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New Performance Indices for 6R Robot Postures

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Singular postures of 6R robots must be avoided because close to singularities an exact tracking of the planned end-effector trajectory requires great angular velocities in the rotary joints. Hence there is an interest in having a distance measure of the instantaneous configuration to the nearest singularity. This article outlines three new methods to measure the closeness to such a singularity. The presented measures are invariant with regard to Euclidean motions and similarities, they have a geometric meaning, and they can be computed in real-time.

1 Introduction

The purpose of this article is to define new measures for $6R$ robots which assign to each posture $\mathcal{K}$ a scalar $D(\mathcal{K})$ obeying the following six properties:

1. $D(\mathcal{K}) \geq 0$ for all $\mathcal{K}$ of the configuration space,
2. $D(\mathcal{K}) = 0$ if and only if $\mathcal{K}$ is singular,
3. $D(\mathcal{K})$ is invariant under Euclidean motions,
4. $D(\mathcal{K})$ is invariant under similarities,
5. $D(\mathcal{K})$ has a geometric meaning,
6. $D(\mathcal{K})$ is computable in real-time.

Many papers have been written on this topic. However, the two best known and most used concepts in this direction are probably the *condition number* and the *manipulability*. Both are based on the Jacobian matrix $\mathcal{J}$ of the manipulator which is defined by

$$\mathcal{J} = \begin{pmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_6 \\ \mathbf{p}_1 \times \mathbf{a}_1 & \dots & \mathbf{p}_6 \times \mathbf{a}_6 \end{pmatrix}, \tag{1}$$

where $\mathbf{a}_i$ describes the unit vector along the i^{th} joint axis and $\mathbf{p}_i$ the position vector of a point on this axis, and this with regard to a common reference frame. The i^{th} column of the Jacobian equals the normalized Plücker vector of the arbitrarily directed i^{th} axis. In the following we denote the moment vector $\mathbf{p}_i \times \mathbf{a}_i$ by $\widehat{\mathbf{a}}_i$ and we use the notation of dual vectors:

$$\underline{\mathbf{x}}_i := \mathbf{x}_i + \varepsilon\, \widehat{\mathbf{x}}_i = (\mathbf{x}_i, \widehat{\mathbf{x}}_i).$$

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1.1 Kinematic meaning of the Jacobian and singular postures

If the joint variables θ_1,..,θ_6 are functions of time t, a constrained motion of the end-effector system Σ_6 against the fixed system Σ_0 is determined. Because of the spatial *Three-Pole-Theorem* the one-parametric motion Σ_6 / Σ_0 can be composed of the relative motions Σ_1 / Σ_0,.., Σ_6 / Σ_5 and therefore the instantaneous screw $\underline{\mathbf{q}}_{6|0} = (\mathbf{q}_{6|0}, \widehat{\mathbf{q}}_{6|0})$ is computable as

$$\underline{\mathbf{q}}_{6|0} = \sum_{i=1}^{6} \underline{\mathbf{a}}_i \, \omega_{i|i-1} = \mathcal{J} \cdot \boldsymbol{\omega} \quad \text{with} \quad \boldsymbol{\omega} = (\omega_{1|0}, .., \omega_{6|5})^T \quad \text{and} \quad \omega_{i|i-1} = \frac{d\theta_i}{dt}. \tag{2}$$

The rank of $\mathcal{J}$ gives the instantaneous degree of freedom (*dof*), i.e., the dimension of the vector space of infinitesimal motions Σ_6 / Σ_0 which is spanned by the relative rotations. This space can be considered as the tangent space of the manipulator's workspace. If the rank drops below 6, then there exists at least one angular velocity ratio $\mathbb{R}\boldsymbol{\omega} \neq \mathbf{o}$ such that the end-effector has an instantaneous standstill while the joints are still moving. This amounts to the loss of the instantaneous *dof*. Therefore we can characterize the singular postures of 6R robots as follows:

Theorem 1. *Any posture of a non-redundant manipulator is singular if and only if the determinant of the Jacobian vanishes. Exactly in this case the axes belong to a linear line complex.*

A consequence of this theorem is the following: If the manipulator is close to a singularity, then great angular velocities in the rotary joints can have only low effects on the instantaneous displacement of the end-effector, insofar as $\mathbb{R}\boldsymbol{\omega}$ is badly chosen.

1.2 Manipulability

This concept was introduced by Yoshikawa (1985). It is based on Theorem 1, because this *Performance Index* equals for non-redundant manipulators nothing else than $|det(\mathcal{J})|$. This index meets obviously properties 1 and 2, but also property 3: Let $\mathcal{R}$ be an orthogonal matrix which induces a rotation ρ of the reference frame and let $\mathbf{t} := (x, y, z)^T$ be the vector of any translation τ. We only have to prove that the determinant of $\mathcal{J}$ is invariant under translations and rotations: The coordinate transformation of spears can be written as

$$(\mathbf{a}, \widehat{\mathbf{a}}) \xrightarrow{\rho} (\mathcal{R}\,\mathbf{a}, \mathcal{R}\,\widehat{\mathbf{a}}) \quad \text{and} \quad (\mathbf{a}, \widehat{\mathbf{a}}) \xrightarrow{\tau} (\mathbf{a}, \widehat{\mathbf{a}} + (\mathbf{t} \times \mathbf{a})) \quad \Longrightarrow \tag{3}$$

$$\mathcal{J} \xrightarrow{\rho} \begin{pmatrix} \mathcal{R} & 0_3 \\ 0_3 & \mathcal{R} \end{pmatrix} \mathcal{J} := \mathcal{J}_\rho \quad \text{and} \quad \mathcal{J} \xrightarrow{\tau} \begin{pmatrix} \mathcal{I}_3 & 0_3 \\ \mathcal{T} & \mathcal{I}_3 \end{pmatrix} \mathcal{J} := \mathcal{J}_\tau \quad \text{with} \quad \mathcal{T} = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

where $\mathcal{I}_3$ is the 3×3-unit matrix. The invariance of $|det(\mathcal{J})|$ follows immediately from $det(\mathcal{R}) = det(\mathcal{I}_3) = 1$ and hence $det(\mathcal{J}_\rho) = det(\mathcal{J}_\tau) = det(\mathcal{J})$. However, this measure does not fulfill property 4, because the determinant is multiplied with μ^3 if μ is the scaling factor. $|det(\mathcal{J})|$ can be interpreted as follows:

In the vector space $\mathbb{R}^6$ of instantaneous screws the subset of all $\underline{\mathbf{q}}_{6|0}$, with $|\omega_{i|i-1}| \leq 1$ for $i = 1, .., 6$, is by (2) a parallelepiped, which has the volume $2^6 \cdot |det(\mathcal{J})|$. Therefore the given index is proportional to this volume.

1.3 Condition Number CDN

According to Salisbury and Craig (1982) the *condition number* is the positive square root of the ratio of the maximal and minimal eigenvalue of the square matrix $\mathcal{J}^T\mathcal{J}$. CDN^{-1} lies in the interval $[0,1]$, and 0 characterizes singular configurations. Postures with the maximum value 1 are called *isotropic*. It should be still noted that the CDN originally has been introduced to indicate numerically how well the numerical procedure of inverting a matrix is conditioned. CDN^{-1} trivially fulfills the properties 1 and 2, but not 3 and 4. Changing the viewpoint allows us to pinpoint the lacking invariances. The latter can be concluded as follows:

1.3.1 Interpretation and variance of the CDN

We start with the following optimization problem: We search for the minimum resp. the maximum of the quadratic objective function $\widehat{\zeta}(\boldsymbol{\omega})$ under the quadratic side condition $\widehat{\nu}(\boldsymbol{\omega})$ with

$$\widehat{\nu}(\boldsymbol{\omega}): \quad \boldsymbol{\omega}^T\boldsymbol{\omega} = \boldsymbol{\omega}^T\mathcal{I}_6\,\boldsymbol{\omega} = 1 \quad \text{and} \quad \widehat{\zeta}(\boldsymbol{\omega}): \quad \underline{\mathbf{q}}_{6|0}^T\,\underline{\mathbf{q}}_{6|0} = \boldsymbol{\omega}^T\mathcal{N}\,\boldsymbol{\omega} \quad \text{whereas} \quad \mathcal{N} = \mathcal{J}^T\mathcal{J}.$$

Then the solutions are the minimal resp. the maximal eigenvalue of $\mathcal{J}^T\mathcal{J}$ (compare *subsection* 2.1). Thus the CDN is the positive square root of the ratio of the maximum and the minimum of the sum $\underline{\mathbf{q}}_{6|0}^T\,\underline{\mathbf{q}}_{6|0}$ under the side condition that the sum of the squared angular velocities is equal to 1. As the objective function is no geometric invariant of a screw, this measure is not invariant under Euclidean motions and similarities. However, the objective function is invariant under rotations around the origin U of the reference frame, because of

$$\underline{\mathbf{q}}_{6|0}^T\underline{\mathbf{q}}_{6|0} = \mathbf{q}_{6|0}^2 + \widehat{\mathbf{q}}_{6|0}^2 = \omega_{6|0}^2 + \left[\widehat{\omega}_{6|0}^2 + \omega_{6|0}^2\,\overline{Up}^2\right], \tag{4}$$

with $\omega_{6|0}$ denoting the angular velocity and $\widehat{\omega}_{6|0}$ the translatory velocity of the screw $\underline{\mathbf{q}}_{6|0}$. $\overline{Up}$ is the perpendicular distance between the origin U and the screw axis p. Therefore the CDN only depends on the choice of the origin U, apart from the scaling factor. In practice U is not selected arbitrarily, but placed in the center of interest, the *tool center point*.

To overcome the dimensional inhomogeneity of the objective function, Tandirci et al. (1992) presented the method of *characteristic length* L. The last three rows of the Jacobian that have units of length are divided by L which is equivalent to the division of $\widehat{\mathbf{q}}^2$ by L^2 in (4). L is chosen such that it minimizes the CDN, which depends on the unknowns $\theta_2, .., \theta_6$ and L.

Gosselin (1990) tried to cope the problem of dimensional inhomogeneity as follows: He described the *global* velocity of the end-effector instead of $\underline{\mathbf{q}}_{6|0}$ by the velocity of three points which are supposed to be noncollinear. In this way the desired quality was achieved. Nevertheless, the choice of the three points is arbitrary, and the CDN is not invariant under changes of these three reference points.

Remark. Summarizing, one can say that both measures have deficiencies. But this does not question their usefulness, because they are very simply computable. On the other hand, any geometrical meaning of a quality measure seems to be important, and this was the main reason to search for a new measure. It should also be mentioned, that the *manipulability* and the *condition number* are used as an index which describes the dexterity of a robot on the one hand and the closeness to the singularity on the other. They share these properties with the following measure $M(\mathcal{K})$ which is introduced next.

2 The new measure $M(\mathcal{K})$

Some authors (see Angeles and Lopez-Cajun (1992); Tandirci et al. (1992)) prefer to use the CDN^{-1} instead of the *manipulability*, because this measure depends on the *tool center point*, as shown above. Basically, one is not interested in the manipulation of a single point, but in that of a complete object $\mathcal{O}$, which can be a welding tool or a load to be transported. More generally, $\mathcal{O}$ can also be defined as an ellipsoid of interest attached to the end-effector system Σ_6. Starting from this point of view we develop the measure $M(\mathcal{K})$.

2.1 Basic idea of $M(\mathcal{K})$

First of all it should be mentioned that $M(\mathcal{K})$ is based on a method, which has its origin in the registration problem with known correspondences.[1] We represent the object $\mathcal{O}$ of interest by a finite number of points $\mathbf{P}_i$ $(i = 1,..,N)$. By means of this point cloud it is possible to define the distance between any two positions of the object, namely $\mathcal{O}(0)$ and $\mathcal{O}(t)$, as follows:

$$d(\mathcal{O}(0), \mathcal{O}(t))^2 := \sum_{i=1}^{N} \|\mathbf{P}_i(0) - \mathbf{P}_i(t)\|^2.$$

This introduces an object-oriented metric in the work space. Because of a well known result from mechanics, we can restrict ourselves to six vertices $\mathbf{V}_i := (v_1^i, v_2^i, v_3^i)$, $i = 1,..,6$, of the inertia ellipsoid of $\mathcal{O}$. The first order Taylor approximation of posture $\mathcal{O}(t)$ yields:

$$d(\mathcal{O}(0), \mathcal{O}(t))^2 = \sum_{i=1}^{6} \|\mathbf{V}_i(0) - \mathbf{V}_i(t)\|^2 \approx \sum_{i=1}^{6} \|\mathbf{v}_F\left(\mathbf{V}_i(0)\right)\|^2 \quad \text{where} \quad \mathbf{v}_F(\mathbf{P}) = \widehat{\mathbf{q}}_{6|0} + (\mathbf{q}_{6|0} \times \mathbf{P})$$

denotes the velocity vector of point $\mathbf{P} \in \Sigma_6$. On the other hand at any singular posture there is at least one $\mathbb{R}\,\boldsymbol{\omega} \neq \mathbf{o}$ with $\underline{\mathbf{q}}_{6|0} = \underline{\mathbf{o}}$, i.e. $\mathbf{v}_F(\mathbf{P}) = \mathbf{o}$ for all $\mathbf{P} \in \Sigma_6$. Hence, we can define a kind of *distance* of $\mathcal{K}$ to a close singularity by solving the following minimization problem: We search for the instantaneous screw $\underline{\mathbf{q}}_{6|0}$, which minimizes the objective function

$$\zeta(\boldsymbol{\omega}): \quad \sum_{i=1}^{6} \|\mathbf{v}_F\left(\mathbf{V}_i(0)\right)\|^2 \quad \text{with} \quad \mathbf{v}_F(\mathbf{V}_i) = \sum_{j=1}^{6} \widehat{\mathbf{a}}_j\, \omega_{j|j-1} + \left(\sum_{j=1}^{6} \mathbf{a}_j\, \omega_{j|j-1} \times \mathbf{V}_i\right)$$

under the side condition

$$\nu(\boldsymbol{\omega}): \quad \boldsymbol{\omega}^T \boldsymbol{\omega} = \boldsymbol{\omega}^T \mathcal{I}_6\, \boldsymbol{\omega} = 1. \tag{5}$$

A preliminary condition is that the inertia ellipsoid must not degenerate to a single line segment, because $\zeta(\boldsymbol{\omega})$ depends on the distances of $\mathbf{V}_i$ from the instantaneous screw axis $p_{6|0}$ and on $\omega_{6|0}$ and $\widehat{\omega}_{6|0}$ according to

$$\sum_{i=1}^{6} \|\mathbf{v}_F\left(\mathbf{V}_i(0)\right)\|^2 = \omega_{6|0}^2 \sum_{i=1}^{6} \overline{p_{6|0}\mathbf{V}_i}^2 + 6\,\widehat{\omega}_{6|0}^2. \tag{6}$$

[1] Let two postures of any object be given, each represented by a finite point cloud $\mathcal{X}$ resp. $\mathcal{Y}$, of corresponding points $\mathbf{x}_i \in \mathcal{X}$ and $\mathbf{y}_i \in \mathcal{Y}$. Now a Euclidean motion is required, which brings each $\mathbf{x}_i$ as close as possible to $\mathbf{y}_i$. For further information on this topic see Hofer et al. (2004).

$\zeta(\boldsymbol{\omega})$ is a quadratic form with the unkowns $\omega_{i|i-1}$, and so we can rewrite the objective function as $\zeta(\boldsymbol{\omega}) = \boldsymbol{\omega}^T \mathcal{Z}\, \boldsymbol{\omega}$, where $\mathcal{Z}$ is given by

$$\mathcal{Z} = \sum_{i=1}^{6} \mathcal{J}^T \begin{pmatrix} \mathcal{V}_i^T \mathcal{V}_i & \mathcal{V}_i^T \\ \mathcal{V}_i & \mathcal{I}_3 \end{pmatrix} \mathcal{J} \quad \text{with} \quad \mathcal{V}_i := \begin{pmatrix} 0 & v_3^i & -v_2^i \\ -v_3^i & 0 & v_1^i \\ v_2^i & -v_1^i & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{V}_i := (v_1^i, v_2^i, v_3^i).$$

We solve the minimization problem by introducing a Lagrange multiplier λ. Then the approach simplifies in consideration of $\nabla\zeta = 2\,\mathcal{Z}\,\boldsymbol{\omega}$ and $\nabla\nu = 2\,\mathcal{I}_6\,\boldsymbol{\omega}$, to the eigenvalue problem $(\mathcal{Z} - \lambda\,\mathcal{I}_6)\,\boldsymbol{\omega} = \mathbf{o}$. This system of linear equations has a nontrivial solution, if and only if the determinant of $(\mathcal{Z} - \lambda\,\mathcal{I}_6)$ vanishes. The characteristic polynomial in λ is of degree ≤ 6. Every eigenvalue λ_i of $\mathcal{Z}$ is linked with an eigenvector $\boldsymbol{\omega}_i$. From

$$\mathcal{Z}\,\boldsymbol{\omega}_i = \lambda_i\,\mathcal{I}_6\,\boldsymbol{\omega}_i \quad \text{and} \quad \boldsymbol{\omega}_i^T\,\mathcal{I}_6\,\boldsymbol{\omega}_i = 1 \quad \text{follows} \quad \zeta(\boldsymbol{\omega}_i) = \boldsymbol{\omega}_i^T\,\mathcal{Z}\,\boldsymbol{\omega}_i = \lambda_i \boldsymbol{\omega}_i^T\,\mathcal{I}_6\,\boldsymbol{\omega}_i = \lambda_i.$$

Therefore the eigenvalue λ_i mirrors the objective function value $\zeta(\boldsymbol{\omega}_i)$ and so the smallest eigenvalue λ_{min} corresponds to the required solution. Due to (5) and (6) the following inequality holds:

$$\lambda_{min} \leq \sum_{i=1}^{6} \overline{p_{6|5}\mathbf{V}_i}^2 =: U \ldots \ \text{const.} \tag{7}$$

2.2 Definition of $M(\mathcal{K})$

Now we define the *distance* of the configuration $\mathcal{K}$ to the closest singularity as

$$M(\mathcal{K}) := +\sqrt{\lambda_{min} U^{-1}} \qquad \text{with} \qquad M(\mathcal{K}) \in [0,1] \qquad \text{because of (7).}$$

The measure $M(\mathcal{K})$ is well defined, because $\mathcal{Z}$ is in the non-singular case positive definite and in the singular case positive semidefinite. The defined measure is also invariant with regard to similarities because $M(\mathcal{K})$ is defined as a ratio. All other claimed properties are trivially fulfilled.

If $M(\mathcal{K})$ is very small the angular velocity ratio of $\mathbb{R}\,\boldsymbol{\omega_{min}}$ should be avoided because it has low effects on the instantaneous displacement of $\mathcal{O}$.

2.3 Weighted measure $\widehat{M}(\mathcal{K})$

We can replace the unit matrix $\mathcal{I}_6$ in (5) by a weighting matrix $\mathcal{D} := diag(d_1, .., d_6)$. How should such weights d_i be defined? The following choices seem to make sense:

We set $d_i = \mathbf{I}_i/2$, where $\mathbf{I}_i > 0$ denotes estimates for the mass moment of inertia of the union of the bodies $\Sigma_i, .., \Sigma_6$, about the axis $\underline{\underline{\mathbf{a}}}_i$. The only difference in the computation is that we require the smallest *general* eigenvalue $\widehat{\lambda}_{min}$. Due to the weighted side condition and (6) the following inequality holds:

$$\widehat{\lambda}_{min} \leq 2\mathbf{I}_6^{-1} \sum_{i=1}^{6} \overline{p_{6|5}\mathbf{V}_i}^2 =: \widehat{U} \ldots \ \text{const.} \quad \Longrightarrow \quad \widehat{M}(\mathcal{K}) := +\sqrt{\widehat{\lambda}_{min} \widehat{U}^{-1}} \in [0,1]\,. \tag{8}$$

The disadvantage of such weights is that one quits the pure kinematic point of view: Robots which only differ in the mass distribution, would give rise to different distances in the same configuration $\mathcal{K}$.

3 The distance measures $D_1(\mathcal{K})$ and $D_2(\mathcal{K})$

It does not make sense to use a purely line geometrical point of view by paying only attention to the instantaneous postures of the six rotary axes. And exactly this is the case if one considers the instantaneous Jacobian only. Of course, it is always possible to find singular postures close to a given configuration. Let's assume that $\mathcal{C}$ is the linear complex spanned by the first five axes. All lines of $\mathcal{C}$ intersecting the sixth axis form in the non-singular case of $\mathcal{C}$ a hyperbolic linear congruence. Every replacement of $\underline{\mathbf{a}}_6$ by an arbitrary line $\underline{\mathbf{l}}$ of this congruence gives a singular configuration. But which $\underline{\mathbf{l}}$ is the closest to $\underline{\mathbf{a}}_6$? How should a distance between the given posture and a close singular one be defined which allows to distinguish between better and worse configurations? Of course, it is obvious using distances and angles of $\underline{\mathbf{a}}_6$ to the second axis of the hyperbolic line congruence, but which function of angle and distance gives a meaningful positive measure, which is invariant under scaling and works in all cases?

Therefore, a reasonable way to come up with a geometrically explicable distance measure is to take the possible variability of the rotary axes into consideration. For a serial robot with fixed $\underline{\mathbf{a}}_1$ the second axis admits only a one-parametric movement against the frame, and its position influences that of the axes $\underline{\mathbf{a}}_3$, $\underline{\mathbf{a}}_4$, et cetera. This concept is advantagous also since for serial robots the direct kinematics is simple to compute.

3.1 Basic idea of $D_1(\mathcal{K})$ and $D_2(\mathcal{K})$

Let's assume, the robot is abruptly stopped when it moves through any configuration $\mathcal{K}$ with the instantaneous angular velocity vector $\boldsymbol{\omega}$. Then it overdrives the posture $\mathcal{K}$, whereas each axis $\underline{\mathbf{a}}_{i+1}$ moves according to the instantaneous screw $(\mathbf{q}_{i|0}, \widehat{\mathbf{q}}_{i|0})$ of the motion of Σ_i/Σ_0 with

$$\underline{\mathbf{q}}_{i|0} = (\mathbf{q}_{i|0}, \widehat{\mathbf{q}}_{i|0}) = \sum_{j=1}^{i} \underline{\mathbf{a}}_j\, \omega_{j|j-1} \qquad \text{for} \qquad i = 1, .., 5. \tag{9}$$

Now let for $i = 1, .., 5$ the instantaneous rotation about $\underline{\mathbf{a}}_i$ with velocity $\omega_{i|i-1}$ operate for a certain time $\delta_i \in \mathbb{R}^+$. Then the finite rotation about $\underline{\mathbf{a}}_i$ through the angle $\Delta_i = \delta_i\, \omega_{i|i-1}$ is performed. How must the rotation angles be chosen to reach the closest singularity?

Since a rotation of the robot about the first axis $\underline{\mathbf{a}}_1$ has no influence on the distance to the closest singular posture, we may assume $\omega_{1|0} = 0$. Depending on the minimization condition which the angular velocities have to fulfill, we will get two different distance measures $D_1(\mathcal{K})$ and $D_2(\mathcal{K})$ for serial robots between the actual posture $\mathcal{K}$ and the closest singularity. Both distances serve as performance index of $\mathcal{K}$.

3.2 Linearized approximation of direct kinematics

We start from the posture $\mathcal{K}(\theta_1, .., \theta_6) =: \mathcal{K}(\theta)$ of the configuration space, and look for a singular $\mathcal{K}(\theta_1, \theta_2 + \Delta_2, .., \theta_5 + \Delta_5, \theta_6) =: \mathcal{K}(\delta)$, where Δ_i should be kept as small as possible. Our two measures are based on a linearized approximation of this problem to reduce the computational costs. This is sufficient for the definition of a distance measure for the following reasons: On the one hand the approximation is sufficiently close to a singularity. On the other hand, the exact value is not interesting as long as it is larger than a critical distance, which must be predefined.[2]

[2]Experiments will prove which value of $D_i(\mathcal{K})$ for given robots can be seen as a sufficient distance to any singularity.

Then the manipulator is not in an imminent closeness of a singular posture.

In order to compute the closest singularity we need an approximation of the axes $\underline{\mathbf{a}}_2, .., \underline{\mathbf{a}}_6$, because the singularity is only determined by the instantaneous postures of the joint axes. As $\underline{\mathbf{a}}_i$ depends on the angles $\theta_j + \Delta_j$, $j = 2, .., i-1$, we obtain by Taylor's formula

$$\underline{\mathbf{a}}_i(\delta) = \underline{\mathbf{a}}_i(\theta_2 + \Delta_2, .., \theta_{i-1} + \Delta_{i-1}) \approx \underline{\mathbf{a}}_i(\theta_2, .., \theta_{i-1}) + \sum_{j=2}^{i-1} \Delta_j \, \frac{\partial \, \underline{\mathbf{a}}_i(\theta_2, .., \theta_{i-1})}{\partial \theta_j} =: \underline{\mathbf{b}}_i(\delta)$$

where $\partial \, \underline{\mathbf{a}}_i(\theta)/\partial\theta_j$ expresses the instantaneous change of $\underline{\mathbf{a}}_i(\theta)$ under the rotation about $\underline{\mathbf{a}}_j(\theta)$ with angular velocity $\omega_{j|j-1} = 1$. The screw associated to this rotation is $\underline{\mathbf{a}}_j$, which implies

$$\frac{\partial \, \underline{\mathbf{a}}_i(\theta)}{\partial \theta_j} = \underline{\mathbf{a}}_j(\theta) \times \underline{\mathbf{a}}_i(\theta).$$

Therefore we get: $\quad \underline{\mathbf{b}}_i(\delta) = \underline{\mathbf{a}}_i(\theta_2, .., \theta_{i-1}) + \sum_{j=2}^{i-1} \Delta_j \, (\underline{\mathbf{a}}_j \times \underline{\mathbf{a}}_i) \quad$ with

the real part $\quad \mathbf{b}_i(\delta) = \mathbf{a}_i(\theta) + \sum_{j=2}^{i-1} \Delta_j \frac{\partial \, \mathbf{a}_i(\theta)}{\partial \theta_j} = \mathbf{a}_i(\theta) + \sum_{j=2}^{i-1} \Delta_j \, (\mathbf{a}_j \times \mathbf{a}_i) \quad$ and

the dual part $\quad \widehat{\mathbf{b}}_i(\delta) = \widehat{\mathbf{a}}_i(\theta) + \sum_{j=2}^{i-1} \Delta_j \frac{\partial \, \widehat{\mathbf{a}}_i(\theta)}{\partial \theta_j} = \widehat{\mathbf{a}}_i(\theta) + \sum_{j=2}^{i-1} \Delta_j \, [(\widehat{\mathbf{a}}_j \times \mathbf{a}_i) + (\mathbf{a}_j \times \widehat{\mathbf{a}}_i)]\,.$

Note that the approximation $\underline{\mathbf{b}}_i(\delta)$ of $\underline{\mathbf{a}}_i(\delta)$, in general, does not satisfy the Plücker condition $\mathbf{b}_i(\delta) \cdot \widehat{\mathbf{b}}_i(\delta) = 0$ and the normalization condition $\underline{\mathbf{b}}_i(\delta) \cdot \underline{\mathbf{b}}_i(\delta) = 1$. But for small Δ_i our approximation meets also these conditions approximativly:

$$\underline{\mathbf{b}}_i(\delta) \cdot \underline{\mathbf{b}}_i(\delta) = \underline{\mathbf{a}}_i \cdot \underline{\mathbf{a}}_i + \sum_{j=2}^{i-1} \Delta_j \frac{\partial (\underline{\mathbf{a}}_i \cdot \underline{\mathbf{a}}_i)}{\partial \theta_j} + \frac{1}{2} \sum_{j,k=2}^{i-1} \Delta_j \Delta_k \frac{\partial^2 (\underline{\mathbf{a}}_i \cdot \underline{\mathbf{a}}_i)}{\partial \theta_j \, \partial \theta_k} \approx 1$$

because the partial derivative of $\underline{\mathbf{a}}_i \cdot \underline{\mathbf{a}}_i = 1$ yields $\dfrac{\partial (\underline{\mathbf{a}}_i \cdot \underline{\mathbf{a}}_i)}{\partial \theta_j} = 2 \underline{\mathbf{a}}_i \dfrac{\partial \underline{\mathbf{a}}_i}{\partial \theta_j} = 0.$

Therefore the approximation of the determinant $\mathcal{J}(\delta) := \mathcal{J}(\theta_2 + \Delta_2, .., \theta_5 + \Delta_5)$ can be written as:

$$det\,[\mathcal{J}(\delta)] = det\,[\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3(\theta_2 + \Delta_2), .., \underline{\mathbf{a}}_6(\theta_2 + \Delta_2, .., \theta_5 + \Delta_5)] \approx \ldots$$

$$\approx det\,[\mathcal{J}(\theta_2, .., \theta_5)] + \sum_{j=2}^{5} \frac{\partial \, det(\mathcal{J})}{\partial \theta_j} \Delta_j \quad \text{with} \quad \frac{\partial \, det(\mathcal{J})}{\partial \theta_j} \Delta_j = \sum_{k=j+1}^{6} det(\underline{\mathbf{a}}_1, .., \frac{\partial \underline{\mathbf{a}}_k}{\partial \theta_j} \Delta_j, ..).$$

We set $det\,[\mathcal{J}(\delta)] = 0$ in order to reach a close singularity, and rewrite this equation by

$$det(\mathcal{J}) + \Delta_2 \cdot \sum_{j=3}^{6} det(\mathcal{J}_{2,j}) + \Delta_3 \cdot \sum_{j=4}^{6} det(\mathcal{J}_{3,j}) + \Delta_4 \cdot \sum_{j=5}^{6} det(\mathcal{J}_{4,j}) + \Delta_5 \cdot det(\mathcal{J}_{5,6}) = 0 \qquad (10)$$

where $\mathcal{J}_{i,j}$ denotes the matrix, where the j^{th} column of the Jacobian matrix is replaced by

$$\underline{\mathbf{a}}_i \times \underline{\mathbf{a}}_j = (\mathbf{a}_i \times \mathbf{a}_j, \, \widehat{\mathbf{a}}_i \times \mathbf{a}_j + \mathbf{a}_i \times \widehat{\mathbf{a}}_j).$$

3.3 Definition of $D_1(\mathcal{K})$ and $D_2(\mathcal{K})$

All quadrupels $(\Delta_2, .., \Delta_5)$, which solve the inhomogenous linear equation (10), are located in a hyperplane $\mathcal{H}$ of the vector space $\mathbb{V}_{\mathcal{K}}$ spanned by $\theta_2, .., \theta_5$. The variables Δ_i serve as local coordinates in $\mathbb{V}_{\mathcal{K}}$ with origin $\mathcal{K}_o := \mathcal{K}(\theta)$. The zero vector is a solution of (10), if and only if $\mathcal{H}$ passes through $\mathcal{K}_o$, which means $det(\mathcal{J}) = 0$ (see *figure 1*).

3.3.1 Method 1 $\Rightarrow D_1(\mathcal{K}) := \Delta_L$

We solve equation (10) under the side condition $|\Delta_i| \leq \Delta_L$ for $i = 2, .., 5$, and Δ_L should be minimal. We get the required quadrupel $(\Delta_2, .., \Delta_5)$ as the nonempty intersection of $\mathcal{H}$ and the smallest hypercube centered in the origin $\mathcal{K}_o$ and with edges parallel to the axes. This nonempty set always contains a vertex $\mathbf{L} = (l_2, .., l_5)$ of the hypercube (*see figure 1*) with

$$l_i = sgn\left(det(\mathcal{J}) \cdot \frac{\partial\, det(\mathcal{J})}{\partial \theta_i}\right) \Delta_L \qquad \text{and} \qquad \Delta_L = \frac{|det(\mathcal{J})|}{\sum_{j=2}^{5} |\frac{\partial\, det(\mathcal{J})}{\partial \theta_j}|}. \tag{11}$$

Theorem 2. $D_1(\mathcal{K})$ *equals approximately the smallest angle* Δ_L (11) *through which one has to rotate simultaneously all joints* Σ_{j+1} *about* $\underline{\mathbf{a}}_j$ $(j = 2, .., 5)$ *with the orientation* $sgn(l_j)$ *to pass through a singular posture.*

It can happen that the smallest nonempty intersection of the hypercube and $\mathcal{H}$ is one-, two- or three-dimensional, because $\mathcal{H}$ can touch the hypercube along an edge, a plane or a hyperplane. These exeptional cases manifest themselves in the above formulas as follows: Depending on the dimension of the intersection some l_i's vanish because the i^{th} coordinate of the normal vector of $\mathcal{H}$ is equal to zero. So the distance to the singularity does not depend on the respective Δ_i's.

3.3.2 Method 2 $\Rightarrow D_2(\mathcal{K}) := \Delta_Q$

As an alternative, we optimize equation (10) such that the sum of the squares of the angles is minimal. That means, that we are searching for the point $\mathbf{Q} = (q_2, .., q_5) \in \mathcal{H}$, for which $q_2^2 + q_3^2 + q_4^2 + q_5^2 = \Delta_Q^2$ is minimal. This side condition seems to be legitimate under the point of view, that all Δ_i should be as small as possible, because of $0 \leq |q_i| \leq \frac{3}{2}\Delta_L$.[3] This relation can be proved as follows: Without loss of generality, we assume that $\mathbf{L}$ is located in the positive sector, to say, $l_2 = .. = l_5 = \Delta_L$. Due to Thales' theorem the foot-points $\mathbf{F} = (f_2, f_3, f_4, f_5)$ of the perpendiculars drawn from the origin to all possible hyperplanes through $\mathbf{L}$ are lying on the hypersphere

$$\Lambda : \quad \sum_{i=2}^{5} \left(f_i - \frac{\Delta_L}{2}\right)^2 = \Delta_L^2.$$

The maximum of f_i is: $\quad f_i = \frac{3}{2}\Delta_L \quad$ when $\quad f_j = \frac{\Delta_L}{2} \quad$ for $\quad j \in \{2, 3, 4, 5\}\backslash\{i\}$. $\square$

$\mathbf{Q}$ is the point of contact between $\mathcal{H}$ and the hypershere $\mathcal{S}$, centered in $\mathcal{K}_o$ (*see figure 1*), with

$$q_i = \frac{-det(\mathcal{J})}{\sum_{j=2}^{5}\left(\frac{\partial\, det(\mathcal{J})}{\partial \theta_j}\right)^2} \cdot \frac{\partial\, det(\mathcal{J})}{\partial \theta_i} \qquad \Longrightarrow \qquad \Delta_Q = \frac{|det(\mathcal{J})|}{+\sqrt{\sum_{j=2}^{5}\left(\frac{\partial\, det(\mathcal{J})}{\partial \theta_j}\right)^2}}. \tag{12}$$

[3] An optimization, which minimizes the sum of the absolute values of the swept angles, does not make sense, because in the worst case one has to assume a factor of 4 instead of 3/2.

Theorem 3. *$D_2(\mathcal{K})$ equals approximately the shortest time Δ_Q (12) which is required to reach the next singularity, provided the sum of the squared angular velocities is bounded by 1. The following relationship holds:*

$$\Delta_L \leq \Delta_Q \leq 2\Delta_L.$$

Proof of the inequality: We have $\Delta_L = \Delta_Q$ when $\mathcal{H}$ touches the hypercube along a hyperplane. And for $\mathbf{L} = \mathbf{Q}$ the upper bound is reached. □

By this method we obtain not only the desired distance measure, but also the worst instantaneous angular velocity ratio $\boldsymbol{\omega}_- := \mathbb{R}^+(q_2, q_3, q_4, q_5)$ i.e., the direction towards the closest singularity.[4] We can even evaluate any instantaneous angular velocity ratio $\boldsymbol{\omega}_o$ at the given non-singular posture $\mathcal{K}_o$. An adequate measure for this seems to be the angle

$$\varepsilon := \sphericalangle(\boldsymbol{\omega}_-, \boldsymbol{\omega}_o) - \frac{\pi}{2} \qquad \text{with} \qquad \sphericalangle(\boldsymbol{\omega}_-, \boldsymbol{\omega}_o) \in [0, \pi] \qquad \text{and} \qquad \varepsilon \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{13}$$

For $\varepsilon = -\frac{\pi}{2}$, one is moving directly to the singularity which is approximated by $\mathcal{H}$. $\varepsilon = \frac{\pi}{2}$ defines the best direction to avoid the singularity, and for $\varepsilon = 0$ one is moving equidistantly to it.

3.4 Discussion of the special cases

(**a**) How can we interpret equation (10), if all coefficients of Δ_i $(i = 2, .., 5)$ are equal to zero and $det(\mathcal{J}) = 0$? In such a case the manipulator is in a singular posture of higher order, because all four partial derivatives

$$\frac{\partial}{\partial\theta_i} det(\mathcal{J})\Big|_{\mathcal{K}_o} = \sum_{j=i+1}^{6} det(\mathcal{J}_{i,j}), \qquad i = 2, .., 5,$$

vanish. Then $D_1(\mathcal{K}) = D_2(\mathcal{K}) = 0$ ($\Leftrightarrow \mathbf{L} = \mathbf{Q} = \mathbf{o}$) as (10) holds for any $(\Delta_2, .., \Delta_5) \in \mathbb{R}^4$.

Remark. 6R robots with generic dimensions do not possess higher-order singularities, because the four unknowns $\theta_2, .., \theta_5$ have to fulfill five equations, namely the four partial derivatives and the Jacobian's determinant are simultaneously equal to zero.

(**b**) A problem occurs if the four partial derivatives of the Jacobian's determinant vanish but $det(\mathcal{J}) \neq 0$. Then equation (10) can never be fulfilled. As a result of the last remark every 6R manipulator has at least such a posture. For a better understanding of this case we start with visualizing the relationship between manipulability, Δ_L, Δ_Q and $\boldsymbol{\omega}_-$:

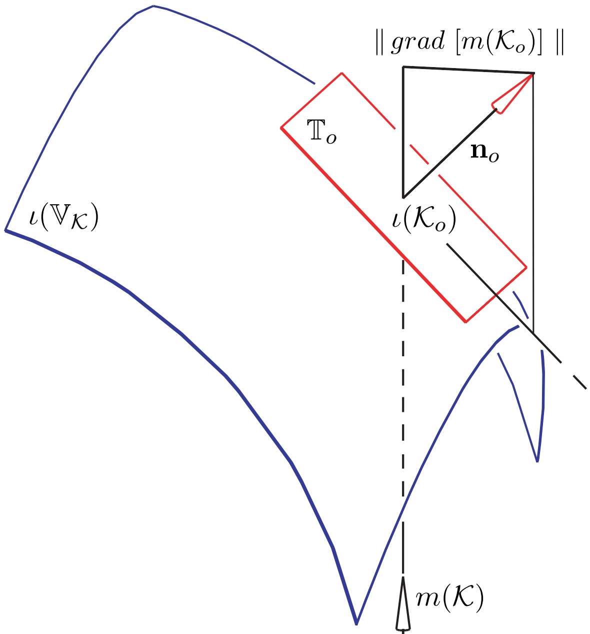
We look at the graph of the function $m(\mathcal{K}) := |det(\mathcal{J})|$ over the space $\mathbb{V}_{\mathcal{K}}$, i.e., the map:

$$\iota: \quad (\theta_2, .., \theta_5) \mapsto (\theta_2, .., \theta_5, |det(\mathcal{J})|).$$

The normal vector $\mathbf{n}_o$ at each point $\mathcal{K}_o$ of the hypersurface $\iota(\mathbb{V}_{\mathcal{K}})$ is given by

$$\mathbf{n}_o = (\alpha\frac{\partial\, det(\mathcal{J})}{\partial\theta_2}, \alpha\frac{\partial\, det(\mathcal{J})}{\partial\theta_3}, \alpha\frac{\partial\, det(\mathcal{J})}{\partial\theta_4}, \alpha\frac{\partial\, det(\mathcal{J})}{\partial\theta_5}, 1) \qquad \text{with} \qquad \alpha := -sgn\left[det(\mathcal{J})\right].$$

[4] With method 1 we were only able to give evidence about the worst instantaneous rotational direction of the joints by the vector $(sgn(l_2), sgn(l_3), sgn(l_4), sgn(l_5))$ resulting from $\mathbf{L}$.



If we project $\mathbf{n}_o$ orthogonally onto $\mathbb{V}_\mathcal{K}$, we obtain the negative gradient of $m(\mathcal{K})$, which points into the direction of the steepest descent. Thus our above defined $\boldsymbol{\omega}_-$ corresponds to $-grad\ [m(\mathcal{K})]$. Moreover, the intersection of the tangent space $\mathbb{T}_o$ at the point $\iota(\mathcal{K}_o)$ with $\mathbb{V}_\mathcal{K}$ is equal to the 3-space $\mathcal{H}$. Hence $D_1(\mathcal{K})$ and $D_2(\mathcal{K})$ can be seen as distances. $\mathcal{K}_o\mathbf{L}$ and $\mathcal{K}_o\mathbf{Q}$ (*see figure 1*) respectivly as explained above. $\mathbf{Q}$ is located on the slope line of $\mathbb{T}_o$. Now we are able to interpret case (**b**) as follows:
If all four partial derivatives of the Jacobian's determinant vanish, $grad\ [m(\mathcal{K}_o)] = \mathbf{o}$ and so $\mathbf{n}_o \perp \mathbb{V}_\mathcal{K}$, which is equivalent to $\mathbb{T}_o \parallel \mathbb{V}_\mathcal{K}$. Consequently, $\mathcal{H}$ is the ideal hyperplane of $\mathbb{V}_\mathcal{K}$. $\mathcal{K}_o$ has the maximal distance $D_1(\mathcal{K}_o) = D_2(\mathcal{K}_o) = \infty$, because $\mathbf{Q}$ and $\mathbf{L}$ are ideal points.

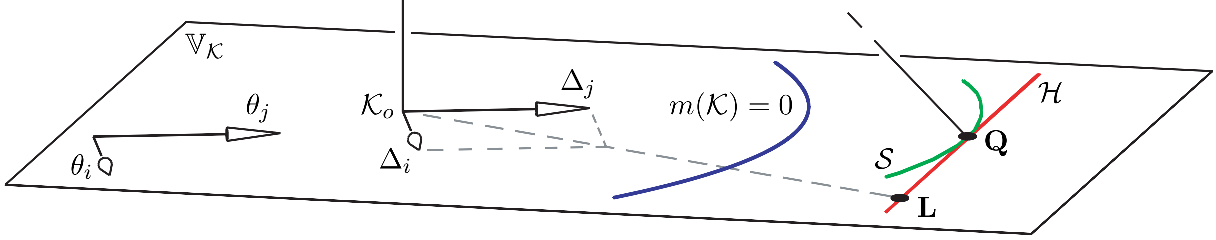


Figure 1: The graph of $|det(\mathcal{J})|$ over $\mathbb{V}_\mathcal{K}$

In cases (**a**) and (**b**) the special configuration $\mathcal{K}_o$ is a stationary point of the vector field $-grad\ [m(\mathcal{K}_o)]$. As a consequence ε is not defined in case (**b**). In order to evaluate $\boldsymbol{\omega}_o$ of $\mathcal{K}_o$, we have to look at the derivatives of order 2, thus the curvature of $\iota(\mathbb{V}_\mathcal{K})$ at $\iota(\mathcal{K}_o)$. So we can define $\boldsymbol{\omega}_-$ in $\mathcal{K}_o$ by the direction $\boldsymbol{\kappa}$ of the principal curvature with the smallest absolute value:

$$\boldsymbol{\omega}_- := \pm\boldsymbol{\kappa} \qquad \Longrightarrow \qquad \sphericalangle(\boldsymbol{\omega}_-, \boldsymbol{\omega}_o) \in \left[0, \frac{\pi}{2}\right] \qquad \Longrightarrow \qquad \varepsilon \in \left[-\frac{\pi}{2}, 0\right].$$

3.5 Does $D_i(\mathcal{K})$ $(i = 1, 2)$ fulfill the six properties $1. - 6.$?

$D_1(\mathcal{K})$ and $D_2(\mathcal{K})$ trivially fulfill the properties 1 and 2, and also the fifth, because the two new distance measures were the result of a geometric reasoning. Now we look at equation (10) to show the remaining properties: The equation's coefficients $det(\mathcal{J}_{i,j})$ are for the same reasons as the Jacobian's determinant invariant with respect to Euclidean motions (see (3)). Furthermore this equation is invariant under similarities, because if we change the scaling by the factor λ the whole equation is multiplied by λ^3. Due to the used approximation of direct kinematics, Δ_L and Δ_Q are computable in real-time, which is an important criterion for practical applications.

3.6 Weighted measures $\widehat{D}_1(\mathcal{K})$ and $\widehat{D}_2(\mathcal{K})$

Obviously equation (10) can also be written as:

$$det(\mathcal{J}) + \omega_{2|1}\sum_{j=3}^{6} det(\mathcal{J}_{2,j}) + \omega_{3|2}\sum_{j=4}^{6} det(\mathcal{J}_{3,j}) + \omega_{4|3}\sum_{j=5}^{6} det(\mathcal{J}_{4,j}) + \omega_{5|4}\, det(\mathcal{J}_{5,6}) = 0. \qquad (14)$$

We can replace the respective side condition by a weighted counterpart based on the rotationary energy, where the mass moments of inertia $\mathbf{I}_i > 0$ are as defined as in *subsection* 2.3. Of course, the posture of the end-effector has some influence on $\widehat{D}_i(\mathcal{K})$ because of the mass distribution.

• **Weighted measure** $\widehat{D}_2(\mathcal{K}) := \mathcal{E}_Q$: Now we want to optimize equation (14) such that the sum of the needed rotationary energy is minimal, i.e.,

$$\omega_{2|1}^2\mathbf{I}_2/2 + \omega_{3|2}^2\mathbf{I}_3/2 + \omega_{4|3}^2\mathbf{I}_4/2 + \omega_{5|4}^2\mathbf{I}_5/2 = \mathcal{E} \quad \longrightarrow \quad min. \qquad (15)$$

All quadrupels $(\omega_{2|1}, .., \omega_{5|4})$ which solve (15) for a fixed $\mathcal{E} \in \mathbb{R}^+$ are located on a hyperellipsoid, centered in $\mathcal{K}_o$. Obviously equation (15) is minimal if the hyperellipsoid touches the hyperplane $\mathcal{H}$ given by (14). So the coordinates of the point of contact $\mathbf{Q}$ yield the required $\mathcal{E}_Q$.

• **Weighted measure** $\widehat{D}_1(\mathcal{K}) := \mathcal{E}_L^2$: We solve (14) under the condition $|\omega_{i|i-1}\sqrt{\mathbf{I}_i/2}| \leq \mathcal{E}_L$, and $\mathcal{E}_L$ should be minimal. We get the required quadrupel $(\omega_{2|1}, .., \omega_{5|4})$ as the nonempty intersection of $\mathcal{H}$ and the smallest hyperbox centered in $\mathcal{K}_o$ with edges parallel to the coordinate axes. The hyperbox is determined by the following ratio of its edge lengths e_i:

$$e_2 : e_3 : e_4 : e_5 = \sqrt{2/\mathbf{I}_2} : \sqrt{2/\mathbf{I}_3} : \sqrt{2/\mathbf{I}_4} : \sqrt{2/\mathbf{I}_5}.$$

The nonempty set always contains the vertex $\mathbf{L}$ of the hyperbox, which coordinates yield the required $\mathcal{E}_L$. So the measure $\widehat{D}_1(\mathcal{K})$ equals approximately the smallest rotationary energy $\mathcal{E}_L^2$, which is needed in every rotary joint j $(j = 2, .., 5)$ to pass through a singular posture.

4 Example

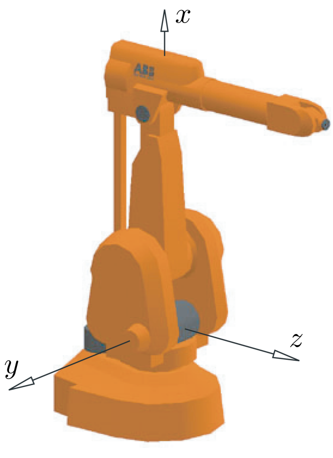


Figure 2: Initial position of the IRB 2000 robot

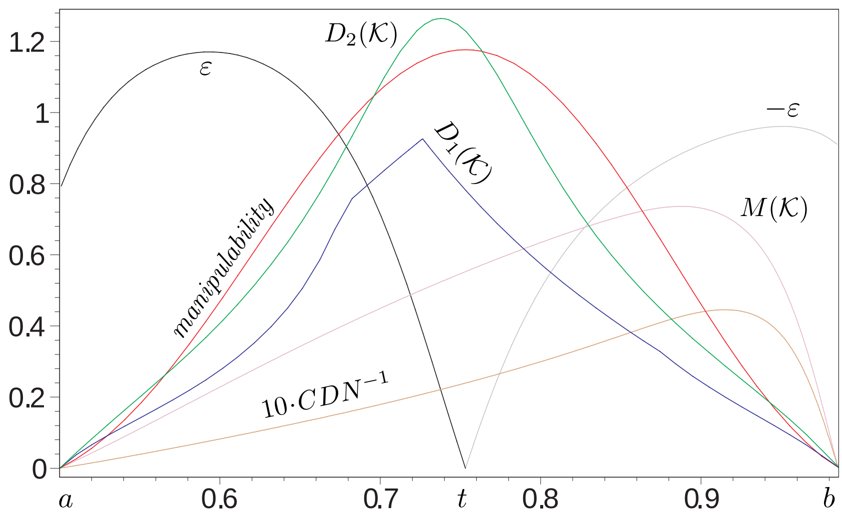


Figure 3: Graphs of manipulability, CDN^{-1}, $D_i(\mathcal{K})$, $M(\mathcal{K})$ and ε

We take the industrial robot *ABB IRB 2000* as a particular example (*see figure 2*). We look at the constrained motion which is determined by:

$$\theta_1 = 0, \quad \theta_2 = t\frac{21\pi}{18} - \frac{11\pi}{18}, \quad \theta_3 = t^2\frac{2\pi}{3} - \frac{\pi}{3}, \quad \theta_4 = t\pi - \frac{\pi}{2}, \quad \theta_5 = t\frac{4\pi}{3} - \frac{2\pi}{3}, \quad \theta_6 = 0.$$

For $t \in [a, b]$ the *manipulability*, CDN^{-1}, $D_i(\mathcal{K})$ and $M(\mathcal{K})$ are displayed in *figure 3*, where $\mathcal{K}(a)$ and $\mathcal{K}(b)$ are singular configurations. CDN^{-1} is computed by the concept of Gosselin (1990). The used reference points $\mathbf{P}_i$ $(i = 1, 2, 3)$ resp. the *tool center point* $\mathbf{P}_o$ have the following coordinates in the initial position:

$$\mathbf{P}_o := (835, 0, 1150) \quad \mathbf{P}_1 := (935, 0, 1150) \quad \mathbf{P}_2 := (785, 50\sqrt{3}, 1150) \quad \mathbf{P}_3 := (785, -50\sqrt{3}, 1150).$$

The inertia ellipsoid of $\mathcal{O}$, which is required for $M(\mathcal{K})$ is a sphere of radius 100 centered in $\mathbf{P}_o$.

5 Conclusion

In this article we presented three new measures $D_1(\mathcal{K})$, $D_2(\mathcal{K})$ and $M(\mathcal{K})$ which reflect the distance of the instantaneous configuration $\mathcal{K}$ to the nearest singularity. Therefore they can be seen as performance indices which obey the initially stated six conditions. The distance measures $D_i(\mathcal{K})$ take the possible variation of the rotary axes into account, because they are based on a linearized approximation of direct kinematics. $M(\mathcal{K})$ depends on the object to manipulate because it is based on an object-oriented metric in the work space.

A similar approach for measures on parallel manipulators is in preparation.

Acknowledgment

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