Planar Stewart Gough platforms with quadratic singularity surface

B. Aigner[1] and G. Nawratil[2]

Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Austria, [1]e-mail: bernd.aigner@gmx.at,[2]e-mail: nawratil@geometrie.tuwien.ac.at

Abstract. We give a detailed study of planar Stewart Gough platforms, which possess a quadratic singularity surface in the space of translations for any orientation of the platform. These manipulators were already characterized by a rank condition of a 5×6 matrix, but a geometric interpretation is still missing until now. We give this geometric criterion based on the useful existence result, that every non-architecturally singular Stewart Gough platform has corresponding triples of anchor points, where the triangles in the platform and the base are not degenerated. Moreover, we present a rational parametrization of the 5-dimensional singularity locus and give an upper bound for the number of solutions of the direct kinematics problem. Finally we remark special properties of the locus of anchor points for singular-invariant leg-replacements.

Key words: Stewart Gough platform, singularities, rational parametrization, direct kinematics, singular-invariant leg-replacement.

1 Introduction

The geometry of a Stewart Gough (SG) platform is given by the six base anchor points M_i with coordinates $\mathbf{M}_i := (A_i, B_i, C_i)^T$ with respect to the fixed frame and by the six platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, b_i, c_i)^T$ with respect to the moving frame (for $i = 1, \ldots, 6$). Each pair $(\mathsf{M}_i, \mathsf{m}_i)$ of corresponding anchor points of the fixed body (base) and the moving body (platform) is connected by an SPS-leg, where only the prismatic joint (P) is active and the spherical joints (S) are passive. Note that for a SG platform, $(\mathsf{M_i}, \mathsf{m_i}) \neq (\mathsf{M_j}, \mathsf{m_j})$ holds for pairwise distinct $i, j \in \{1, \ldots, 6\}$. Moreover, a SG manipulator is called planar, if $\mathsf{M}_1, \ldots, \mathsf{M}_6$, as well as $\mathsf{m}_1, \ldots, \mathsf{m}_6$ are coplanar ($\Leftrightarrow c_i = C_i = 0$ for $i = 1, \ldots, 6$).

The necessary and sufficient condition for the infinitesimal flexibility of a SG platform is that the carrier lines of the six legs belong to a linear line complex [9]. The corresponding configurations of the manipulator are called singular (or shaky). Parallel manipulators of SG type, which are singular in every possible configuration, are called architecturally singular. They are well studied and classified (for a review see [13, Section 3.1]).

1.1 Review and outline

Based on the articles [10, 11] the following result is already known:

Theorem 1. *A planar SG manipulator possesses a quadratic singularity surface in the space of translations for any orientation of the platform if and only if* $rk(\mathbf{N}) = 4$ *holds with*

$$\mathbf{N} := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{pmatrix}. \tag{1}$$

During the study of planar SG manipulators with a simplified solution for the direct kinematics, Karger [8] has also been interested in the geometric interpretation of the different ranks of $\mathbf{N}$. He obtained the following results:

Theorem 2. $rk(\mathbf{N}) = 3 \Leftrightarrow$ *base and platform are affinely equivalent.*

Moreover Karger [8] noted that $rk(\mathbf{N}) < 3$ is not of interest as this condition implies trivial cases of architecturally singular designs and that for $rk(\mathbf{N}) = 4$ "*no special properties are known so far*".

Based on an useful existence theorem proven in Section 2, we provide a geometric characterization for planar SG manipulators with $rk(\mathbf{N}) = 4$ in Section 3. In Section 4 we present a rational parametrization of the 5-dimensional singularity set of these manipulators. In Section 5 we give an upper bound for the number of solutions of the direct kinematics problem and note special properties of the locus of anchor points for singular-invariant leg-replacements.

2 Existence theorem

The following existence theorem is implied by the Lemmata 1-3:

Theorem 3. *Every non-architecturally singular SG manipulator possesses triples of anchor points* $(\mathsf{M}_i, \mathsf{M}_j, \mathsf{M}_k)$ *and* $(\mathsf{m}_i, \mathsf{m}_j, \mathsf{m}_k)$ *with* $i, j, k \in \{1, \dots, 6\}$ *in a way that the triangles* $\triangle(\mathsf{M}_i\mathsf{M}_j\mathsf{M}_k)$ *and* $\triangle(\mathsf{m}_i\mathsf{m}_j\mathsf{m}_k)$ *are not degenerated; i.e. they do not collapse into a line or even a point.*

Lemma 1. *Every non-architecturally singular SG manipulator possesses corresponding pairs of anchor points* $(\mathsf{M}_i, \mathsf{M}_j)$ *and* $(\mathsf{m}_i, \mathsf{m}_j)$ *with* $\mathsf{M}_i \neq \mathsf{M}_j$ *and* $\mathsf{m}_i \neq \mathsf{m}_j$ *for* $i, j \in \{1, \dots, 6\}$.

Proof. Note that not more than three platform (resp. base) anchor points can collapse into one point, as otherwise we get the architectural singularity [7, Theorem 3(5)]. Therefore there exists a point m_i, which does not coincide with $\mathsf{m}_k, \mathsf{m}_l, \mathsf{m}_m$ with pairwise distinct $i, k, l, m \in \{1, \dots, 6\}$. As no four base points are allowed to coincide, there exists at least one index $j \in \{k, l, m\}$ with $\mathsf{M}_i \neq \mathsf{M}_j$. □

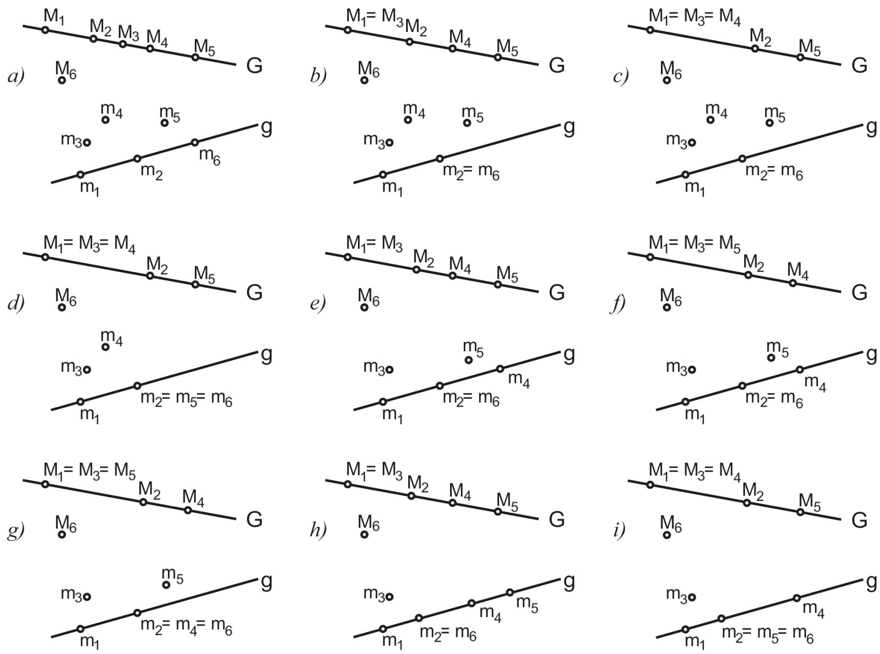

Fig. 1 Sketches for the proof of Lemma 3, case B.

In the following we denote corresponding pairs of triangles $\triangle(\mathsf{M}_i\mathsf{M}_j\mathsf{M}_k)$ and $\triangle(\mathsf{m}_i\mathsf{m}_j\mathsf{m}_k)$ just by $\triangle_{ijk}$. We call $\triangle_{ijk}$ non-degenerated if both involved triangles have this property; otherwise $\triangle_{ijk}$ is a degenerated one.

Lemma 2. *Every non-architecturally singular SG manipulator with no four anchor points collinear possesses a non-degenerated $\triangle_{ijk}$ for $i,j,k \in \{1,\dots,6\}$.*

Proof. Due to Lemma 1 we can assume without loss of generality (w.l.o.g.) that $\mathsf{M}_1 \neq \mathsf{M}_2$ and $\mathsf{m}_1 \neq \mathsf{m}_2$ hold. We denote by $\mathscr{I}$ the set of indices such that $\triangle_{12i}$ is degenerated. As no four points are assumed to be collinear we have $\#\mathscr{I} < 3$. Therefore there exist an $i \in \{3,\dots,6\} \setminus \mathscr{I}$ in a way that $\triangle_{12i}$ is not degenerated. □

Lemma 3. *Every non-architecturally singular SG manipulator with four anchor points collinear possesses a non-degenerated $\triangle_{ijk}$ for $i,j,k \in \{1,\dots,6\}$.*

Proof. The proof is done by contradiction; i.e. we assume that all $\triangle_{ijk}$ degenerate und show that we end up with an architectural singularity (a contradiction).

Due to Lemma 1 we can assume w.l.o.g. that $\mathsf{M}_1 \neq \mathsf{M}_2$ and $\mathsf{m}_1 \neq \mathsf{m}_2$ hold. These points span the lines $\mathsf{g} := [\mathsf{m}_1, \mathsf{m}_2]$ and $\mathsf{G} := [\mathsf{M}_1, \mathsf{M}_2]$, respectively. All $\triangle_{12i}$ degenerate if either $\mathsf{M}_i \in \mathsf{G}$ or $\mathsf{m}_i \in \mathsf{g}$ hold for $i = 3,\dots,6$. This results in the following cases (up to relabeling of anchor points and exchange of platform and base):

A. $\mathsf{M}_3,\dots,\mathsf{M}_6 \in \mathsf{G}$: We get the architectural singularity [7, Theorem 3(1)].

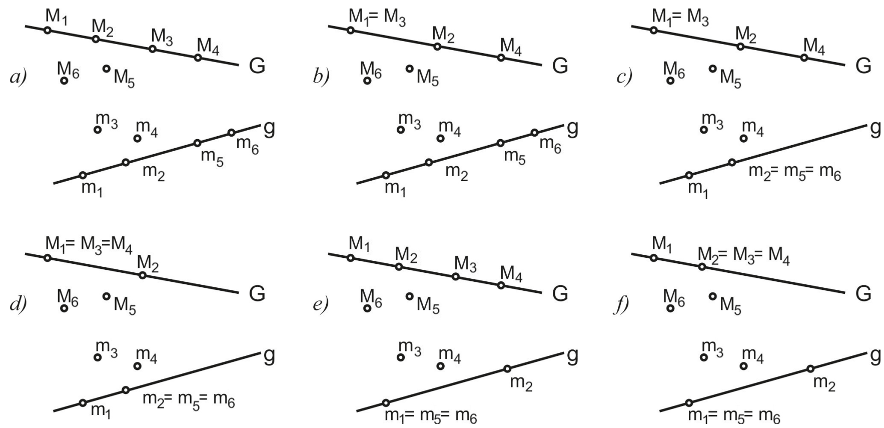

Fig. 2 Sketches for the proof of Lemma 3, case C.

B. $\mathsf{M}_3, \mathsf{M}_4, \mathsf{M}_5 \in \mathsf{G}$, $\mathsf{m}_6 \in \mathsf{g}$ and $\mathsf{M}_6 \notin \mathsf{G}$ (Fig. 1a): We can assume that one of the three points $\mathsf{m}_3, \mathsf{m}_4, \mathsf{m}_5$ is not located on g, as otherwise we get again the architectural singularity [7, Theorem 3(1)]. W.l.o.g. let this point be m_3. Now $\triangle_{136}$ and $\triangle_{236}$ can only degenerate for $\mathsf{M}_i = \mathsf{M}_3$ and $\mathsf{m}_j = \mathsf{m}_6$ for $i \neq j \in \{1,2\}$. W.l.o.g. we can set $i = 1$ and $j = 2$ (Fig. 1b). Now $\triangle_{146}$ can only degenerate in one of the following two cases:

   a. $\mathsf{M}_1 = \mathsf{M}_3 = \mathsf{M}_4$ (Fig. 1c): $\triangle_{156}$, $\triangle_{356}$ and $\triangle_{456}$ cannot degenerate by a condition on the base as this is only possible for $\mathsf{M}_1 = \mathsf{M}_3 = \mathsf{M}_4 = \mathsf{M}_5$, which yields a contradiction (cf. [7, Theorem 3(5)]). Therefore the platform triangles have to degenerate, which can only be the case for $\mathsf{m}_5 = \mathsf{m}_6$ (Fig. 1d) yielding the architectural singularity [7, Theorem 3(2)].
   b. $\mathsf{m}_4 \in \mathsf{g}$ (Fig. 1e): Now $\triangle_{156}$ degenerates in one of the following two cases:
      i. $\mathsf{M}_1 = \mathsf{M}_3 = \mathsf{M}_5$ (Fig. 1f): Then $\triangle_{346}$ can only degenerate for either $\mathsf{m}_2 = \mathsf{m}_4 = \mathsf{m}_6$ (Fig. 1g) or $\mathsf{M}_1 = \mathsf{M}_3 = \mathsf{M}_4 = \mathsf{M}_5$, which yields in both cases architecture singularities [7, Theorem 3(2 resp. 5)].
      ii. $\mathsf{m}_5 \in \mathsf{g}$ (Fig. 1h): Now $\triangle_{3i6}$ with $i \in \{4,5\}$ degenerates for $\mathsf{M}_1 = \mathsf{M}_3 = \mathsf{M}_i$ or $\mathsf{m}_2 = \mathsf{m}_i = \mathsf{m}_6$. All possible cases, which arise for $i = 4,5$ (e.g. Fig. 1i), yield architecture singularities [7, Theorem 3(2 resp. 5)].

C. $\mathsf{M}_3, \mathsf{M}_4 \in \mathsf{G}$, $\mathsf{m}_5, \mathsf{m}_6 \in \mathsf{g}$ and $\mathsf{M}_5\mathsf{M}_6 \notin \mathsf{G}$ (Fig. 2a): In addition we can assume $\mathsf{m}_3, \mathsf{m}_4 \notin \mathsf{g}$ as otherwise we also get case B. Therefore $\triangle_{135}$ and $\triangle_{136}$ can only degenerate in the following two cases:

   a. $\mathsf{M}_1 = \mathsf{M}_3$ (Fig. 2b): Now $\triangle_{235}$ and $\triangle_{236}$ can only degenerate for $\mathsf{m}_2 = \mathsf{m}_5 = \mathsf{m}_6$ (Fig. 2c). Then $\triangle_{145}$ can only degenerate for $\mathsf{M}_1 = \mathsf{M}_3 = \mathsf{M}_4$ (Fig. 2d), which yields the architecture singularity [7, Theorem 3(2)].
   b. $\mathsf{m}_1 = \mathsf{m}_5 = \mathsf{m}_6$ (Fig. 2e): Now $\triangle_{235}$ and $\triangle_{245}$ can only degenerate for $\mathsf{M}_2 = \mathsf{M}_3 = \mathsf{M}_4$ (Fig. 2f) yielding the architecture singularity [7, Theorem 3(2)]. $\square$

3 Geometric interpretation of $rk(\mathbf{N}) = 4$

In a first step we can always choose the fixed frame and moving frame in a way that the first base and platform anchor point are located in their origins; i.e. $a_1 = b_1 = A_1 = B_1 = 0$. Therefore $rk(\mathbf{N}) = 4$ is equivalent to $rk(\mathbf{n}) = 3$ with

$$\mathbf{n} := \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ b_2 & b_3 & b_4 & b_5 & b_6 \\ A_2 & A_3 & A_4 & A_5 & A_6 \\ B_2 & B_3 & B_4 & B_5 & B_6 \end{pmatrix}. \tag{2}$$

Lemma 4. *The rank of* $\mathbf{n}$ *is invariant under regular affinities of the platform and the base.*

Proof. We apply regular affine transformations τ_0 to the base and τ to the platform, respectively, where

$$\mathbf{T} := \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{t} := \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \tag{3}$$

with $\det(\mathbf{T}) \neq 0 \neq \det(\mathbf{t})$ are the matrices of these transformations; i.e.

$$\tau_0 : \mathbf{M}_i \mapsto \mathbf{T}\mathbf{M}_i \quad \text{and} \quad \tau : \mathbf{m}_i \mapsto \mathbf{t}\mathbf{m}_i. \tag{4}$$

Now we build the analogous matrix as given in Eq. (2) with respect to the coordinates of $\tau_0(\mathsf{M}_i)$ and $\tau(\mathsf{m}_i)$ computed by Eq. (4). This 4×5 matrix is denoted by $\widetilde{\mathbf{n}}$. Then the determinant of the 4×4 submatrix $\widetilde{\mathbf{n}}_i$ of $\widetilde{\mathbf{n}}$, which is obtained by removing the ith column of $\widetilde{\mathbf{n}}$, factors into $\det(\mathbf{T})\det(\mathbf{t})\det(\mathbf{n}_i)$. □

Based on Lemma 4 and Theorem 3 we can prove the following theorem:

Theorem 4. $rk(\mathbf{N}) = 4 \Leftrightarrow$ *There exists a regular affinity* α *from the platform to the base in a way that* $\alpha(\mathsf{m}_i)$ *and* M_i *are located on lines of a parallel line pencil with vertex* P *at infinity.*

Proof. Due to Theorem 3 there exists a non-degenerated $\triangle_{ijk}$. W.l.o.g. we can assume that $i = 1,\ j = 2$ and $k = 3$ holds. Moreover due to Lemma 4 we can also assume w.l.o.g. that

$$B_2 = A_3 = b_2 = a_3 = 0 \quad \text{and} \quad A_2 = B_3 = a_2 = b_3 = 1 \tag{5}$$

hold. Based on this preparatory work we prove both directions separately:

"$\Rightarrow$" By basic operations on columns of the matrix $\mathbf{n}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & A_4 - a_4 & A_5 - a_5 & A_6 - a_6 \\ 0 & 0 & B_4 - b_4 & B_5 - b_5 & B_6 - b_6 \end{pmatrix}, \tag{6}$$

which implies that

$$(A_i - a_i)(B_j - b_j) = (A_j - a_j)(B_i - b_i),$$

has to hold for pairwise distinct $i, j \in \{4,5,6\}$, which already proves "$\Rightarrow$".

"$\Leftarrow$" Under our assumptions the coordinates of the anchor points can be written as:

$$\mathbf{M}_1 = \mathbf{m}_1 = (0,0,0)^T, \qquad \mathbf{M}_4 = \mathbf{m}_4 + \xi_4 \mathbf{p}, \tag{7}$$
$$\mathbf{M}_2 = \mathbf{m}_2 = (1,0,0)^T, \qquad \mathbf{M}_5 = \mathbf{m}_5 + \xi_5 \mathbf{p}, \tag{8}$$
$$\mathbf{M}_3 = \mathbf{m}_3 = (0,1,0)^T, \qquad \mathbf{M}_6 = \mathbf{m}_6 + \xi_6 \mathbf{p}, \tag{9}$$

where $\mathbf{p} = (p_1, p_2, 0)^T$ is the direction of the parallel line pencil (= direction of ideal point P). It can easily be seen by applying analogous column operations as in "$\Rightarrow$" that $rk(\mathbf{n}) = 3$ holds. □

Remark 1. Note that due to Theorem 4 a planar SG manipulator with $rk(\mathbf{N}) = 4$ possesses a so-called *similarity bond* (cf. [4]). Conversely, for every planar SG manipulator with $rk(\mathbf{N}) > 3$ the existence of a similarity bond implies $rk(\mathbf{N}) = 4$. ⋄

4 Rational parametrization of the singularity locus

Due to the results of Section 3 we can coordinatize each planar SG manipulator with $rk(\mathbf{N}) = 4$ as follows: We take the coordinates of Eqs. (7-9) and apply affine transformations as given in Eq. (4), where one can assume w.l.o.g. that $T_{21} = t_{21} = 0$ and $T_{11} > 0 < t_{11}$ holds. If one wants to eliminate the factor of similarity one can set e.g. t_{11} equal to 1.

Based on this coordinatization we can compute the Plücker coordinates of the carrier lines l_i of the six legs by $(\mathbf{l}_i, \widehat{\mathbf{l}}_i) := (\mathbf{m}'_i - \mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$, where $\mathbf{m}'_i$ is the location-vector of $\mathbf{m}_i$ with respect to the fixed system; i.e. $\mathbf{m}'_i = N^{-1}\mathbf{R}\mathbf{m}_i + \mathbf{s}$ with

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix},$$

$N = e_0^2 + e_1^2 + e_2^2 + e_3^2$ and $\mathbf{s} = (s_1, s_2, s_3)^T$.

Now the lines $\mathsf{l}_1, \ldots, \mathsf{l}_6$ belong to a linear line complex if and only if $S = 0$ holds with

$$S := \begin{vmatrix} \mathbf{l}_1 & \ldots & \mathbf{l}_6 \\ \widehat{\mathbf{l}}_1 & \ldots & \widehat{\mathbf{l}}_6 \end{vmatrix}. \tag{10}$$

Therefore the singularity locus Σ, which is only quadratic in the translation parameters s_1, s_2, s_3 according to Theorem 1, is given by $S = 0$ (can be computed explicitly by e.g. MAPLE).

For the parametrization of the singularity locus we follow the idea of [3]. We homogenize $S=0$ by replacing s_i by ς_i/ς_0 for $i=1,2,3$ and multiply the resulting equation by ς_0^2. We denote the obtained equation by $S_h=0$. Then it can be checked by direct computations that the ideal point W with $(\varsigma_0:\ldots:\varsigma_3)=(0:r_{23}:-r_{13}:0)$ is located on Σ_h: $S_h=0$. Now the parallel line bundle $\mathscr{B}$ through W given by:

$$\mathscr{B}: \begin{pmatrix} r_{13}u \\ r_{23}u \\ v \end{pmatrix} + w \begin{pmatrix} r_{23} \\ -r_{13} \\ 0 \end{pmatrix} \tag{11}$$

can be used for the rational parametrization. We only have to plug these coordinates of $\mathscr{B}$ into $S=0$ and obtain an equation of the form $p+qw=0$. If we insert $w=-p/q$ back into Eq. (11) we get the desired rational parametrization of Σ in dependency of u, v and the homogenous Euler parameters $e_0:\ldots:e_3$, which is well defined for all orientations with $q\neq 0$. This result corresponds to [3, Corollary 7].

Remark 2. A special case of $q=0$ is given if W is not defined; i.e. $r_{13}=r_{23}=0$. As W is the ideal point of the intersection line of the platform and the base, this can only be the case if the platform and the base are parallel to each other; i.e. $e_0=e_3=0$ or $e_1=e_2=0$. Therefore these manipulators are also Schönflies-singular with respect to the rotation axis orthogonal to both planes (cf. [12]). $\diamond$

5 Direct kinematics

Finally we want to give some remarks concerning the direct kinematics of planar SG manipulators with $rk(\mathbf{N})=4$. The condition that the point m_i is located on a sphere with center M_i and radius d_i is a quadratic condition $K_i=0$ (e.g. Husty [5]) in $e_0,\ldots,e_3,s_1,s_2,s_3$. Then one considers the linear combination:

$$Q := \kappa_1K_1+\kappa_2K_2+\kappa_3K_3+\kappa_4K_4+\kappa_5K_5+\kappa_6K_6. \tag{12}$$

According to [8] there exists a $(5-rk(\mathbf{N}))$-dimensional solution set for $(\kappa_1:\ldots:\kappa_6)\neq(0:\ldots:0)$ in a way that Q is free of translation parameters (but still quadratic in the Euler parameters).

Therefore $rk(\mathbf{N})=4$ implies the existence of two linear independent linear combinations Q_1 and Q_2. Moreover one can solve the system of equations $K_2-K_1=K_3-K_1=K_4-K_1=0$, which is linear in s_1,s_2,s_3 for these unknowns. Inserting the resulting expressions into K_1 implies a condition O of degree 8 in the Euler parameters (see also [5]). Therefore the direct kinematics reduces to the intersection of an octic surface $O=0$ and two quadrics $Q_1=Q_2=0$ in the Euler parameter space, which shows the following result considering Bezout's theorem.

Theorem 5. *A planar SG manipulator with $rk(\mathbf{N})=4$ has not more than 32 solutions (over $\mathbb{C}$) for the direct kinematics problem.*

An example verifying these upper bound of 32 (over $\mathbb{C}$) is given in [1, Section 4.1].

5.1 Leg-replacement

Now we consider the set $\mathscr{L}$ of legs, which can be added to a SG manipulator without changing neither the direct kinematics [6] nor the set of singular configurations [2]. From the referred two papers it is known that for a general planar SG manipulator a one-parametric set $\mathscr{L}$ exists, where the platform (resp. base) anchor points are located on a planar cubic curve c (resp. C) on the platform (resp. base).

It can be shown by direct computation (see [1, Theorem 8]) that P is located on C and $\alpha^{-1}(\mathsf{P})$ on c with P and α of Theorem 4. In the general case the corresponding platform anchor point of P differs from $\alpha^{-1}(\mathsf{P})$. As the leg-replacement is singular-invariant, $\alpha(\mathsf{m}) \in \alpha(\mathsf{c})$ and the corresponding base point $\mathsf{M} \in \mathsf{C}$ have to be located on a line through P.[1] This gives a nice geometric relation between platform cubic and base cubic. Illustrating examples are given in [1, Sections 4.3.1 and 4.3.2].

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[1] The corresponding platform point $\in \alpha(\mathsf{c})$ (resp. base point $\in \mathsf{C}$) of $\mathsf{P} \in \mathsf{C}$ (resp. $\mathsf{P} \in \alpha(\mathsf{c})$) is located on the asymptote of C (resp. $\alpha(\mathsf{c})$) through P.