Self-motions of TSSM manipulators with two parallel rotary axes

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In this paper we determine all non-trivial self-motions of TSSM manipulators with two parallel rotary axes which equals the determination of all flexible octahedra where one vertex is an ideal point. This study also closes the classification of these motions for the whole set of parallel manipulators of TSSM type. Our approach is based on Kokotsakis meshes and the reducible compositions of spherical coupler motions with a spherical coupler component.

Keywords: TSSM manipulator, self-motion, flexible octahedra, Kokotsakis mesh, spherical coupler motion.

I. INTRODUCTION

A parallel manipulator of TSSM type (Triangular Symmetric Simplified Manipulator) consists of a platform, which is connected via three Spherical-Prismatical-Rotational (SPR) legs l_i with the base (see FIG. 1), where the axes r_i $(i = 1,2,3)$ of the rotational joints are coplanar.

In general, the manipulator is rigid if all three leg lengths are fixed, but in some special cases, the platform can even perform a continuous motion. Such motions, which also yield solutions to the famous Borel-Bricard problem (cf. HUSTY [1]), are called self-motions. As we can replace each SPR leg l_i by two SPS legs p_i and q_i (as shown in FIG. 1 for $i = 1$) the determination of TSSM self-motions can be traced back to those of planar 6-3 parallel manipulators of Stewart Gough type (SG type).

Therefore TSSM self-motions must be contained in the singularity loci of the 6-3 manipulators, which have been analyzed in previous works [2–6]. Hence, the geometric interpretation of the singularities of a 6-3 manipulator must also hold for each pose of its self-motion. One possible geometric characterization was given in [5] and reads as follows: A singular configuration of a 6-3 manipulator occurs when the moving platform and three planes, each created by a pair of intersecting legs, intersect in at least one point.

In spite of all the work done on singularities of these manipulators a complete classification of all TSSM designs with self-motions is missing until now. The presented paper closes this gap, where we distinguish four subcases of TSSM manipulators in order to classify their self-motions:

(1) TSSM manipulator with intersecting axes: In this case the corresponding planar 6-3 SG platform can easily be transformed by so-called Δ-transforms (cf. [7]) into an octahedral manipulator by choosing the base anchor points as the intersection points of the axes r_i. Therefore TSSM manipulators with non-trivial self-motions correspond to the three types of *Bricard's flexible octahedra*:

type 1 All three pairs of opposite vertices are symmetric with respect to a common line.

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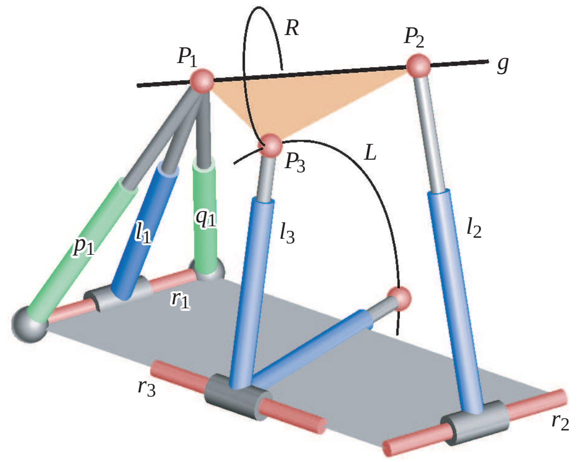

FIG. 1: TSSM manipulator with parallel rotary axes r_1, r_2. Moreover, the substitution of l_1 by p_1 and q_1 is illustrated.

type 2 Two pairs of opposite vertices are symmetric with respect to a common plane which passes through the remaining two vertices.

type 3 For a detailed discussion of this type we refer to [8]. We only want to mention that these flexible octahedra possess two flat poses.

These are all flexible octahedra as long as we assume that no two faces coincide permanently during the flex. Without this assumption we get two more cases (cf. STACHEL [9]) with so-called trivial self-motions, which are also known as *butterfly motion* and *spherical four-bar motion*, respectively (cf. KARGER [10]).

But TSSM manipulators can also have configurations which are not considered in the theory of flexible octahedra, as one can generate poses (by changing the leg lengths) where faces degenerate into lines. Then at least one platform anchor point P_i has to be located on the rotary axis r_i. As a consequence, the self-motion can only be a spherical motion with center P_i. It can easily be seen, that this can only yield special cases of the *butterfly motion* and *spherical four-bar motion*, respectively.

Finally, one also has to consider the TSSM manipulator with 3 collinear platform anchor points. In this case we trivially get an architectural singular manipulator (cf. [11, 12]).

(2) TSSM manipulators with 3 parallel axes: For TSSM manipulators with 3 parallel axes (which can also coincide) the problem reduces to a planar one, as these manipulators possess a cylindrical singularity surface (cf. NAWRATIL [13]). Therefore the self-motions correspond to those of the 3-dof Revolute-Prismatic-Revolute manipulator (with three collinear base anchor points) which are well known.

(3) TSSM manipulators with 2 coinciding axes: The determination of all self-motions of TSSM manipulators with two coinciding axes $r := r_1 = r_2$ can be reduced to the following geometric problem: If we disconnect the third leg from the platform, the anchor point P_3 describes a so-called fourth order cyclide of revolution Φ. This surface is generated by the rotation of the circle R about r, where R is the path of P_3 during its rotation about $g := [P_1P_2]$ (see FIG. 1). Now there exists a self-motion if the circle L (or a segment of it) is located on Φ, where L is generated by the end point of the disconnected leg during the rotation about r_3. Therefore the problem reduces to the determination of all circle sections on Φ. In the following we sum up the results of this well studied geometric problem (e.g. KRAMES [14]):

a. If g and r are parallel then Φ is a part of a plane. If they intersect then Φ is a part of a sphere (or a whole sphere). In these cases the determination of circles (or circle segments) is trivial.

b. g and r are skew and R is located in a meridian plane: In this case Φ is a torus. It is well known that each torus has two one-parametric sets of circles, namely one generated by R and the other set is given by the meridian circles of the torus. Only in the case of a ring torus we get two further sets additionally, namely those generated by the well known *Villarceau circles*.

c. g and r are skew and R is not located in a meridian plane: If we reflect the generating circle R on a meridian plane we get the circle $\overline{R}$. Therefore Φ has at least three one-parametric sets of circles, namely the meridian set and the sets generated by R and $\overline{R}$, respectively. Only in the case where R and r have no point in common, there exist two further one-parametric sets of circles which can be constructed similarly to the *Villarceau circles* by intersecting Φ with a double tangent plane (which is not orthogonal to r).

(4) TSSM manipulators with 2 parallel axes: Similar considerations as in item (1) yield that the non-trivial self-motions of TSSM manipulators with two parallel rotary axes correspond with flexible octahedra where one vertex is an ideal point.

Therefore the determination of all flexible octahedra with one vertex in the plane at infinity is the goal of this article. As a consequence, this work can be regarded as the continuation of [15], where a conjecture about the solution of this problem was formulated and which is proved within this article. In [15] we conjectured that the only flexible octahedra with one vertex in the plane at infinity are Bricard octahedra of type 2 and type 3 (with one vertex in the plane at infinity).

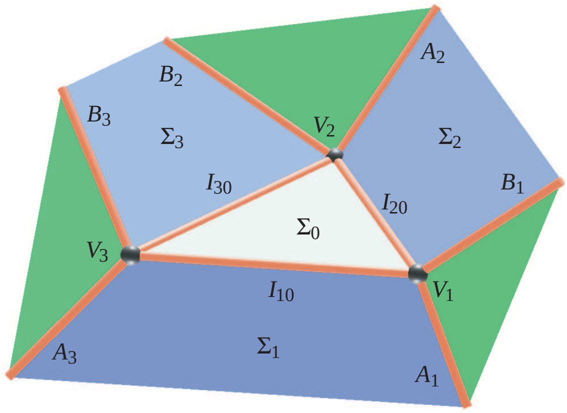

FIG. 2: A *Kokotsakis mesh* is a polyhedral structure consisting of a n-sided central polygon $\Sigma_0 \in E^3$ surrounded by a belt of polygons in the following way: Each side I_{i0} of Σ_0 is shared by an adjacent polygon Σ_i, and the relative motion between cyclically consecutive neighbor polygons is a spherical coupler motion. Here the *Kokotsakis mesh* for $n = 3$ is given, which determines an octahedron.

It is not obvious that these flexible octahedra are the only ones where one vertex is an ideal point, as there could even exist flexible octahedra which do not have flexible counterparts with finite vertices. For example, it was shown by the author in [16] that there exist two types of flexible octahedra with two opposite vertices being ideal points, which do not possess flexible counterparts with six finite vertices. Therefore this article can also be seen as the first part of a classification of all flexible octahedra in the projective extension of the Euclidean 3-space E^3 which is completed in [16].

Moreover, it should be mentioned that we denote the projective extension of E^3 by $E^\star$ in the remaining paper.

A. Related work and overview

In 1897 all types of flexible octahedra in E^3 were firstly classified by BRICARD [17]. In 1978 CONNELLY [18] sketched a further algebraic method for the determination of all flexible octahedra in E^3.

STACHEL [9] presented a new proof which uses mainly arguments from projective geometry beside the converse of *Ivory's Theorem*, which limits this approach also to flexible octahedra in E^3. But STACHEL [8] also gave the construction of flexible octahedra of type 3 with one vertex at infinity.

KOKOTSAKIS [19] discussed the flexible octahedra as special cases of a sort of meshes named after him (see FIG. 2). His very short and elegant proof for *Bricard octahedra* is also valid for type 3 in $E^\star$ if no opposite vertices are ideal points. Up to recent, to the author's best knowledge this fact was not recognized before, not even by KOKOTSAKIS [19].

Moreover, there are no proofs for Bricard's famous statement known to the author which enclose the projective extension of E^3 although these flexible structures attracted many prominent mathematicians; e.g. BENNETT [20], BLASCHKE [21], BOTTEMA [22], LEBESGUE [23] and WUNDERLICH [24].

We tackle this problem by considering an octahedron as a *Kokotsakis mesh* (see FIG. 2) with a 3-sided central polygon Σ_0. In the next step we investigate the spherical image of such *Kokotsakis meshes*; i.e. we translate each face of the mesh through the point M and intersect this translated face with the sphere S^2 (with radius 1) centered in M. Then the relative motion Σ_i/Σ_{i+1} (mod 3) appears as a spherical coupler motion (cf. section II). Based on reducible compositions of spherical coupler motions with spherical coupler components (cf. [25]) which meet the so-called closure condition (cf. section III) the set of flexible octahedra with one vertex in the plane at infinity can be classified into three types (type A, B, C). This classification (cf. Lemma 1) as well as the discussion of type A and type C is given in section IV. After some additional preparatory work done in section V, we solve the missing type B in section VI.

II. NOTATION AND RELATED RESULTS

As already mentioned, our approach is based on a kinematic analysis of *Kokotsakis meshes* (see FIG. 2) as composition of spherical coupler motions given by STACHEL [26], which is repeated in this section.

A. Transmission by a spherical four-bar mechanism

We start with the analysis of the first spherical four-bar linkage $\mathscr{C}$ (cf. [27, 28]) with the frame link $I_{10}I_{20}$ and the coupler A_1B_1 according to STACHEL [26] (see FIG. 2 and FIG. 3).

We set $\alpha_1 := \overline{I_{10}A_1}$ for the spherical length of the driving arm, $\beta_1 := \overline{I_{20}B_1}$ for the output arm, $\gamma_1 := \overline{A_1B_1}$ and $\delta_1 := \overline{I_{10}I_{20}}$. We may suppose $0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < \pi$.

The coupler motion remains unchanged when A_1 is replaced by its antipode $\overline{A}_1$ and at the same time α_1 and γ_1 are substituted by $\pi - \alpha_1$ and $\pi - \gamma_1$, respectively. The same holds for the other vertices. When I_{10} is replaced by its antipode $\overline{I}_{10}$, then also the sense of orientation changes, when the rotation of the driving bar $I_{10}A_1$ is inspected from outside of S^2 either at I_{10} or at $\overline{I}_{10}$.

We use a cartesian coordinate frame with I_{10} on the positive x-axis and $I_{10}I_{20}$ in the xy-plane such that I_{20} has a positive y-coordinate (see FIG. 3). The input angle τ_1 is measured between $I_{10}I_{20}$ and the driving arm $I_{10}A_1$ in counter-clock orientation. The output angle $\tau_2 = \sphericalangle \overline{I}_{10}I_{20}B_1$ is the oriented exterior angle at vertex I_{20}. As given in [26] the constant spherical length γ_1 of the coupler implies the following equation

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0 \tag{1}$$

with $t_i = \tan(\tau_i/2)$, $c_{11} = 4\,\mathrm{s}\alpha_1\,\mathrm{s}\beta_1 \neq 0$,

$$\begin{aligned} c_{00} &= N_1 - K_1 + L_1 + M_1, \quad & c_{02} &= N_1 + K_1 + L_1 - M_1, \\ c_{20} &= N_1 - K_1 - L_1 - M_1, \quad & c_{22} &= N_1 + K_1 - L_1 + M_1, \end{aligned} \tag{2}$$

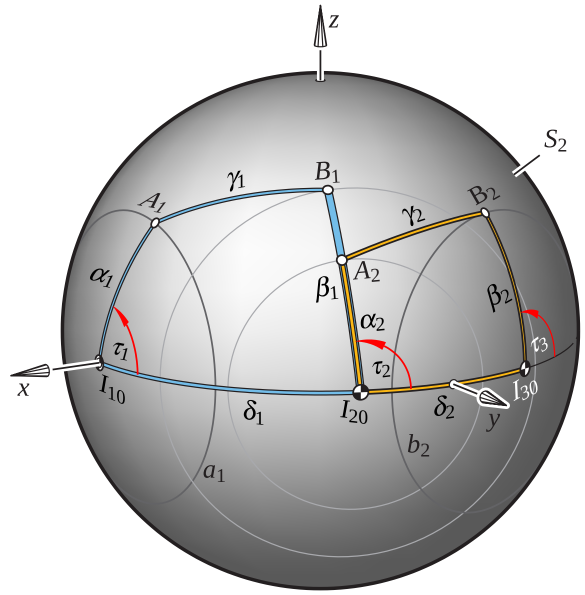

FIG. 3: Composition of the two spherical four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ with spherical side lengths $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1, 2$ (Courtesy of H. Stachel).

and

$$\begin{aligned} K_1 &= \mathrm{c}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{s}\delta_1\,, \quad & M_1 &= \mathrm{s}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{c}\delta_1\,, \\ L_1 &= \mathrm{s}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{s}\delta_1\,, \quad & N_1 &= \mathrm{c}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{c}\delta_1 - \mathrm{c}\gamma_1\,. \end{aligned} \tag{3}$$

In this equation s and c are abbreviations for the sine and cosine function, respectively, and the spherical lengths α_1, β_1 and δ_1 are signed.

Note that the biquadratic equation Eq. (1) describes a *2-2-correspondence* between points A_1 on the circle $a_1 = (I_{10}; \alpha_1)$ and B_1 on $b_1 = (I_{20}; \beta_1)$ (see FIG. 3). Moreover, it should be said that this 2-2-correspondence only depends on the ratio of the coefficients $c_{22} : \cdots : c_{00}$ (cf. Lemma 1 of NAWRATIL AND STACHEL [29]).

In this section we only summed up the information which is necessary to understand the remaining part of the paper. A more detailed explanation of this basic considerations can be found in [26].

B. Composition of two spherical four-bar linkages

Now we use the output angle τ_2 of the first four-bar linkage $\mathscr{C}$ as input angle of a second four-bar linkage $\mathscr{D}$ with vertices $I_{20}A_2B_2I_{30}$ and consecutive spherical side lengths α_2, γ_2, β_2 and δ_2 (FIG. 3). The two frame links are assumed in aligned position. In the case $\sphericalangle I_{10}I_{20}I_{30} = \pi$ the spherical length δ_2 is positive, otherwise negative. Analogously, a negative α_2 expresses the fact that the aligned bars $I_{20}B_1$ and $I_{20}A_2$ are pointing to opposite sides. Changing the sign of β_2 means replacing the output angle τ_3 by $\tau_3 - \pi$. The sign of γ_2 has no influence on the transmission and therefore we can assume without loss of generality (w.l.o.g.) that $\gamma_2 > 0$ holds.

Due to (1) the transmission between the angles τ_1, τ_2 and the output angle τ_3 of the second four-bar with $t_3 := \tan(\tau_3/2)$ can be expressed by the two biquadratic equations

$$\begin{aligned} c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0\,, \\ d_{22}t_2^2t_3^2 + d_{20}t_2^2 + d_{02}t_3^2 + d_{11}t_2t_3 + d_{00} = 0\,. \end{aligned} \tag{4}$$

The d_{ik} are defined by equations analogue to Eqs. (2) and (3).

After this preparatory work we can go in medias res: Clearly, if a *Kokotsakis mesh* for $n = 3$ ($\Leftrightarrow$ octahedron) is flexible then also its spherical image has to be flexible. A necessary condition for that is that the motion transmission from Σ_1 to Σ_3 via Σ_2 ($\Leftrightarrow$ composition of two spherical four-bar linkages) can also be produced by a single spherical four-bar mechanism ($\Leftrightarrow$ the motion transmission from Σ_1 to Σ_3 via V_3, see FIG. 2). These are so-called reducible compositions of spherical four-bar linkages with a spherical coupler component, which were already determined by the author in [25] and can be summarized as follows (cf. Corollary 1 of [25]):

Theorem 1. *If a reducible composition of two spherical four-bar linkages with a spherical coupler component is given, then it is one of the following cases or a special case of them, respectively:*

(a) One spherical coupler is a spherical isogram which happens in one of the following four cases:

$$c_{00} = c_{22} = 0,\; d_{00} = d_{22} = 0,\; c_{20} = c_{02} = 0,\; d_{20} = d_{02} = 0,$$

(b) The spherical couplers form a spherical focal mechanism, which is analytically given for $F \in \mathbb{R}\setminus\{0\}$ by:

$$\begin{aligned} c_{00}c_{20} = Fd_{00}d_{02}, \quad c_{22}c_{02} = Fd_{22}d_{20}, \\ c_{11}^2 - 4(c_{00}c_{22} + c_{20}c_{02}) = F[d_{11}^2 - 4(d_{00}d_{22} + d_{20}d_{02})], \end{aligned} \tag{5}$$

(c) One of the following two cases hold:

$$c_{22} = c_{02} = d_{00} = d_{02} = 0, \quad d_{22} = d_{20} = c_{00} = c_{20} = 0,$$

(d) One of the following two cases hold for $A \in \mathbb{R}\setminus\{0\}$ and $B \in \mathbb{R}$:

- $c_{20} = Ad_{02},\; c_{22} = Ad_{22},\; c_{02} = Bd_{22},\; c_{00} = Bd_{02},$ $d_{00} = d_{20} = 0,\; d_{02}d_{22} \neq 0,$
- $d_{02} = Ac_{20},\; d_{22} = Ac_{22},\; d_{20} = Bc_{22},\; d_{00} = Bc_{20},$ $c_{00} = c_{02} = 0,\; c_{20}c_{22} \neq 0.$

C. Geometric aspects of Theorem 1

In the following we point out the already known geometric aspects of Theorem 1. For a more detailed geometric explanation of these cases please see the corresponding publications [19, 25, 26, 29].

Spherical isogram: The geometric difference between the two spherical isograms given by $c_{00} = c_{22} = 0$ and $c_{20} = c_{02} = 0$, respectively, is as follows:

(i) It was already shown in [26] that $c_{00} = c_{22} = 0$ is equivalent with the conditions $\beta_1 = \alpha_1$ and $\delta_1 = \gamma_1$ which determine a spherical isogram.

(ii) In contrary, $c_{20} = c_{02} = 0$ is equivalent with the conditions $\beta_1 = \pi - \alpha_1$ and $\delta_1 = \pi - \gamma_1$ (cf. [25]). Note that the couplers of both isograms have the same movement because we get item (ii) by replacing either I_{10} or I_{20} of item (i) by its antipode.

Moreover, it should be noted that the cosines of opposite angles in the spherical isograms (of both types) are equal (cf. §8 of [19]).

Spherical focal mechanism: Here also two cases can be distinguished:

(i) In [29] it was shown that the characterization of the spherical focal mechanism given in Theorem 1 is equivalent with the condition:

$$\begin{aligned} \mathrm{s}\alpha_1\,\mathrm{s}\gamma_1 : \mathrm{s}\beta_1\,\mathrm{s}\delta_1 : (\mathrm{c}\alpha_1\,\mathrm{c}\gamma_1 - \mathrm{c}\beta_1\,\mathrm{c}\delta_1) = \\ \mathrm{s}\beta_2\,\mathrm{s}\gamma_2 : \mathrm{s}\alpha_2\,\mathrm{s}\delta_2 : (\mathrm{c}\alpha_2\,\mathrm{c}\delta_2 - \mathrm{c}\beta_2\,\mathrm{c}\gamma_2). \end{aligned} \tag{6}$$

Moreover, it should be noted that in this case $\mathrm{c}\chi_1 = -\mathrm{c}\psi_2$ holds with $\chi_1 = \sphericalangle I_{10}A_1B_1$ and $\psi_2 = \sphericalangle I_{30}B_2A_2$.

(ii) But in the algebraic characterization of the spherical focal mechanism (5) also a second possibility is hidden, namely:

$$\begin{aligned} \mathrm{s}\alpha_1\,\mathrm{s}\gamma_1 : \mathrm{s}\beta_1\,\mathrm{s}\delta_1 : (\mathrm{c}\alpha_1\,\mathrm{c}\gamma_1 - \mathrm{c}\beta_1\,\mathrm{c}\delta_1) = \\ \mathrm{s}\beta_2\,\mathrm{s}\gamma_2 : \mathrm{s}\alpha_2\,\mathrm{s}\delta_2 : (\mathrm{c}\beta_2\,\mathrm{c}\gamma_2 - \mathrm{c}\alpha_2\,\mathrm{c}\delta_2). \end{aligned} \tag{7}$$

In this case $\mathrm{c}\chi_1 = \mathrm{c}\psi_2$ holds. Note that we get this case from the first one by replacing either I_{30} or I_{10} by its antipode. Moreover, it should be mentioned that a replacement of I_{20} by its antipode would not have any effect on the given relation (6) of the first case.

<u>Remark 1.</u> It should be noted that in item (c) and item (d) of Theorem 1 both couplers are so-called orthogonal spherical four-bar mechanisms, as the diagonals of the spherical quadrangles are orthogonal (cf. [26]). ◇

But not all cases where the relation between the input angle τ_1 of the arm $I_{10}A_1$ and the output angle τ_3 of $I_{30}B_2$ can be produced by a single spherical four-bar linkage $\mathscr{R}$ (= spherical quadrangle $I_{10}I_{r0}B_3A_3$), yield a flexible octahedron, as the resulting configuration of the three spherical four-bar linkages $\mathscr{C}, \mathscr{D}, \mathscr{R}$ has not to be closed; i.e. $I_{r0} \neq I_{30}$.

Therefore the *Kokotsakis mesh* for $n = 3$ is flexible if and only if the transmission of the composition of the two spherical four-bar linkages $\mathscr{C}$ and $\mathscr{D}$ equals the one of the single spherical four-bar linkage $\mathscr{R}$ which meets the so-called closure condition $I_{r0} = I_{30}$.

If we denote the edge lengths of the corresponding spherical quadrangle $I_{10}I_{r0}B_3A_3$ of $\mathscr{R}$ by $\alpha_3 := \overline{I_{10}A_3}$, $\beta_3 := \overline{I_{r0}B_3}$, $\gamma_3 := \overline{A_3B_3}$ and $\delta_3 := \overline{I_{10}I_{r0}}$, we can rewrite the closure condition in terms of oriented enclosed angles δ_i of the central triangle according to FIG. 4 as $\delta_3 = \delta_1 + \delta_2$. Therefore all

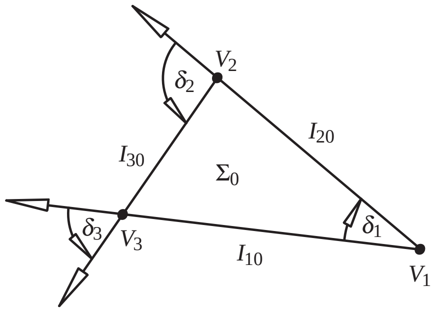

FIG. 4: Oriented sides of the central triangle Σ_0 with the oriented enclosed angles δ_i.

reducible compositions with a spherical coupler component given in Theorem 1 fulfilling $\delta_3 = \delta_1 + \delta_2$ imply a flexible octahedron. This result is the base for the understanding of the next section.

III. PREPARATORY WORK

In order that we can also use the results produced in this section for [16], where the determination of all flexible octahedra in $E^\star$ is completed, we generalize our approach in the way that we study a superset of the set of all flexible octahedra with one vertex on the plane at infinity. This superset contains all flexible octahedra where no pair of opposite vertices are ideal points. Such octahedra possess at least one face where all three vertices are in E^3. We can assume w.l.o.g. that this face corresponds to Σ_0 in FIG. 2.

In the following two subsections we show that the reducible composition of item (c) and item (d) of Theorem 1 cannot fulfill the closure condition $\delta_3 = \delta_1 + \delta_2$ without any contradiction:

ad Theorem 1 (c)

Here we only discuss the case $c_{22} = c_{02} = d_{00} = d_{02} = 0$. For the other case of item (c) we refer to analogy.

The case under consideration stem from the excluded cases of the symmetric reducible composition (cf. section 4 of [25]). In this case the transmission of the spherical four-bar linkage $\mathscr{R}$ is given by

$$r'_{22}t_1^2t_3^2 + r'_{20}t_1^2 + r'_{02}t_3^2 + r'_{11}t_1t_3 + r'_{00} = 0 \tag{8}$$

with

$$\begin{gathered} r'_{00} = -c_{00}d_{20}, \quad r'_{20} = -c_{20}d_{20}, \\ r'_{02} = -c_{00}d_{22}, \quad r'_{22} = -c_{20}d_{22}, \quad r'_{11} = c_{11}d_{11}. \end{gathered} \tag{9}$$

If we substitute (3) into (2) the conditions $c_{22} = c_{02} = 0$ imply:

$$\mathrm{c}\delta_1 := \mathrm{s}\delta_1\mathrm{c}\beta_1/\mathrm{s}\beta_1, \quad \mathrm{c}\gamma_1 := \mathrm{s}\delta_1\mathrm{c}\alpha_1/\mathrm{s}\beta_1. \tag{10}$$

Analogous considerations for $d_{00} = d_{02} = 0$ yield:

$$\mathrm{c}\delta_2 := \mathrm{s}\delta_2\mathrm{c}\alpha_2/\mathrm{s}\alpha_2, \quad \mathrm{c}\gamma_2 := \mathrm{s}\delta_2\mathrm{c}\beta_2/\mathrm{s}\alpha_2. \tag{11}$$

Moreover, we also express the remaining c_{ij} and d_{ij} in dependency of the angles $\alpha_i, \beta_i, \gamma_i, \delta_i$ with $i = 1, 2$ and substitute the obtained expressions into Eq. (8). As Eq. (8) gives the transmission of a spherical four-bar mechanism the coefficients r'_{ij} of this equation must be proportional to $r_{11} = 4\mathrm{s}\alpha_3\mathrm{s}\beta_3 \neq 0$ and

$$\begin{aligned} r_{00} &= N_3 - K_3 + L_3 + M_3, \quad & r_{02} &= N_3 + K_3 + L_3 - M_3, \\ r_{20} &= N_3 - K_3 - L_3 - M_3, \quad & r_{22} &= N_3 + K_3 - L_3 + M_3, \end{aligned} \tag{12}$$

with K_3, L_3, M_3, N_3 according to Eq. (3) for $\alpha_3, \beta_3, \gamma_3, \delta_3$.

A flipped over spherical deltoid (cf. Remark 2) is the only spherical four-bar mechanism where an input angle or an output angle remains constant (0° or 180°) under the motion. We are not interested in such assembly modes of spherical deltoids, as they correspond to flexible octahedra where at least two pairs of neighboring faces coincide ($\Rightarrow$ trivial self-motion, cf. section I). Therefore we can assume that the t_i's are not constant and hence the comparison of coefficients imply 5 equations eq_{ij}, which are defined as the numerator of $r_{ij} - \lambda r'_{ij}$ with $\lambda \in \mathbb{R} \setminus \{0\}$. Beside these 5 equations also $\delta_3 = \delta_1 + \delta_2$ has to hold. Therefore the addition theorems for the sine and cosine function imply:

$$\mathrm{s}\delta_3 := \mathrm{s}\delta_1\mathrm{c}\delta_2 + \mathrm{c}\delta_1\mathrm{s}\delta_2, \quad \mathrm{c}\delta_3 := \mathrm{c}\delta_1\mathrm{c}\delta_2 - \mathrm{s}\delta_1\mathrm{s}\delta_2. \tag{13}$$

After expressing λ from $eq_{11} = 0$ we compute the following linear combination

$$eq_{22} - eq_{20} - eq_{02} + eq_{00} = -4\mathrm{s}\alpha_1\mathrm{s}\alpha_2\mathrm{s}\alpha_3\mathrm{s}\beta_1\mathrm{s}\beta_2\mathrm{s}\beta_3\mathrm{s}\delta_1\mathrm{s}\delta_2$$

which cannot vanish without contradiction (w.c.).

ad Theorem 1 (d)

Here we only discuss the case $c_{20} = Ad_{02}$, $c_{22} = Ad_{22}$, $c_{02} = Bd_{22}$, $c_{00} = Bd_{02}$, $d_{00} = d_{20} = 0$, $d_{02}d_{22} \neq 0$. For the second possibility we refer again to analogy.

This case is the only special asymmetric case (cf. Theorem 6 of [25]). The transmission of the resulting spherical four-bar linkage $\mathscr{R}$ is given by

$$r'_{20}t_1^2 + r'_{11}t_1t_3 + r'_{00} = 0 \tag{14}$$

with

$$r'_{00} = -d_{11}B, \quad r'_{20} = -d_{11}A, \quad r'_{11} = c_{11}. \tag{15}$$

In this case the conditions $d_{00} = d_{20} = 0$ imply:

$$\mathrm{c}\delta_2 := -\mathrm{s}\delta_2\mathrm{c}\beta_2/\mathrm{s}\beta_2, \quad \mathrm{c}\gamma_2 := -\mathrm{s}\delta_2\mathrm{c}\alpha_2/\mathrm{s}\beta_2. \tag{16}$$

Moreover, we also express d_{22}, d_{02} and d_{11} in dependency of the angles $\alpha_2, \beta_2, \gamma_2, \delta_2$ and substitute the obtained expressions into Eq. (14). As Eq. (14) gives the transmission of a spherical four-bar mechanism the coefficients r'_{ij} ($r'_{22} = r'_{02} = 0$) of this equation have to be proportional to the r_{ij} of Eq. (12). Analogous considerations as in the above discussed case (c) yield again 5 equations eq_{ij} which are defined as the numerator of $r_{ij}-\lambda r'_{ij}$ with $\lambda \in \mathbb{R}\setminus\{0\}$. Moreover, again $\delta_3=\delta_1+\delta_2$ must hold which yields Eq. (13). From eq_{11} we can express λ w.l.o.g.. Then we compute the following two linear combinations

$$eq_{22}+eq_{20}-eq_{02}-eq_{00}, \quad eq_{22}-eq_{20}+eq_{02}-eq_{00}, \tag{17}$$

which are linear in A and B. Moreover, these two equations can be solved for A and B w.l.o.g.. Then we are left with the equations $eq_{22}+eq_{20}=0$ and $eq_{22}-eq_{20}=0$, where we can solve $eq_{22}+eq_{20}=0$ for $\mathrm{c}\gamma_3$ w.l.o.g.. Now we distinguish the following cases:

1. $\mathrm{s}\delta_1\mathrm{c}\beta_2-\mathrm{c}\delta_1\mathrm{s}\beta_2\neq 0$: Under this assumption we can solve $eq_{22}-eq_{20}=0$ for $\mathrm{c}\beta_3$ w.l.o.g.. Now we only have to solve the four remaining equations

$$\begin{aligned} ex_{20} &:= c_{20}-Ad_{02}, & ex_{22} &:= c_{22}-Ad_{22}, \\ ex_{02} &:= c_{02}-Bd_{22}, & ex_{00} &:= c_{00}-Bd_{02}, \end{aligned} \tag{18}$$

with c_{ij} in dependency of the angles $\alpha_1,\beta_1,\gamma_1,\delta_1$ according to Eqs. (3) and (2). Then we compute the following four linear combinations:

$$\begin{aligned} \nu_1 &:= ex_{00}-ex_{20}, & \nu_2 &:= ex_{00}+ex_{20}, \\ \mu_1 &:= ex_{22}-ex_{02}, & \mu_2 &:= ex_{22}+ex_{02}. \end{aligned} \tag{19}$$

As $\mathrm{s}\alpha_2\mathrm{c}\beta_2+\mathrm{c}\alpha_2\mathrm{s}\beta_2=0$ implies $d_{02}=0$, a contradiction, we can assume $\mathrm{s}\alpha_2\mathrm{c}\beta_2+\mathrm{c}\alpha_2\mathrm{s}\beta_2\neq 0$. Therefore we can compute $\mathrm{c}\alpha_3$ from $\nu_2=0$ w.l.o.g..

   a. $\mathrm{c}\beta_2\neq 0$: Now we can express $\mathrm{c}\gamma_1$ from $\mu_2=0$ w.l.o.g.. Then we apply the half angle substitution

$$\mathrm{s}\delta_1 := 2d_1/(1+d_1^2), \quad \mathrm{c}\delta_1 := (1-d_1^2)/(1+d_1^2),$$

   and eliminate the parameter d_1 from the linear combinations $\nu_1+\mu_1$ and $\nu_1-\mu_1$ with the resultant method. The resulting expression can only vanish w.c. for

$$\mathrm{c}\beta_1\mathrm{s}\alpha_2\mathrm{s}\delta_2^2\mathrm{c}\beta_2^2-\mathrm{c}\beta_1\mathrm{s}\alpha_2\mathrm{s}\beta_2^2+\mathrm{s}\delta_2^2\mathrm{s}\beta_1\mathrm{s}\beta_2^2\mathrm{c}\alpha_2=0.$$

   Under consideration of $\mathrm{s}\delta_2^2+\mathrm{c}\delta_2^2=1$ this yields $\mathrm{s}\beta_1\mathrm{c}\alpha_2-\mathrm{s}\alpha_2\mathrm{c}\beta_1=0$. This condition implies $\alpha_2=\beta_1$ or $\alpha_2=\beta_1+\pi$. In both cases the back-substitution into $\nu_1+\mu_1=0$ and $\nu_1-\mu_1=0$, respectively, implies $\mathrm{s}\delta_1\mathrm{c}\beta_2-\mathrm{c}\delta_1\mathrm{s}\beta_2=0$, a contradiction.

   b. $\mathrm{c}\beta_2=0$: Now $\mu_2=0$ can only vanish w.c. for $\mathrm{c}\alpha_1=0$. Then $\nu_1+\mu_1=0$ cannot vanish w.c..

2. $\mathrm{s}\delta_1\mathrm{c}\beta_2-\mathrm{c}\delta_1\mathrm{s}\beta_2=0$: W.l.o.g. we can express $\mathrm{c}\beta_2$ from this condition. Then $eq_{22}-eq_{20}=0$ cannot vanish w.c..

ad Theorem 1 (b)

It was already remarked in [29] that the spherical focal mechanism (i) cannot imply a solution to our problem. The geometric reasoning for this is hidden in §9 of [19] and reads as follows: If one computes in general the relation between χ_1 and ψ_2 one will end up with Eq. (11) of [19]:

$$\begin{aligned} (\mathrm{c}\alpha_1\mathrm{c}\gamma_1-\mathrm{c}\delta_1\mathrm{c}\beta_1)\mathrm{s}\delta_2\mathrm{s}\alpha_2-(\mathrm{c}\beta_2\mathrm{c}\gamma_2-\mathrm{c}\delta_2\mathrm{c}\alpha_2)\mathrm{s}\delta_1\mathrm{s}\beta_1 \\ +\mathrm{s}\alpha_1\mathrm{s}\gamma_1\mathrm{s}\delta_2\mathrm{s}\alpha_2\mathrm{c}\chi_1-\mathrm{s}\delta_1\mathrm{s}\beta_1\mathrm{s}\gamma_2\mathrm{s}\beta_2\mathrm{c}\psi_2=0, \end{aligned} \tag{20}$$

and Eq. (13) of [19]:

$$\begin{aligned} (\mathrm{c}\alpha_1\mathrm{c}\gamma_1-\mathrm{c}\delta_1\mathrm{c}\beta_1)\mathrm{s}\delta_2\mathrm{s}\alpha_2+(\mathrm{c}\beta_2\mathrm{c}\gamma_2-\mathrm{c}\delta_2\mathrm{c}\alpha_2)\mathrm{s}\delta_1\mathrm{s}\beta_1 \\ +\mathrm{s}\alpha_1\mathrm{s}\gamma_1\mathrm{s}\delta_2\mathrm{s}\alpha_2\mathrm{c}\chi_1+\mathrm{s}\delta_1\mathrm{s}\beta_1\mathrm{s}\gamma_2\mathrm{s}\beta_2\mathrm{c}\psi_2=0, \end{aligned} \tag{21}$$

respectively, depending on the circumstance if the arms I_{10} and I_{30} lie on the same side or different sides, respectively, with respect to the line $[V_1,V_2]$ (see FIG. 2).

If we plug the relations of Eq. (6) under consideration of $\mathrm{c}\chi_1=-\mathrm{c}\psi_2$ into Eq. (20) we will see that χ_1 and ψ_2 must be constant. This already yields a contradiction as the corresponding octahedron is rigid. On the other hand, Eq. (21) is fulfilled identically for the relations of Eq. (6) and $\mathrm{c}\chi_1=-\mathrm{c}\psi_2$. But this also yields a contradiction for flexible octahedra as any two sides of the central triangle always lie on the same side with respect to the third remaining side.

For the spherical focal mechanism of type (ii) Eq. (20) is fulfilled identically and therefore it is still a possible solution of our problem.

ad Theorem 1 (a)

It should be noted that the composition of two spherical isograms of any type also forms a (special) spherical focal mechanism as Eq. (5) holds. Moreover, we know that this is a reducible composition with a spherical coupler component. As this spherical four-bar mechanism has to have two folded positions, it can only be a spherical isogram.

Remark 2. Note that beside the spherical isogram the following spherical four-bars also have two folded positions:

$$\begin{aligned} &\alpha_3=\gamma_3 \wedge \beta_3=\delta_3, \quad \alpha_3+\gamma_3=\pi \wedge \beta_3+\delta_3=\pi, \\ &\alpha_3=\delta_3 \wedge \beta_3=\gamma_3, \quad \alpha_3+\delta_3=\pi \wedge \beta_3+\gamma_3=\pi. \end{aligned} \tag{22}$$

It can easily be seen, that these mechanisms which are summarized under the notation of *spherical deltoids*, do not fit with both folded positions of the spherical focal mechanism composed of two spherical isograms.

Moreover, it should be noted that the cosines of one pair of opposite angles in the spherical deltoid are equal. ⋄

IV. CLASSIFICATION OF FLEXIBLE OCTAHEDRA

Now we emphasis on the combinatorial aspect of this problem because due to the last four subsections a reducible composition with a spherical coupler component can only be of *isogram* type or *focal* type (ii). Together with Theorem 1 this yields the following lemma:

Lemma 1. *If an octahedron in the projective extension of E^3 is flexible where no pair of opposite vertices are ideal points, then its spherical image is a composition of spherical four-bar linkages $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{R}$ of the following type:*

A. $\mathscr{C}$ and $\mathscr{D}$, $\mathscr{C}$ and $\mathscr{R}$ as well as $\mathscr{D}$ and $\mathscr{R}$ form a spherical focal mechanism of type (ii),

B. $\mathscr{C}$ and $\mathscr{D}$ form a spherical focal mechanism of type (ii) and $\mathscr{R}$ is a spherical isogram,

C. $\mathscr{C}$, $\mathscr{D}$ ($\Rightarrow$ and $\mathscr{R}$) are spherical isograms.

Remark 3. Note that type C can be seen as special case of type A and type B, respectively, namely if the involved spherical focal mechanisms of type (ii) are composed of two spherical isograms (cf. section III). From this point of view there only exist two classes of flexible octahedra whereby their common elements are the octahedra of type C. ⋄

The fact that exactly the three types of Lemma 1 yield those of BRICARD [17] (A yields type 1, B yields type 2 and C yields type 3) if all vertices are in E^3 was already proved by KOKOTSAKIS [19]. His proof that item A yields type 1 and item B yields type 2 was based on a theorem (*Satz über zwei Vierkante*) given in §12 of [19] which is limited to E^3. Moreover, he also argued with the help of edge lengths relations, which can not be done just like that if one vertex is an ideal point.

A. Flexible octahedra of type C with one vertex at infinity

In contrast to type A and type B, KOKOTSAKIS showed without any limiting argumentation with respect to $E^\star$ that item C of Lemma 1 corresponds with Bricard flexible octahedra of type 3. As already mentioned this fact was not recognized before, not even by KOKOTSAKIS.

Moreover, STACHEL [8] also proved the existence of type 3 octahedra with one vertex on the plane at infinity. In this article also the construction of these octahedra is given. Therefore the following theorem holds:

Theorem 2. *There exist flexible octahedra of type C with one vertex at infinity. These flexible octahedra are nothing else than Bricard's flexible octahedra of type 3 where one vertex is an ideal point.*

B. Flexible octahedra of type A with one vertex at infinity

W.l.o.g. we can assume that U is the ideal point and that its opposite vertex is denoted by V. Now the spherical coupler at any of the five vertices $\in E^3$ forms with the spherical coupler of every neighboring vertex $\in E^3$ a spherical focal mechanism of type (ii). This already implies that the cosines of opposite dihedral angles of the octahedron are equal.

Now we assume that there exists a configuration where only one of these angles equals zero or π. Then we get a 3-sided prism through U and a 3-sided pyramid with apex V. The sum of the dihedral angles of the 3-sided prism equals always π. In contrast, the sum of the dihedral angles of the 3-sided pyramid is always greater than π if $V \in E^3$. Therefore this already implies the contradiction.

Now we assume that there do not exist a configuration where only one of these dihedral angles equals zero or π. It can easily be seen that this can only be the case if the spherical image of the faces through V is a spherical isogram ($\Rightarrow$ orthogonal cross section of the prism through U is a parallelogram or antiparallelogram). As this already yields a type B octahedron (cf. section VI) the following theorem is proven:

Theorem 3. *In the projective extension of E^3 there do not exist flexible octahedra of type A where only one vertex is an ideal point.*

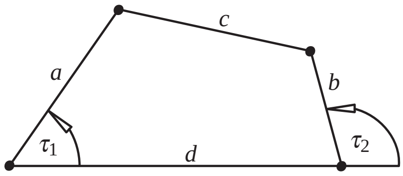

FIG. 5: Planar four-bar mechanism with driving arm a, follower b, coupler c and base d.

V. CENTRAL TRIANGLES WITH ONE IDEAL POINT

For the discussion of flexible octahedra of type B we need some additional considerations, which are prepared in this section.

Given is an octahedron where one vertex is an ideal point and the remaining five vertices are in E^3. Now we consider one of the four faces through the ideal point as central triangle Σ_0. Moreover, these four faces built a 4-sided prism where the motion transmission between opposite faces equals the one of the corresponding planar four-bar mechanism (orthogonal cross section of the prism). It can easily be seen that the input angle τ_1 and the output angle τ_2 of a planar four-bar linkage (see FIG. 5) are related by:

$$p_{22}t_1^2t_2^2 + p_{20}t_1^2 + p_{02}t_2^2 + p_{11}t_1t_2 + p_{00} = 0 \tag{23}$$

with $t_i := \tan(\tau_i/2)$, $p_{11} = -8ab$ and

$$\begin{aligned} p_{22} &= (a-b+c+d)(a-b-c+d), \\ p_{20} &= (a+b+c+d)(a+b-c+d), \\ p_{02} &= (a+b+c-d)(a+b-c-d), \\ p_{00} &= (a-b+c-d)(a-b-c-d). \end{aligned} \tag{24}$$

W.l.o.g. we can assume $a, b, c, d > 0$ which implies $p_{11} \neq 0$. Therefore using the abbreviations p_{ij} the formula of the 2-2-correspondence of a planar four-bar mechanism equals that of a spherical one (cf. Eq. (1)). Moreover, one only has to check the technical detail that the factor W_6 of [25] with

$$W_6 = p_{11}^4 - 8p_{11}^2(p_{00}p_{22} + p_{20}p_{02}) + 16(p_{00}p_{22} - p_{20}p_{02})^2$$

is also different from zero for the planar four-bar mechanism as for the spherical one. As we get $W_6 = 2^{12}a^2b^2c^2d^2 \neq 0$, this already proves the following lemma (see also Corollary 1 of NAWRATIL [25]):

Lemma 2. *If a reducible composition of one planar and one spherical four-bar linkage with a spherical coupler component is given (cf. Eq. (4)), then it is one of the following cases or a special case of them, respectively:*

(a) One of the following four cases hold:

$$c_{00} = c_{22} = 0,\ d_{00} = d_{22} = 0,\ c_{20} = c_{02} = 0,\ d_{20} = d_{02} = 0,$$

(b) The following algebraic conditions hold for $F \in \mathbb{R}\setminus\{0\}$:

$$c_{00}c_{20} = Fd_{00}d_{02}, \quad c_{22}c_{02} = Fd_{22}d_{20},$$
$$c_{11}^2 - 4(c_{00}c_{22} + c_{20}c_{02}) = F[d_{11}^2 - 4(d_{00}d_{22} + d_{20}d_{02})],$$

(c) One of the following two cases hold:

$$c_{22} = c_{02} = d_{00} = d_{02} = 0, \quad d_{22} = d_{20} = c_{00} = c_{20} = 0,$$

(d) One of the following two cases hold for $A \in \mathbb{R}\setminus\{0\}$ and $B \in \mathbb{R}$:

- $c_{20} = Ad_{02},\ c_{22} = Ad_{22},\ c_{02} = Bd_{22},\ c_{00} = Bd_{02},$ $d_{00} = d_{20} = 0,\ d_{02}d_{22} \neq 0,$
- $d_{02} = Ac_{20},\ d_{22} = Ac_{22},\ d_{20} = Bc_{22},\ d_{00} = Bc_{20},$ $c_{00} = c_{02} = 0,\ c_{20}c_{22} \neq 0.$

Remark 4. Note that the projection onto the sphere of the *Kokotsakis mesh* where the center triangle has one point at infinity can be done analogously to the general case (see last paragraph of section I). Therefore the spherical image of the prism is not a spherical four-bar mechanism but consists of four great circles which intersect each other in one pair of antipodal points. Moreover, the relative motion of the two great circles which correspond with the input and output face of the prism, respectively, is determined by Eq. (23). ◇

For the following closer study of the items (a)–(d) of Lemma 2 we can assume w.l.o.g. that V_1 denotes the ideal point. Moreover, for the remainder of this article we use the systematic notation of dihedral angles according to FIG. 6.

ad Lemma 2 (c)

The case $d_{22} = d_{20} = c_{00} = c_{20} = 0$ does not yield a solution because $c_{00} = c_{20} = 0$ cannot be fulfilled for $a, b, c, d > 0$.

The other case $d_{00} = d_{02} = c_{22} = c_{02} = 0$ can be done analogously to the corresponding case of section III if one substitutes the c_{ij}'s by the p_{ij}'s of Eq. (24). The conditions $c_{22} = c_{02} = 0$ imply $a = c$ and $b = d$. Therefore the corresponding planar four-bar mechanism is a deltoidal linkage with $c\varphi_1 = c\psi_1$ (cf. FIG. 6).

Then we compute the five equations eq_{ij} (cf. section III) where $eq_{11} = 0$ can be solved for λ w.l.o.g.. Then the linear

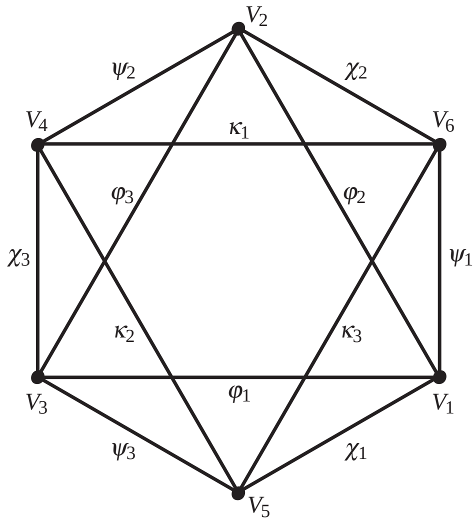

FIG. 6: Schematic sketch of the octahedron $V_1,\dots,V_6$. The dihedral angles are denoted by $\varphi_i, \psi_i, \chi_i, \kappa_i$ with $i = 1, 2, 3$.

combination $eq_{22} - eq_{20} - eq_{02} + eq_{00}$ can only vanish w.c. for $c\alpha_2 = c\delta_3 s\alpha_2/s\delta_2$. Now $eq_{22} - eq_{20} + eq_{02} - eq_{00} = 0$ implies $c\delta_3 = c\alpha_3 s\delta_3 c/(s\alpha_3 d)$. Moreover, from $eq_{22} + eq_{20} - eq_{02} - eq_{00} = 0$ we get $c\beta_2 = s\delta_3 c\beta_3 s\beta_2/(s\delta_2 s\beta_3)$.

Then we compute the difference of the two necessary conditions $c\delta_i^2 + s\delta_i^2 - 1 = 0$ with $i = 2, 3$. The resulting expression can only vanish w.c. for $s\delta_2 = \pm s\delta_3$.

1. $\delta_2 = \delta_3$: In this case $c\beta_i^2 + s\beta_i^2 - 1 = 0$ with $i = 2, 3$ can only vanish for $\beta_3 = \beta_2$ or $\beta_3 = \beta_2 + \pi$. Both conditions imply that the opposite vertex of V_1 is also an ideal point. As this contradicts our assumptions we are done.

2. $\delta_2 = -\delta_3$: Now the two arms can only be parallel if δ_2 and δ_3 are right angles. This implies that the angles α_2 and α_3 also have to be orthogonal. At the very end of section VI we show that this case also implies a contradiction.

Remark 5. As preparatory work for [16] we continue the discussion of case 1 under consideration of $\beta_3 = \beta_2$ and $\beta_3 = \beta_2 + \pi$, respectively. Then $s\alpha_2^2 + c\alpha_2^2 - 1 = 0$ and $s\delta_2^2 + c\delta_2^2 - 1 = 0$ can only vanish for $s\alpha_2 = \pm s\delta_2$. Moreover, $s\beta_2^2 + c\beta_2^2 - 1 = 0$ and $s\gamma_2^2 + c\gamma_2^2 - 1 = 0$ imply $s\beta_2 = \pm s\gamma_2$. Therefore the spherical four-bar linkage $\mathscr{D}$ is a spherical deltoid (cf. Remark 2) with $c\chi_2 = c\varphi_3$ (cf. FIG. 6). ◇

ad Lemma 2 (d)

- $c_{20} = Ad_{02}$, $c_{22} = Ad_{22}$, $c_{02} = Bd_{22}$, $c_{00} = Bd_{02}$, $d_{00} = d_{20} = 0$, $d_{02}d_{22} \neq 0$: This case can again be done analogously to the corresponding case of section III if one substitutes the c_{ij}'s by the p_{ij}'s of Eq. (24).

  We compute the five equations eq_{ij} (cf. section III) where $eq_{11} = 0$ can be solved for λ w.l.o.g.. Now we compute the two linear combinations given in Eq. (17) which can again be solved for A and B w.l.o.g.. We are left with the equations $eq_{22} + eq_{20} = 0$ and $eq_{22} - eq_{20} = 0$ which can only vanish w.c. for $c\beta_3 = c\delta_3 s\beta_3/s\delta_3$ and $c\gamma_3 = s\beta_3 c\alpha_3/s\delta_3$.

Now the four equations given in Eq. (17) remain. Again we compute the four linear combinations μ_i, ν_i ($i = 1,2$) of Eq. (19). Then $\nu_1 + \mu_1 = 0$ and $\nu_1 - \mu_1 = 0$ can only vanish w.c. for $s\beta_2 = -s\delta_2 c\delta_3 c\beta_2$ and $s\alpha_2 = -c\alpha_2 c\delta_3 s\delta_2 b/d$. Now it can easily be seen that the sum of the two necessary conditions $c\delta_i^2 + s\delta_i^2 - 1 = 0$ for $i = 2,3$ cannot vanish w.c..

- $d_{02} = Ac_{20}$, $d_{22} = Ac_{22}$, $d_{20} = Bc_{22}$, $d_{00} = Bc_{20}$, $c_{00} = c_{02} = 0$, $c_{20}c_{22} \neq 0$: The conditions $c_{00} = c_{02} = 0$ imply $c = b$ and $a = d$. Therefore the corresponding planar four-bar mechanism is a deltoidal linkage, which already yields $c\varphi_2 = c\chi_1$ (cf. FIG. 6). Moreover, the remaining c_{ij}'s are set to the values of the corresponding p_{ij}'s of Eq. (24).

  In this case (cf. [25]) the transmission of the resulting spherical four-bar linkage $\mathscr{R}$ is given by

$$r'_{02}t_3^2 + r'_{11}t_1t_3 + r'_{00} = 0 \tag{25}$$

  with

$$r'_{00} = -c_{11}B, \quad r'_{02} = -c_{11}A, \quad r'_{11} = d_{11}. \tag{26}$$

  As Eq. (25) gives the transmission of a spherical four-bar mechanism the coefficients r'_{ij} ($r'_{22} = r'_{20} = 0$) of this equation have to be proportional to the r_{ij} of Eq. (12). Therefore a comparison of coefficients imply the 5 equations eq_{ij} which are defined as the numerator of $r_{ij} - \lambda r'_{ij}$ with $\lambda \in \mathbb{R} \setminus \{0\}$.

  From $eq_{11} = 0$ we can express λ w.l.o.g.. Now we compute the two linear combinations given in Eq. (17) which can again be solved for A and B w.l.o.g.. Then $eq_{22} - eq_{20} = 0$ can only vanish w.c. for $c\alpha_3 = -c\delta_3 s\alpha_3/s\delta_3$ and $eq_{22} + eq_{20} = 0$ implies $c\gamma_3 = -c\beta_3 s\alpha_3/s\delta_3$.

  Now we only have to solve the four remaining equations

$$\begin{aligned} ex_{20} &:= d_{20} - Bc_{22}, & ex_{22} &:= d_{22} - Ac_{22}, \\ ex_{02} &:= d_{02} - Ac_{20}, & ex_{00} &:= d_{00} - Bc_{20}, \end{aligned} \tag{27}$$

  with d_{ij} in dependency of the angles $\alpha_2, \beta_2, \gamma_2, \delta_2$ according to Eqs. (3) and (2). Then we compute the four linear combinations of Eq. (19). Now $\mu_1 + \nu_1 = 0$ implies $c\delta_3 = c\delta_2$ and $\mu_2 - \nu_2 = 0$ yields $c\delta_2 = -c\alpha_2 s\delta_2 b/(s\alpha_2 d)$. Finally, $\mu_1 - \nu_1 = 0$ can be solved w.l.o.g. for $c\beta_2 = c\beta_3 s\beta_2/s\beta_3$.

  Then we compute the difference of the two necessary conditions $c\delta_i^2 + s\delta_i^2 - 1 = 0$ with $i = 2,3$. The resulting expression can only vanish w.c. for $s\delta_2 = \pm s\delta_3$.

  1. $\delta_2 = \delta_3$: In this case $c\beta_i^2 + s\beta_i^2 - 1 = 0$ with $i = 2,3$ can only vanish if the opposite vertex of V_1 is also an ideal point. As this contradicts our assumptions we are done.
  2. $\delta_2 = -\delta_3$: Now the two arms can only be parallel if δ_2 and δ_3 are right angles. This implies that the angles α_2 and α_3 also have to be orthogonal. This is the same case as the one given in item 2 of the last subsection, if one interchanges the vertices V_2 and V_3.

<u>*Remark 6.*</u> Analogous considerations as in Remark 5 yield that the spherical four-bar linkage $\mathscr{R}$ has to be a spherical deltoid with $c\varphi_3 = c\psi_3$ (cf. FIG. 6), if the opposite vertex of V_1 is an ideal point as well. Therefore this case is the same as the one given in Remark 5, if one interchanges the vertices V_2 and V_3.

In the preprint version of this paper on which [16] is based, a computational error yields for this case the wrong deltoidal condition $c\varphi_1 = c\chi_3$ instead of $c\varphi_3 = c\psi_3$. As a consequence, we can skip the first part of the proof of Theorem 6 of [16], as this case has not to be discussed. ◇

ad Lemma 2 (a)

The conditions $c_{00} = c_{22} = 0$ imply $a = b$ and $c = d$, i.e. the planar four-bar mechanism is a parallelogram or an antiparallelogram. Note that the opposite angles in the parallelogram and in the antiparallelogram are equal.

The conditions $c_{20} = c_{02} = 0$ have no solution under the assumption $a, b, c, d > 0$.

ad Lemma 2 (b)

Analogous considerations as in [29] yield for this case that one of the following two relations has to hold:

$$(i) \quad \begin{aligned} &2ac : 2bd : (a^2 - b^2 + c^2 - d^2) = \\ &s\beta_2 s\gamma_2 : s\alpha_2 s\delta_2 : (c\alpha_2 c\delta_2 - c\beta_2 c\gamma_2), \end{aligned} \tag{28}$$

$$(ii) \quad \begin{aligned} &2ac : 2bd : (a^2 - b^2 + c^2 - d^2) = \\ &s\beta_2 s\gamma_2 : s\alpha_2 s\delta_2 : (c\beta_2 c\gamma_2 - c\alpha_2 c\delta_2). \end{aligned} \tag{29}$$

Moreover, the conditions of Eq. (28) imply again $c\chi_1 = -c\psi_2$ and from those of Eq. (29) we get $c\chi_1 = c\psi_2$. Again the arms I_{10} and I_{30} can lie on the same side or different sides with respect to the line $[V_1, V_2]$. Therefore only type (ii) yields a possible solution.

VI. FLEXIBLE OCTAHEDRA OF TYPE B WITH ONE VERTEX AT INFINITY

W.l.o.g. we can assume that the spherical image of the motion transmission from the face $[V_1, V_3, V_5]$ to $[V_2, V_3, V_4]$ via V_1 and V_2 corresponds to a spherical focal mechanism of type (ii). Consequently the motion transmission from $[V_1, V_3, V_5]$ to $[V_2, V_3, V_4]$ via V_3 corresponds to a spherical isogram. Therefore the following relations hold (cf. FIG. 6):

$$c\chi_1 = c\psi_2, \quad c\chi_3 = c\varphi_1, \quad c\varphi_3 = c\psi_3. \tag{30}$$

Due to the symmetry we can assume w.l.o.g. that V_4 is no ideal point. Now the spherical image of the faces through V_4 cannot be a spherical isogram because otherwise we would end up with a flexible octahedron of type 3. Therefore the spherical image of the motion transmission from the face $[V_3, V_4, V_5]$ to $[V_1, V_2, V_3]$ via V_4 and V_2 has to be a spherical focal mechanism of type (ii). This implies $c\varphi_2 = c\kappa_2$ (cf. FIG. 6).

Now we have to distinguish the following cases:

1. V_5 is an ideal point: There are the following subcases:

   a. The spherical image of the faces through V_6 is a spherical isogram. This implies the relation:

   $$c\chi_2 = c\kappa_3, \quad c\kappa_1 = c\psi_1. \tag{31}$$

   As a consequence the cosines of angles along the four edges through V_2 and V_5 are pairwise the same. Then analogous considerations as in subsection IV B yield the contradiction.

   b. The spherical coupler at the vertex V_6 forms with every spherical coupler of the neighboring vertices V_1, V_2, V_4 a spherical focal mechanism of type (ii). Moreover, we can assume w.l.o.g. that the spherical image of the faces through V_6 is no spherical isogram. This implies:

   $$c\varphi_3 = c\kappa_3, \quad c\varphi_1 = c\kappa_1 = c\psi_1, \tag{32}$$

   with κ_1 and κ_3 according to FIG. 6. Due to section V the motion transmission from $[V_3, V_4, V_5]$ to $[V_2, V_4, V_6]$ via V_5 and V_6 is reducible in one of the following two cases:

      i. As the spherical image of the faces through V_6 is no spherical isogram, item (a) can only imply the case $c\kappa_2 = c\chi_1$. Therefore the spherical images of the faces through V_4 and V_5 are isograms. As V_3, V_4 and V_5 are the vertices of one face, this implies a type 3 octahedron.

      ii. Item (b) implies $c\chi_2 = c\psi_3$ which already yields that the spherical image of the faces through V_6 is a spherical isogram (a contradiction).

2. If V_6 is an ideal point: We will show that this case yields the following solution:

Theorem 4. *In the projective extension of E^3 there exists the following flexible octahedron of type B where only one vertex is an ideal point: The two pairs (V_1, V_4) and (V_2, V_5) of opposite vertices are symmetric with respect to a common plane which passes through the vertices V_3 and the ideal point V_6. This flexible octahedron is nothing else than Bricard's flexible octahedron of type 2 where one vertex located in the plane of symmetry is an ideal point.*

Proof: The proof of this theorem is split into two parts. In the first one we show that the motion transmission of the faces through V_6 corresponds to the one of a parallelogram or antiparallelogram. In the second step the given geometric characterization of flexible octahedra of type B is proven.

Step 1) The proof is done by contradiction, i.e. we assume that the motion transmission of the faces through V_6 does not correspond to the one of a parallelogram or antiparallelogram. Due to this assumption and section V the motion transmission from $[V_3, V_4, V_5]$ to $[V_2, V_4, V_6]$ (or from $[V_1, V_3, V_5]$ to $[V_1, V_2, V_6]$) via V_5 and V_6 is reducible for $c\psi_3 = c\chi_2$, as the spherical image of the faces through V_5 has to differ from a spherical isogram (otherwise we get a type 3 octahedron).

Analogous considerations for V_1 and V_6 yield $c\varphi_1 = c\kappa_1$.

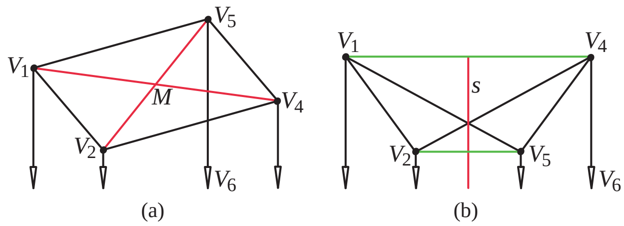

FIG. 7: The two cases implied by the conditions $\overline{V_2V_4} = \overline{V_1V_5}$ and $\overline{V_4V_5} = \overline{V_1V_2}$.

Analogous considerations for V_2 and V_6 yield $c\varphi_3 = c\kappa_3$. Analogous considerations for V_4 and V_6 yield $c\chi_3 = c\psi_1$. These four conditions imply under consideration of Eq. (30) the following contradiction:

$$c\chi_2 = c\kappa_3 \quad \text{and} \quad c\kappa_1 = c\psi_1. \tag{33}$$

Step 2) Due to the above considerations, we can assume w.l.o.g. that Eq. (33) holds. As a consequence, the cosines of the angles along the four edges through V_1 and V_4 are pairwise the same. As V_1 and V_4 are no ideal points, we can apply Kokotsakis theorem (*Satz über zwei Vierkante*) given in §12 of [19] which implies:

$$\begin{aligned} \sphericalangle V_3V_1V_5 &= \sphericalangle V_2V_4V_3, & \sphericalangle V_2V_1V_6 &= \sphericalangle V_5V_4V_6, \\ \sphericalangle V_5V_1V_6 &= \sphericalangle V_2V_4V_6, & \sphericalangle V_2V_1V_3 &= \sphericalangle V_3V_4V_5, \end{aligned} \tag{34}$$

if the spherical image of the faces through V_i $(i = 1, 4)$ is no spherical isogram (otherwise we get a type 3 octahedron). Moreover, the cosines of the angles along the four edges through V_2 and V_5 are pairwise the same. Analogous considerations yield:

$$\begin{aligned} \sphericalangle V_1V_2V_6 &= \sphericalangle V_4V_5V_6, & \sphericalangle V_3V_2V_4 &= \sphericalangle V_1V_5V_3, \\ \sphericalangle V_4V_2V_6 &= \sphericalangle V_1V_5V_6, & \sphericalangle V_1V_2V_3 &= \sphericalangle V_3V_5V_4. \end{aligned} \tag{35}$$

As a consequence the triangles $\triangle(V_3, V_2, V_4)$ and $\triangle(V_3, V_5, V_1)$ are similar as well as the triangles $\triangle(V_3, V_1, V_2)$ and $\triangle(V_3, V_4, V_5)$. Moreover, these triangles must not only be similar but even congruent in order that the 4-sided pyramid with apex V_3 can be assembled. Therefore $\overline{V_2V_4} = \overline{V_1V_5}$ and $\overline{V_4V_5} = \overline{V_1V_2}$ hold. As a consequence $\triangle(V_1, V_3, V_4)$ is an isosceles triangle which implies that the plane Γ_1 orthogonal to $[V_1, V_4]$ through the midpoint of V_1 and V_4 contains V_3. Clearly, the same holds for the isosceles triangle $\triangle(V_2, V_3, V_5)$ and the plane Γ_2 orthogonal to $[V_2, V_5]$ through the midpoint of V_2 and V_5. The intersection line of Γ_1 and Γ_2 is denoted by s.

In the following we prove that the points V_1, V_2, V_4, V_5 have to be coplanar. This can be done as follows: We assume that the prism is in one of its two flat poses. Now there exist two configurations such that $\overline{V_2V_4} = \overline{V_1V_5}$ and $\overline{V_4V_5} = \overline{V_1V_2}$ is fulfilled (see FIG. 7):

1. $[V_1, V_2] \parallel [V_4, V_5]$ and $[V_1, V_5] \parallel [V_2, V_3]$ (cf. FIG. 7 (a)): If the motion transmission of the faces through V_6 corresponds to the one of a parallelogram then the points V_1, V_2, V_4, V_5 are

always coplanar during the flex. If this transmission corresponds to the one of an antiparallelogram we distinguish the following two cases:

a. If $[V_1V_2]$ and $[V_1V_5]$ are orthogonal to the edges of the prism then V_1, V_2, V_4, V_5 remain coplanar during the flex.
b. In any other case V_1, V_2, V_4, V_5 form at least in one of the two possible flat poses of the prism a parallelogram. The intersection point of its diagonals is denoted by M. Then V_3 has to be located on the line s orthogonal to the carrier plane Ω of V_1, V_2, V_4, V_5, V_6 through M. We denote the point which results from reflecting V_3 on Ω by $\overline{V}_3$.

   If a point $V_3 \neq M$ exists such that the structure is flexible then the octahedron $V_1, V_2, V_3, V_4, V_5, \overline{V}_3$ also has to be flexible due to the symmetry. As this octahedron is convex, we get a contradiction with *Cauchys Theorem*.

   For the special case $V_3 = M$ the structure V_1, V_2, V_3, V_4, V_5 can only flex either along the diagonal $[V_1, V_4]$ or along the diagonal $[V_2, V_5]$. This yields a trivial self-motion as some faces coincide permanently during the flex.

2. In the second case we get the points V_4 and V_5 by reflecting V_1 and V_2, respectively, on the line s (cf. FIG. 7 (b)). As $[V_1, V_2] \parallel [V_4, V_5]$ implies item (1a) we can assume $[V_1, V_2] \nparallel [V_4, V_5]$ w.l.o.g.. If the motion transmission of the faces through V_6 corresponds to the one of an antiparallelogram then the points V_1, V_2, V_4, V_5 are always coplanar during the flex.

   Now we assume that this transmission corresponds to the one of a parallelogram. We consider the two planes Γ_1 and Γ_2 which are always parallel to the edges of the prism. As in all poses with $\Gamma_1 \neq \Gamma_2$, the intersection line s contains V_6 during the flex, this also has to hold in the limit where Γ_1 and Γ_2 coincide.

   Now we consider both flat poses V_1, V_2, V_4, V_5 and V_1', V_2', V_4', V_5' of the prism, which are illustrated in FIG. 8: Due to the above considerations s is spanned by the midpoints of V_1V_4 and V_2V_5. The condition that V_3 has to be located on s and V_3' on s', respectively, already determines these points uniquely as they have to be related by a reflection on $[V_1, V_5]$ (which neither cannot be parallel nor orthogonal to s).

   Trivial computations (which are left to the reader) show that e.g. the angles $\sphericalangle V_3V_4V_2$ and $\sphericalangle V_6V_4V_5$ are equal, which already yields a contradiction as the spherical image of the faces through V_4 is an isogram.

As a consequence of this case study we can assume w.l.o.g. that the points V_1, V_2, V_4, V_5 are planar during the flex of the prism. Now the proof is closed by footnote 6 of STACHEL [9], which says that a planar base polygon V_1, V_2, V_4, V_5 of a 4-sided pyramid remains planar during the flex if and only if the quadrilateral is an antiparallelogram and its plane of symmetry contains the apex V_3. □

It remains to show that case 2 of item (c) of section V (resp. its corresponding case of item (d) of section V) does not imply a solution: In this case, again all vertices apart from the ideal point and its opposite vertex are coplanar during the flex. Therefore we can also apply footnote 6 of STACHEL [9]. Due to $a = c$ and $b = d$ (resp. $b = c$ and $a = d$) two vertices of the antiparallelogram always coincide (= a flipped over rhombus). This can only yield a trivial self-motion as some faces coincide permanently during the flex.

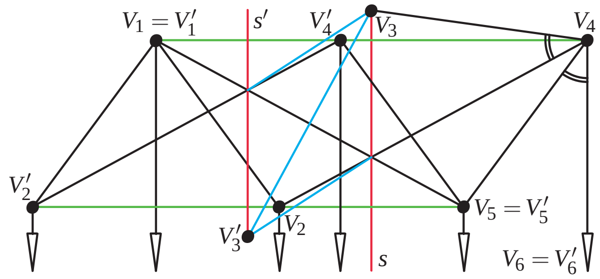

FIG. 8: The carrier plane of both flat poses V_1, V_2, V_4, V_5 and V_1', V_2', V_4', V_5' of the prism is spanned by $[V_1, V_5, V_6]$.

This closes the determination of all non-trivial self-motions of TSSM manipulators with parallel rotary axes. Again it is not difficult to verify that the trivial self-motions of these manipulators can only be the *butterfly motion* or the *spherical four-bar motion*, respectively.

Note, as we have completed the classification of all TSSM manipulators with self-motions, it is an easy task to compute the associated self-motions themselves. The simplest way for doing this is to consider the corresponding 6-3 parallel manipulators of SG type and to run the algorithm for the solution of the forward kinematics of SG platforms given by HUSTY [30], where n-dimensional self-motions appear as n-dimensional solutions of the direct kinematic problem.

<u>*Remark 7.*</u> Finally, we consider a polyhedron consisting of two pyramids Λ_1 and Λ_2 and a cylindrical middle part Π_0 (with an antiparallelogram as orthogonal cross section). This polyhedron is flexible if Λ_i and Π_0 form a flexible octahedron of type 2 or type 3 with one vertex in the plane at infinity for $i = 1, 2$. This construction is a generalization of the one given by STACHEL in the concluding remarks of [8]. ◇

VII. CONCLUSION

In this article we determined the whole set of non-trivial self-motions of TSSM manipulators with two parallel rotary axes which equals the determination of all flexible octahedra where one vertex is an ideal point. In Theorem 2, 3 and 4 it was shown that there are only two such flexible octahedra which proves the conjecture formulated by the author in [15]. This study also closes the classification of self-motions for parallel manipulators of TSSM type and of planar 6-3 SG platforms, respectively.

As a byproduct of this work we also gained a deeper insight into the classical *Bricard octahedra* (cf. Lemma 1 and Remark 3). Moreover, this article is the fist part of a classification of all flexible octahedra in the projective extension of the Euclidean 3-space which is completed in [16].

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