Fundamentals of Quaternionic Kinematics in Euclidean 4-Space*

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The elegance of the quaternion based analytical treatment of kinematics in Euclidean spaces of dimension 2 and 3 was pointed out and used by various authors (e.g. Blaschke [1], Müller [2], Ströher [3]). The quaternionic approach does not only yield a more compact notation in comparison with matrices (which also implies some computational advantages used in robotics), but it also provides an easier access to the geometry of motions.

Motivated by this circumstance, we extend this quaternionic kinematic to the Euclidean 4-space E^4. A first step in this direction was already done by the speaker in [4], where a kinematic mapping of E^4 was introduced, which can be seen as the generalization of the Blaschke-Grünwald parameters of E^2 and the Study parameters of E^3. These quaternion based kinematic parameters of E^4 are repeated in the first part of the talk, where also the notation of [4] is slightly modified in order to get more suitable formulas and representations.

Based on this kinematic mapping, we show in the second part of the presentation that the displacement δ (= orientation preserving congruence transformation) of basic geometric elements in E^4 can be treated in a unified way using the compact and elegant notation of (triangular) 2×2 quaternionic matrices. Under consideration that δ is represented by the lower triangular 2×2 quaternionic matrix $\underline{\mathbf{D}}$, the odd-dimensional elements (oriented hyperplanes, oriented lines) of E^4 are mapped by the sandwich operation $\underline{*} \mapsto \underline{\mathbf{D}}\,\underline{*}\,\widetilde{\underline{\mathbf{D}}}^T$ and the even-dimensional elements (points, oriented planes) of E^4 are mapped by the sandwich operation $\underline{*} \mapsto \widetilde{\underline{\mathbf{D}}}^{-T}\,\underline{*}\,\widetilde{\underline{\mathbf{D}}}^T$. In the latter way also the instantaneous screws are displaced, which are investigated in detail in the third part of the talk on infinitesimal kinematics.

Finally it should be noted that the 2×2 quaternionic matrix algebra is isomorphic to the Clifford Algebra with signature $(1_+, 3_-, 0_0)$, which shows the difference to the other existing kinematic mapping of E^4 (cf. Klawitter and Hagemann [5]), based on the Spin group of the Clifford Algebra with signature $(4_+, 0_-, 1_0)$.

References

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