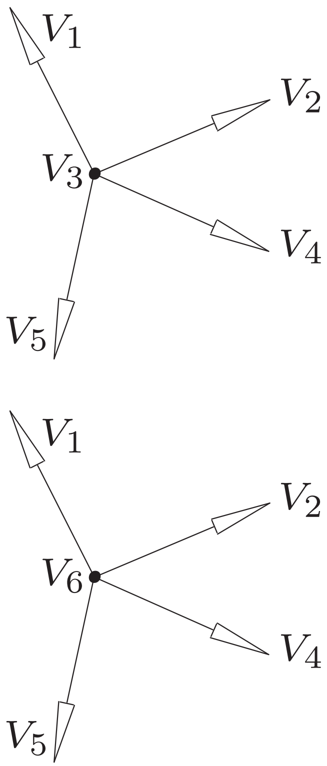


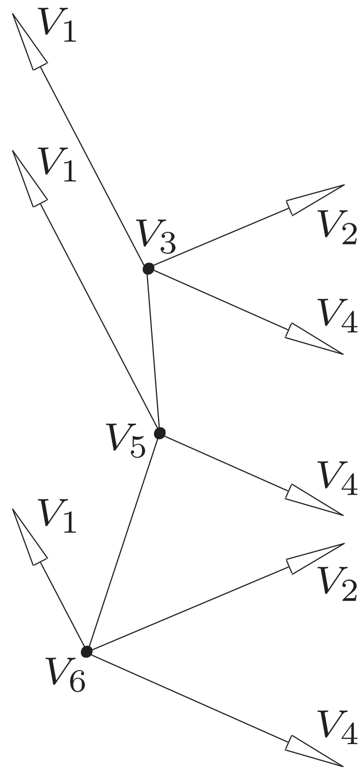


1c. Trivial cases

Theorem 9 NAWRATIL [C]
In the projective extension of E^3, any octahedron is flexible, where at least two edges are ideal lines but no face coincides with the plane at infinity.

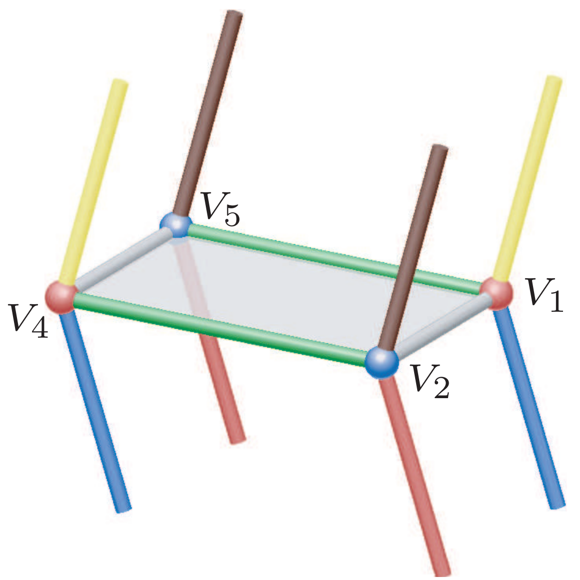
There are only two types of octahedra fulfilling the requirements of theorem 9.

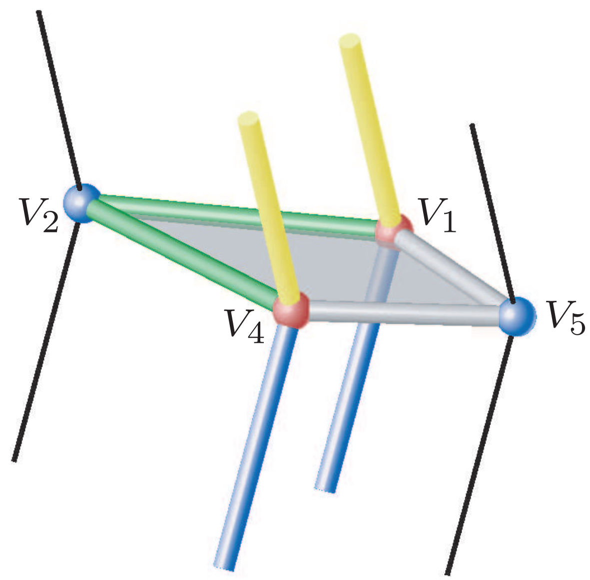




1d. Application in school

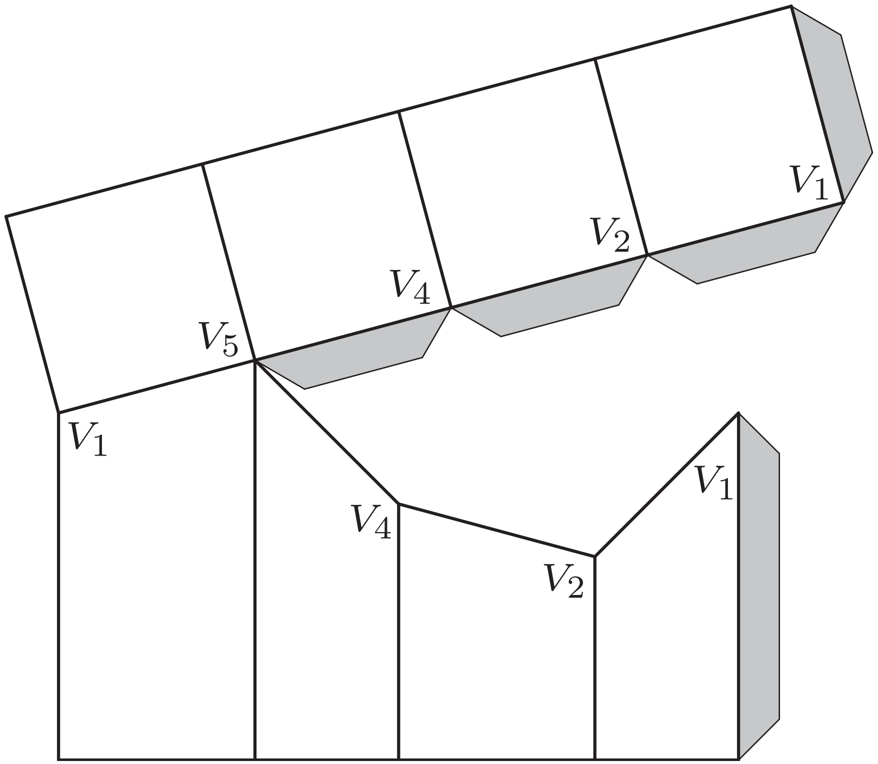
Trivially, the two types of flexible octahedra, given in Theorem 7 and Theorem 8, which do not have flexible analogue Bricard octahedra, can be built as paper models without self-intersections.

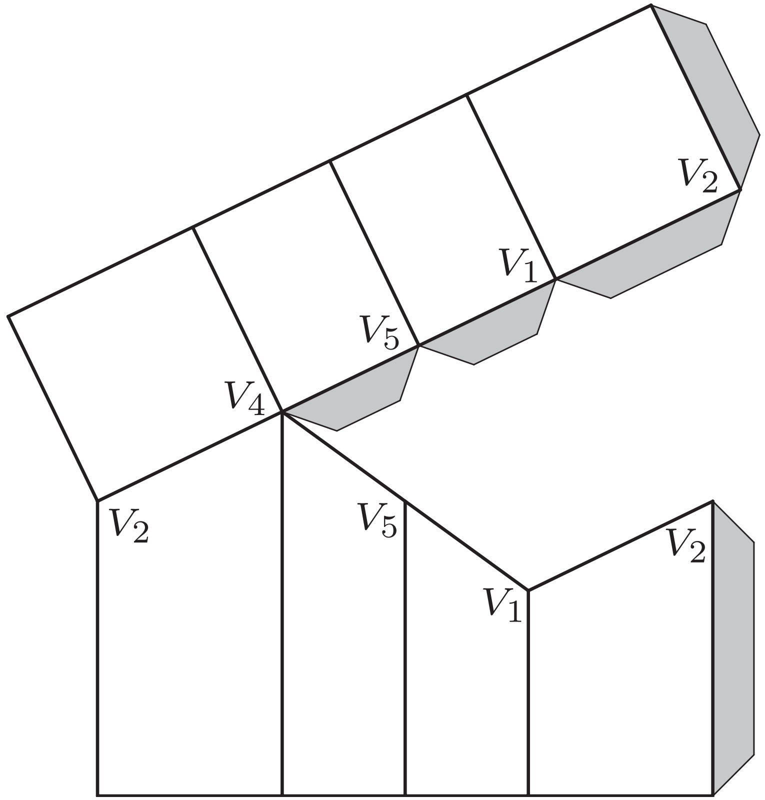




1d. Nets for paper models

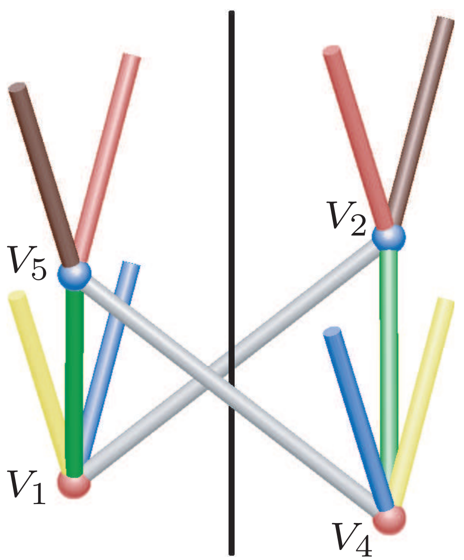
All nets can be downloaded from www.geometrie.tuwien.ac.at/nawratil/talks.html

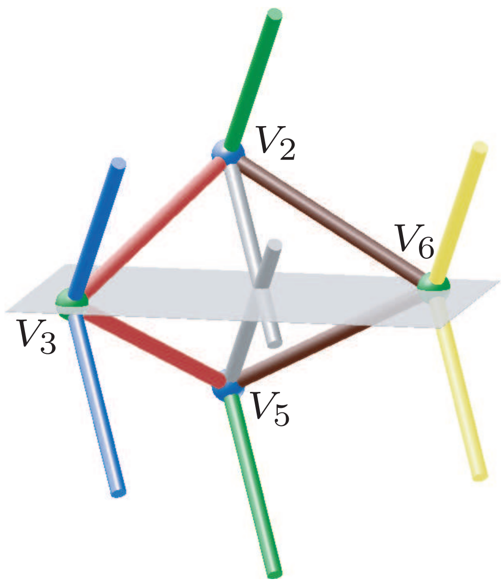




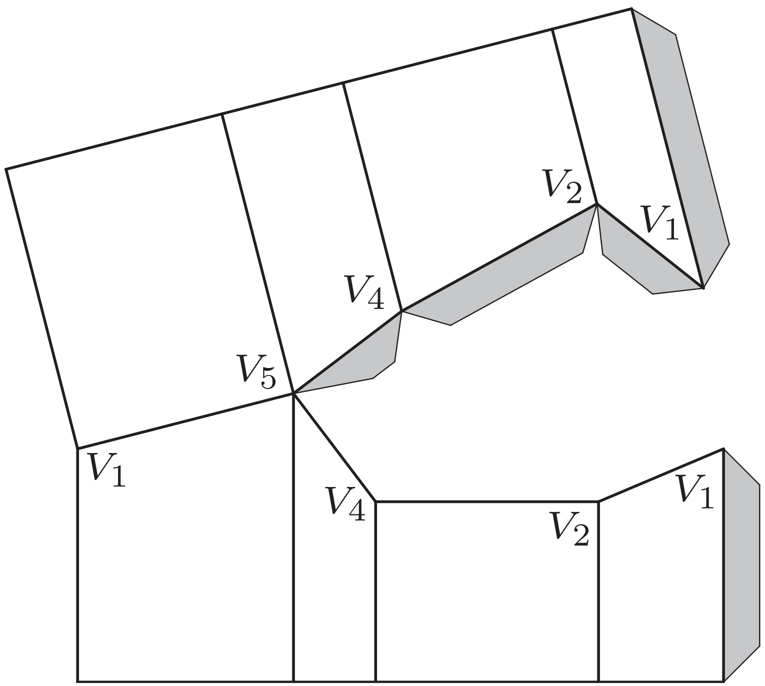
1d. Application in school

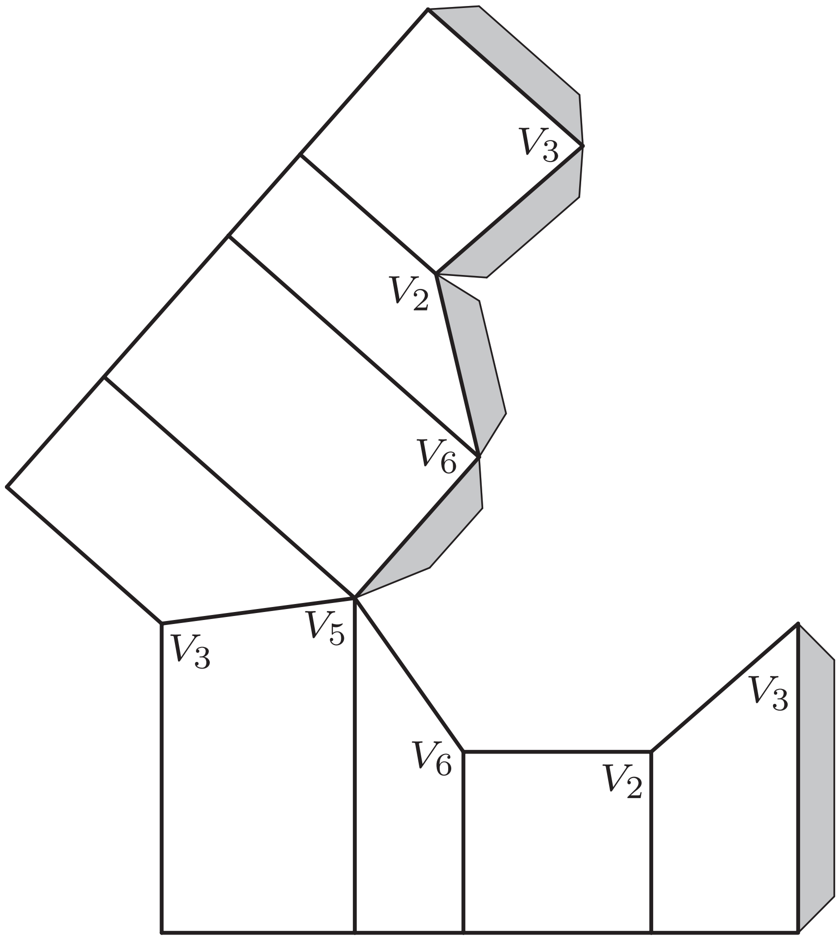
In addition the flexible octahedra of Theorem 3 and Theorem 4 can also be built as paper models without self-intersections, if we "materialize" the complementary part of one of the displayed prisms.





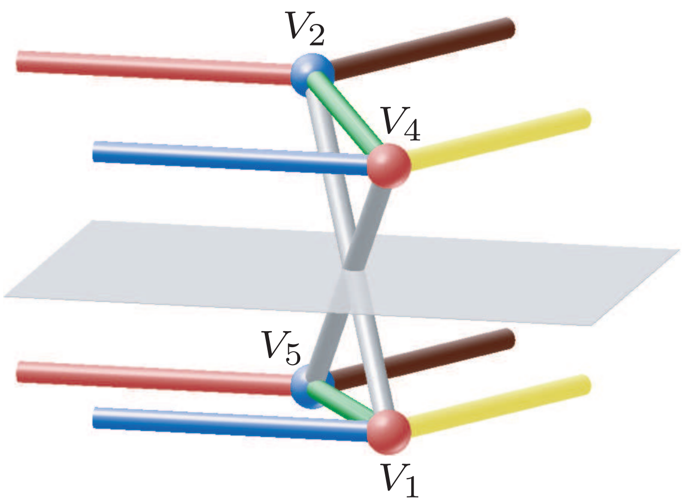
1d. Nets for paper models

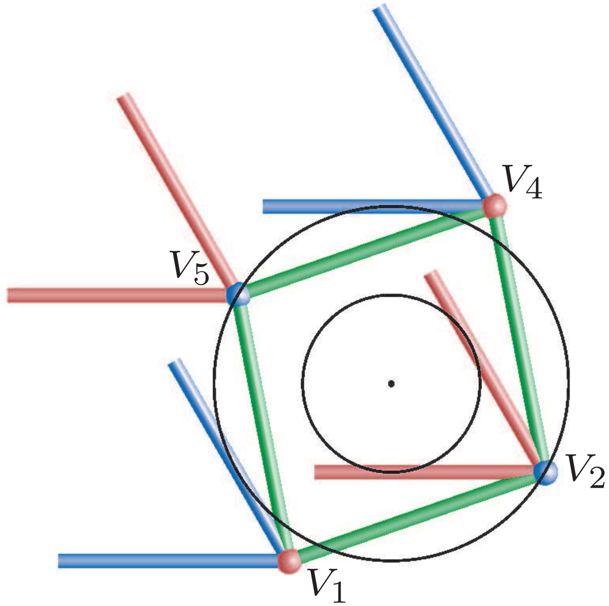




1d. Application in school

The remaining two flexible octahedra of Theorem 5 and Theorem 6, respectively, are not suited to be built as paper models, as the orthogonal section of each involved prism is an antiparallelogram.





Self-movable SGPs implied by Bricard octahedra I

Presented results are published in:

[D] Types of self-motions of planar Stewart Gough platforms. Meccanica, conditionally accepted.

[E] Necessary conditions for type II DM self-motions of planar Stewart Gough platforms. Journal for Geometry and Graphics, conditionally accepted.

[F] Planar Stewart Gough platforms with a type II DM self-motion. Journal of Geometry 102 (1) 149-169 (2011)

Acknowledgements
The research reported in [D,E,F] was supported by Grant No. I 408-N13 of the Austrian Science Fund FWF within the project “Flexible polyhedra and frameworks in different spaces”, an international cooperation between FWF and RFBR, the Russian Foundation for Basic Research.

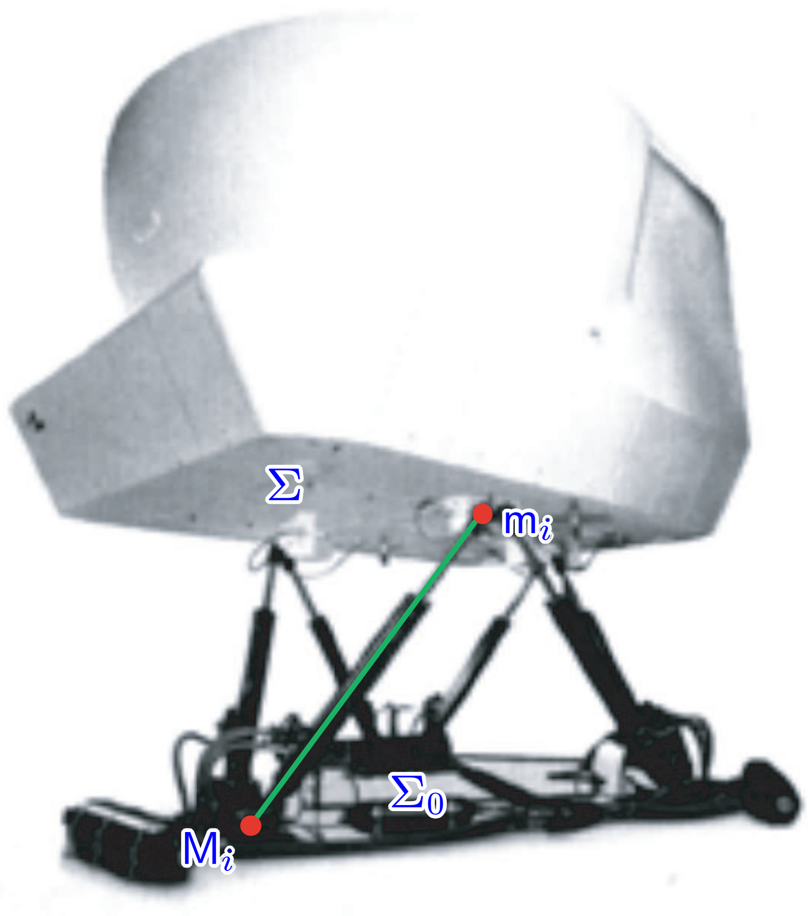
2a. Singular configurations of SGPs

The geometry of a SGP is given by the six base anchor points $\mathsf{M}_i \in \Sigma_0$ and by the six platform points $\mathsf{m}_i \in \Sigma$.

A SGP is called planar, if $\mathsf{M}_1, \ldots, \mathsf{M}_6$ are coplanar and $\mathsf{m}_1, \ldots, \mathsf{m}_6$ are coplanar.

M_i and m_i are connected with a S<u>P</u>S leg.

Theorem 10 MERLET [6]
A SGP is *singular* (*infinitesimal flexible*, *shaky*), if and only if, the carrier lines of the six S<u>P</u>S legs belong to a linear line complex.

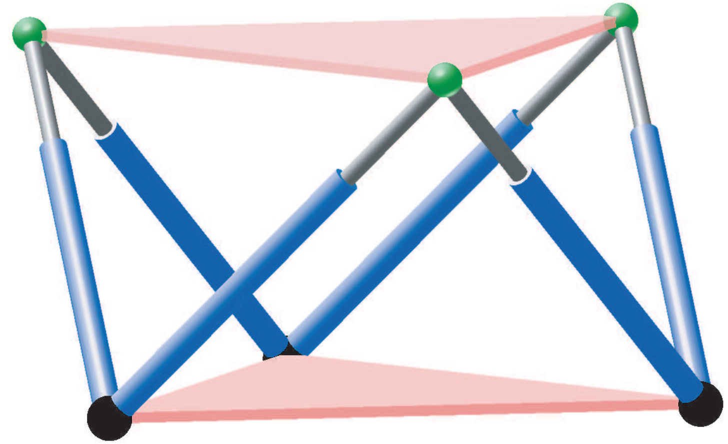


2a. Self-motions and the Borel Bricard problem

If all P-joints are locked, a SGP is in general rigid. But, in some special cases the SGP can perform an n-parametric motion ($n > 0$), which is called *self-motion*.

It should be noted, that in each pose of the self-motion, the SGP has to be singular.

Moreover, all self-motions of SGPs are solutions to the problem posed by the French Academy of Science for the *Prix Vaillant* (1904), which is also known as

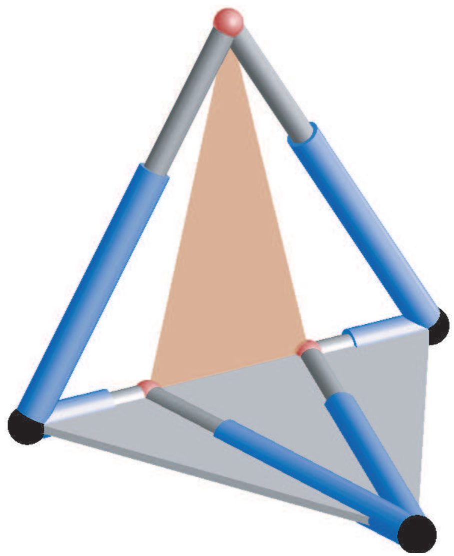


Borel Bricard problem (still unsolved)
Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.

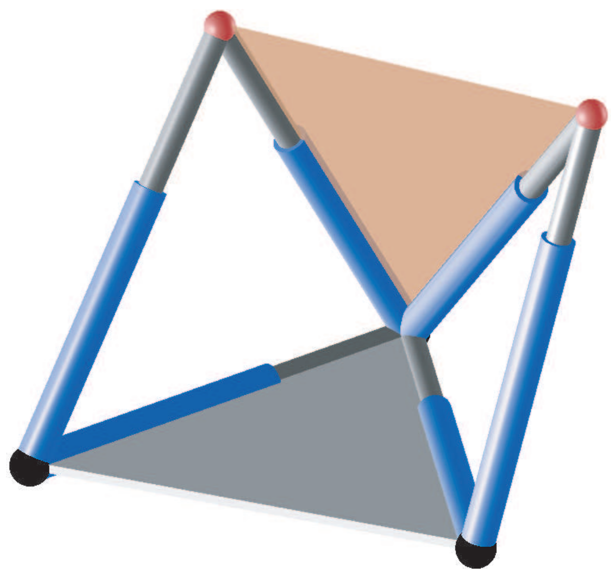
2a. Self-motions of octahedral SGPs

Clearly, all Bricard octahedra imply self-motions of octahedral SGPs. But, if we allow two faces to coincide permanently during the flex, there exist two more types of octahedral self-motions (cf. Stachel [10], Karger [4]):

Butterfly motion



Spherical four-bar motion



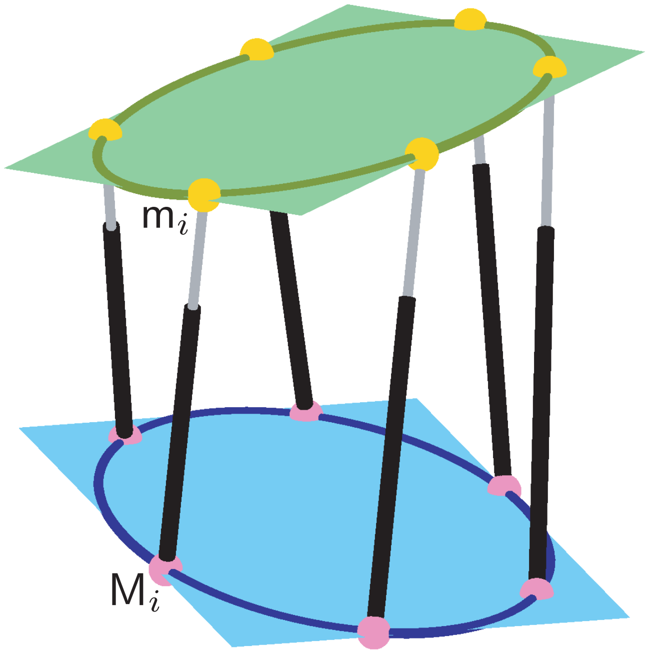
2a. Architecturally singular SGPs

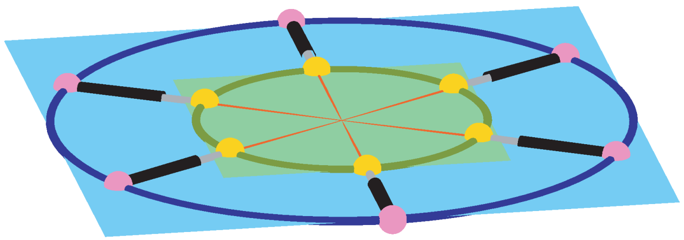
SGPs, which are singular in every configuration, are called *architecturally singular*.

Architecturally singular SGPs are well studied:

- ⋆ For the planar case see Röschel & Mick [9], Karger [2], Nawratil [7] and Wohlhart [13].
- ⋆ For the non-planar case see Karger [3] and Nawratil [8].

Architecturally singular SGPs possess self-motions in each pose (over $\mathbb{C}$).

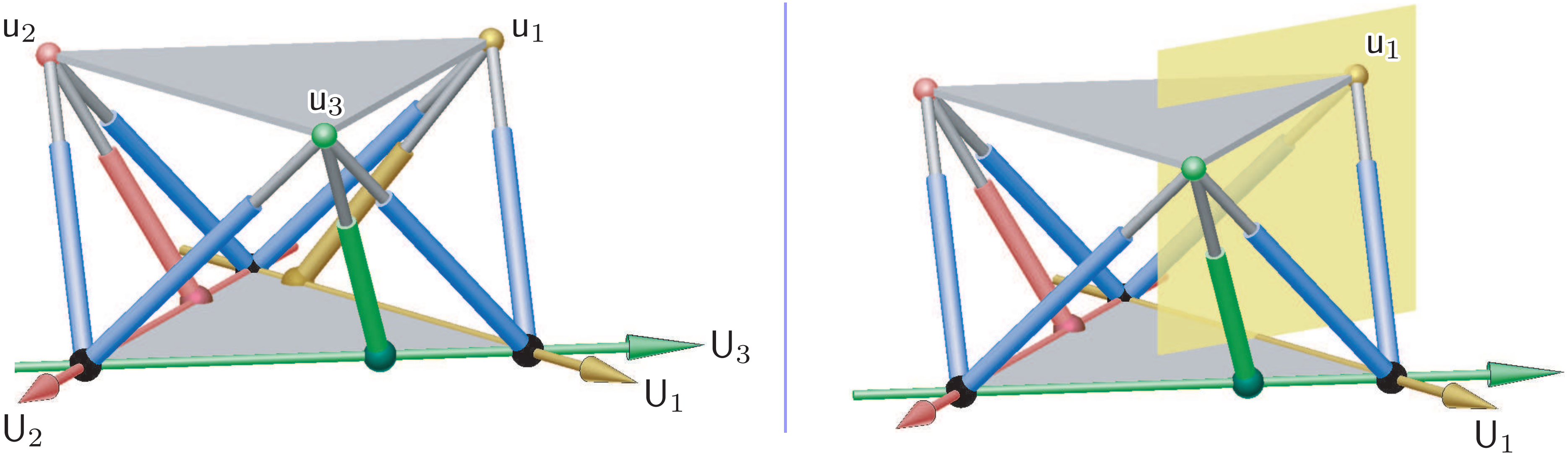




$\Rightarrow$ We are interested in non-architecturally singular SGPs with self-motions.

2b. Darboux constraints of octahedral SGPs

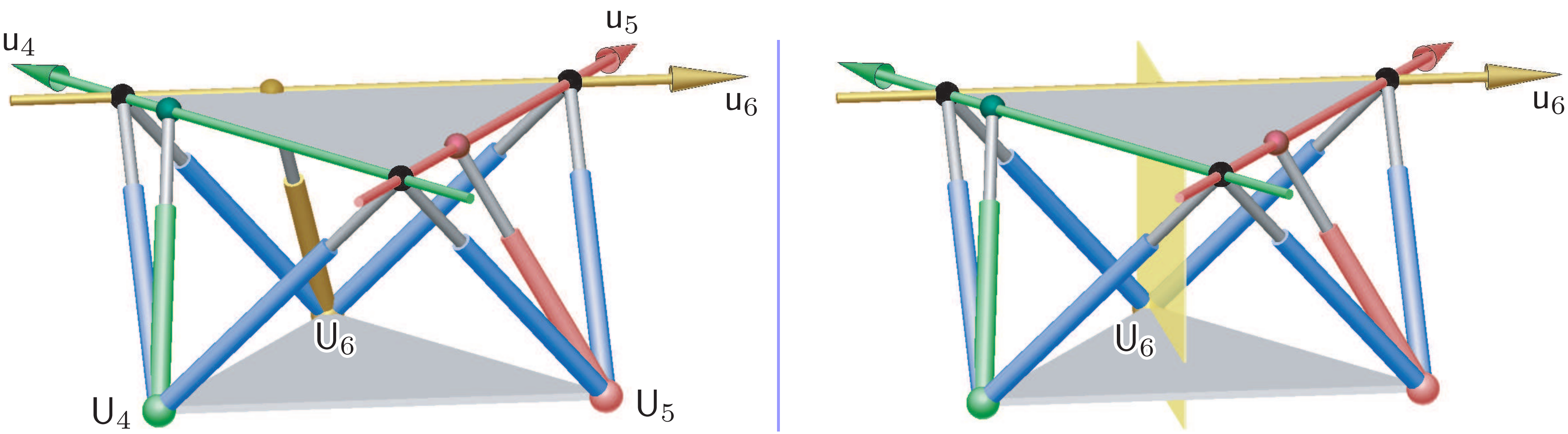
We can attach the following 3 pencils of additional legs to an octahedral SGP without changing the direct kinematics and the set of singular configurations:



Each leg $\overline{u_iU_i}$ for $i = 1, 2, 3$ imply a so-called Darboux motion, which means that u_i moves in a plane $\in \Sigma_0$ orthogonal to the direction of the ideal point U_i.

2b. Mannheim constraints of octahedral SGPs

We can attach the following 3 pencils of additional legs to an octahedral SGP without changing the direct kinematics and the set of singular configurations:



Each leg $\overline{\mathsf{u}_i\mathsf{U}_i}$ for $i = 4, 5, 6$ imply a so-called Mannheim motion, which means that a plane of Σ orthogonal to u_j slides through the point $\mathsf{U}_j \in \Sigma_0$.

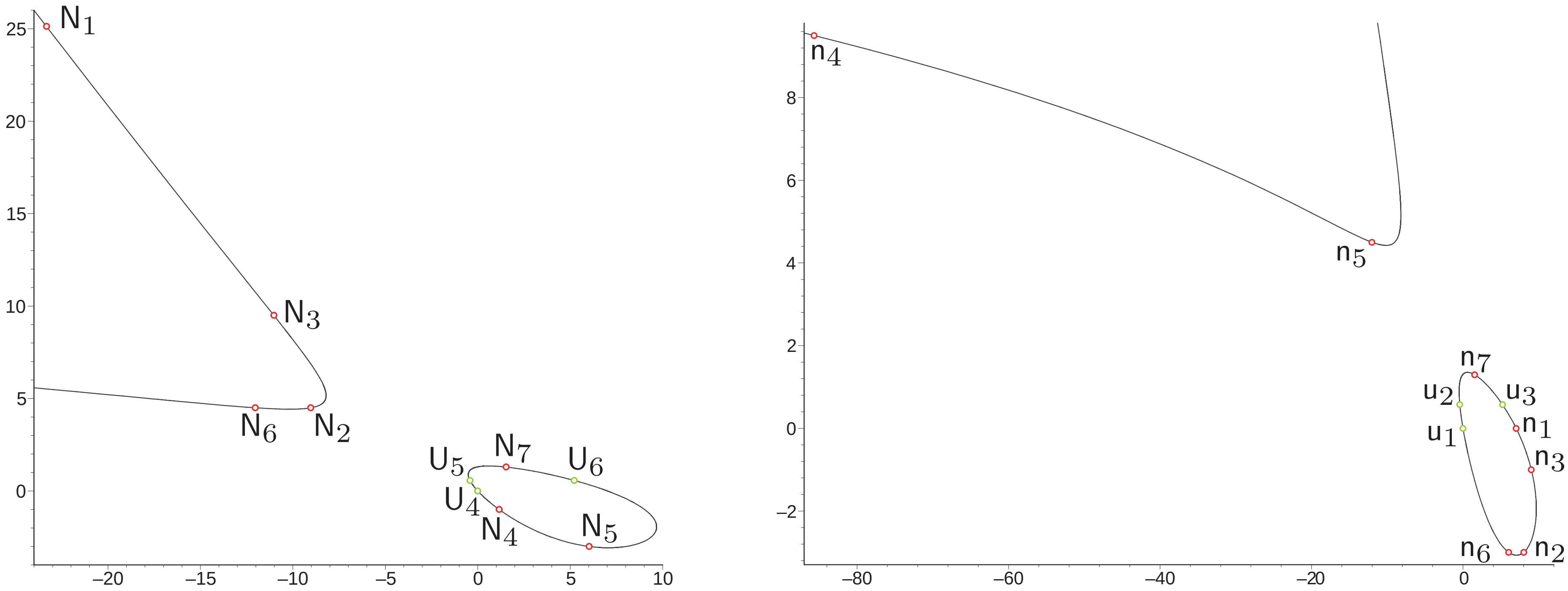
2c. Self-motions implied by Bricard octahedra I

We attach the six special legs $\overline{\mathsf{u}_i\mathsf{U}_i}$ for $i = 1, \ldots, 6$ to an octahedral SGP $\mathsf{m}_1, \ldots, \mathsf{M}_6$, which is a Bricard octahedron of type I. These additional legs do not disturb the self-motion $\mathcal{M}$ of the line-symmetric octahedron.

We remove the original six legs $\overline{\mathsf{m}_i\mathsf{M}_i}$ and remain with the manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$. It can be shown, that the manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ is architecturally singular and that it has a 2-parametric self-motion $\mathcal{U}$, which contains $\mathcal{M}$ (cf. Nawratil [D]).

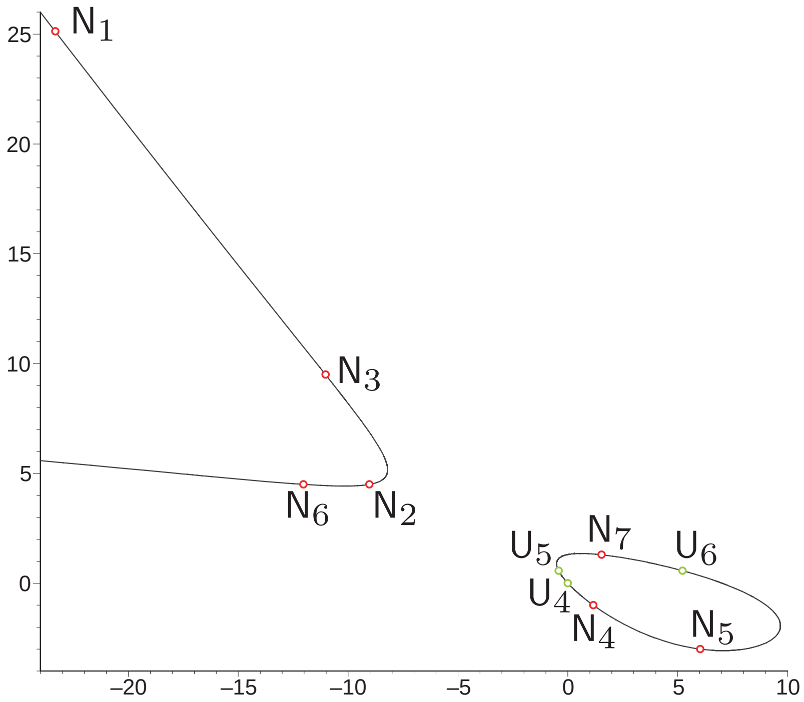
Now, we can add any leg $\overline{\mathsf{nN}}$, where the finite points n and N are located in the planar platform and planar base, respectively. This leg restricts the 2-parametric self-motion $\mathcal{U}$ to a 1-parametric self-motion $\mathcal{N}$.

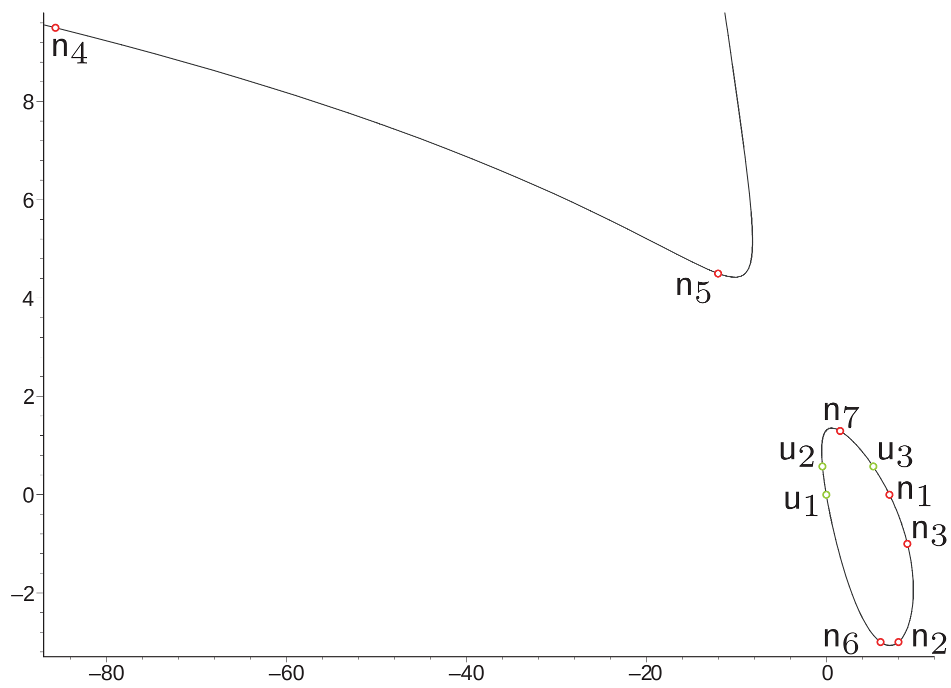
2c. Self-motions implied by Bricard octahedra I



Moreover, the points n and N determine a 1-parametric set of legs, which can be attached to the manipulator $u_1, \ldots, U_6, n, N$ without disturbing the self-motion $\mathcal{N}$. The anchor points of these legs are located on congruent cubics (cf. NAWRATIL [D]).

2c. Self-motions implied by Bricard octahedra I





Any six finite pairs of anchor points, which do not yield an architecturally singular SGP, are a solution to our problem (e.g. $n_1, \ldots, N_6$).
Note that $n_2, \ldots, N_7$ is an example for an architecturally singular SGP.

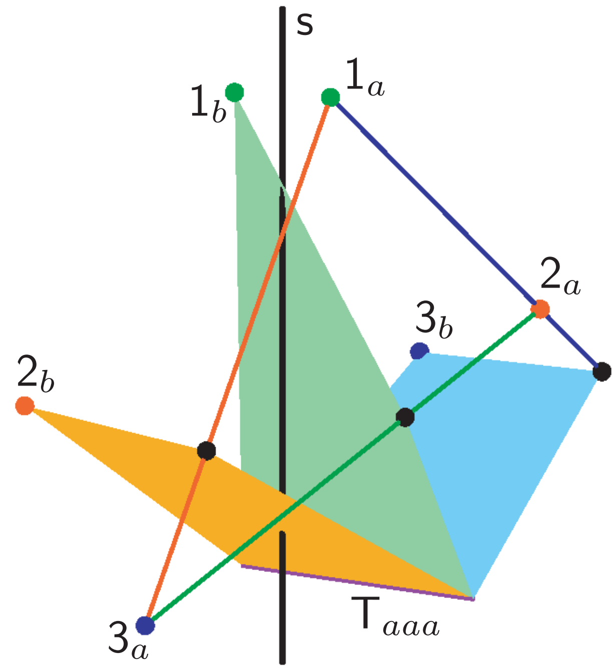
2d. New property of Bricard octahedra I

The presented construction was studied in a more general setting in NAWRATIL [E,F]. Within this generalization, it turned out, that each Bricard octahedron of type I with vertices $1_a, 1_b, 2_a, 2_b, 3_a, 3_b$, where v_a and v_b are symmetric with respect to the line s for $\mathsf{v} \in \{1, 2, 3\}$, possesses the following property:

Theorem 11 NAWRATIL [E]
The following 3 planes have a common line T_{ijk}:

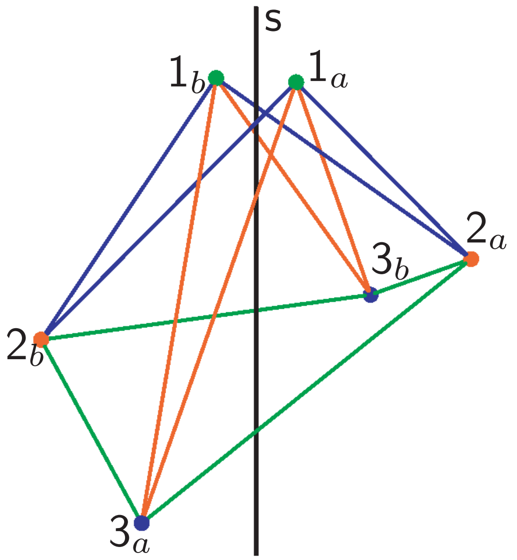
- ⋆ plane orthogonal to $[1_i, 2_j]$ though $3_{k'}$
- ⋆ plane orthogonal to $[2_j, 3_k]$ though $1_{i'}$
- ⋆ plane orthogonal to $[3_k, 1_i]$ though $2_{j'}$

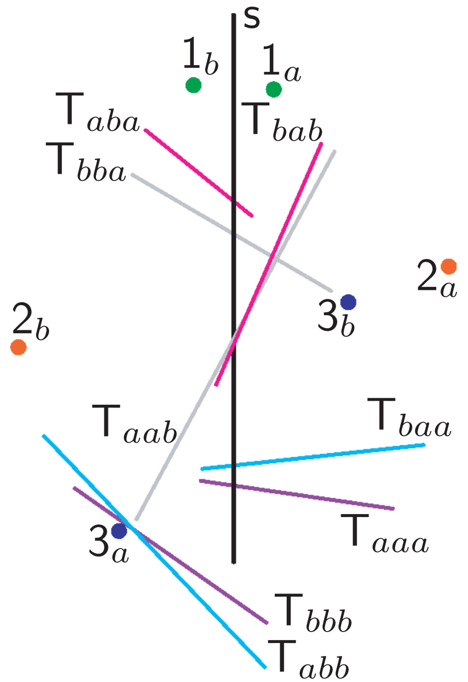
with $i \neq i'$, $j \neq j'$, $k \neq k' \in \{a, b\}$.



2d. Example

All eight possible axes T_{ijk} of the following Bricard octahedron I are drawn: $1_a = (1, 0, 0)$, $2_a = (5, 3, -6)$, $3_a = (-2, -7, -9)$, line of symmetry s is the z-axis.





References

[1] BRICARD R (1897) Mémoire sur la théorie de l'octaèdre articulé. Journal de Mathématiques pures et appliquées, Liouville 3:113–148

[2] KARGER A (2003) Architecture singular planar parallel manipulators. Mech Mach Theory 38(11):1149–1164

[3] KARGER A (2008) Architecturally singular non-planar parallel manipulators. Mech Mach Theory 43(3):335–346

[4] KARGER A (2010) Self-Motions of 6-3 Stewart Gough Type Parallel Manipulators, In: Lencarcic J, Stanisic MM (eds) Advances in Robot Kinematics: Motion in Man and Machine, Springer, pp 359–366

[5] KOKOTSAKIS A (1932) Über bewegliche Polyeder. Math Ann 107:627–647

[6] MERLET J-P (1989) Singular configurations of parallel manipulators and Grassmann geometry. Int J Rob Res 8(5) 45–56

[7] NAWRATIL G (2008) On the degenerated cases of architecturally singular planar parallel manipulators. J Geom Graphics 12(2):141–149

[8] NAWRATIL G (2009) A new approach to the classification of architecturally singular parallel manipulators. In: Kecskemethy A, Müller A (eds) Computational Kinematics, Springer, pp 349–358

[9] RÖSCHEL O, MICK S (1998) Characterisation of architecturally shaky platforms. In: Lenarcic J, Husty ML (eds) Advances in Robot Kinematics: Analysis and Control, Kluwer, pp 465–474

[10] STACHEL H (1999) Higher order flexibility of octahedra. Period Math Hungar 39:225–240

[11] STACHEL H (2002) Remarks on Bricard's flexible octahedra of type 3. Proc. 10th International Conference on Geometry and Graphics, Kiev, Ukraine, 1:8–12

[12] STACHEL H (2010) A kinematic approach to Kokotsakis meshes. Comput Aided Geom Des 27:428-437

[13] WOHLHART K (2010) From higher degrees of shakiness to mobility. Mech Mach Theory 45(3):467–476