# Special cases of Schönflies-singular planar Stewart Gough platforms

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**Abstract.** Parallel manipulators which are singular with respect to the Schönflies motion group $\mathsf{X}(\mathsf{a})$ are called Schönflies-singular, or more precisely $\mathsf{X}(\mathsf{a})$-singular, where $\mathsf{a}$ denotes the rotary axis. A special class of such manipulators are architecturally singular ones because they are singular with respect to any Schönflies group. Another remarkable set of Schönflies-singular planar parallel manipulators of Stewart Gough type was already presented by the author in [6]. Moreover the main theorem on these manipulators was given in [7]. In this paper we give a complete discussion of the remaining special cases which also include so-called Cartesian-singular planar manipulators as side-product.

**Key words:** Schönflies-singular, Schönflies motion group, Stewart Gough platform, singularities

## 1 Introduction

The Schönflies motion group $\mathsf{X}(\mathsf{a})$ consists of three linearly independent translations and all rotations about a fixed axis $\mathsf{a}$. This 4-dimensional group is of importance in practice because it is well adapted for pick-and-place operations.

The geometry of a planar parallel manipulator of Stewart Gough type (SG type) is given by the six base anchor points $\mathsf{M}_i \in \Sigma_0$ with coordinates $\mathbf{M}_i := (A_i, B_i, 0)^T$ and by the six platform anchor points $\mathsf{m}_i \in \Sigma$ with coordinates $\mathbf{m}_i := (a_i, b_i, 0)^T$. By using Euler Parameters $(e_0, e_1, e_2, e_3)$ for the parametrization of the spherical motion group $\mathsf{SO}(3)$ the coordinates $\mathbf{m}_i'$ of the platform anchor points with respect to the fixed space can be written as $\mathbf{m}_i' = K^{-1}\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t}$ with

$$\mathbf{R} := (r_{ij}) = \begin{pmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2-e_0e_3) & 2(e_1e_3+e_0e_2) \\ 2(e_1e_2+e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3-e_0e_1) \\ 2(e_1e_3-e_0e_2) & 2(e_2e_3+e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{pmatrix}, \qquad (1)$$

the translation vector $\mathbf{t} := (t_1, t_2, t_3)^T$ and $K := e_0^2+e_1^2+e_2^2+e_3^2$.

It is well known that a SG platform is singular if and only if $Q := det(\mathbf{Q}) = 0$ holds, where the $i^{th}$ row of the $6\times 6$ matrix $\mathbf{Q}$ equals the Plücker coordinates $\underline{\mathbf{l}}_i := (\mathbf{l}_i, \widehat{\mathbf{l}}_i) := (\mathbf{m}'_i - \mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$ of the $i^{th}$ carrier line.

### 1.1 Related work and notation

For the determination of $\mathsf{X}(\mathsf{a})$-singular planar parallel manipulators we distinguish the following cases depending on the angle $\alpha \in [0, \pi/2]$ between the axis $\mathsf{a}$ and the carrier plane $\Phi$ of the base anchor points and the angle $\beta \in [0, \pi/2]$ between $\mathsf{a}$ and the carrier plane $\varphi$ of the platform anchor points:

1. $\alpha \neq \beta$: (a) $\alpha = \pi/2$, $\beta \in [0, \pi/2[$ (b) $\alpha, \beta \in [0, \pi/2[$
2. $\alpha = \beta$: (a) $\alpha = \pi/2$ (b) $\alpha = ]0, \pi/2[$ (c) $\alpha = 0$

Every $\mathsf{X}(\mathsf{a})$-singular manipulator belongs to one of these 5 cases (after exchanging platform and base). Due to the theorem given in [7] the manipulators of the solution set of case (1a) presented in [6] are the only $\mathsf{X}(\mathsf{a})$-singular ones with $\alpha \neq \beta$ which are not architecturally singular. In this paper we discuss the remaining special cases with $\alpha = \beta$.

In the following we use the notation introduced in [6]. We denote the determinant of certain $j \times j$ matrices as follows:

$$|\mathbf{X}, \mathbf{y}, \ldots, \mathbf{Xy}|_{(i_1, i_2, \ldots, i_j)} := det(\mathbf{X}_{(i_1, i_2, \ldots, i_j)}, \mathbf{y}_{(i_1, i_2, \ldots, i_j)}, \ldots, \mathbf{Xy}_{(i_1, i_2, \ldots, i_j)}) \tag{2}$$

$$\text{with} \quad \mathbf{X}_{(i_1, i_2, \ldots, i_j)} = \begin{bmatrix} X_{i_1} \\ X_{i_2} \\ \vdots \\ X_{i_j} \end{bmatrix}, \mathbf{y}_{(i_1, i_2, \ldots, i_j)} = \begin{bmatrix} y_{i_1} \\ y_{i_2} \\ \vdots \\ y_{i_j} \end{bmatrix}, \mathbf{Xy}_{(i_1, i_2, \ldots, i_j)} = \begin{bmatrix} X_{i_1} y_{i_1} \\ X_{i_2} y_{i_2} \\ \vdots \\ X_{i_j} y_{i_j} \end{bmatrix} \tag{3}$$

and $(i_1, i_2, \ldots, i_j) \in \{1, \ldots, 6\}$ with $i_1 < i_2 < \ldots < i_j$. Moreover it should be noted that we write $|\mathbf{X}, \mathbf{y}, \ldots, \mathbf{Xy}|_{i_1}^{i_j}$ if $i_{k+1} = i_k + 1$ for $k = 1, \ldots, j-1$ hold.

The algebraic condition that $\mathsf{M}_i, \mathsf{M}_j, \mathsf{M}_k$ or $\mathsf{m}_i, \mathsf{m}_j, \mathsf{m}_k$ are collinear is denoted by $C_{(i,j,k)} := |\mathbf{1}, \mathbf{A}, \mathbf{B}|_{(i,j,k)} = 0$ and $c_{(i,j,k)} := |\mathbf{1}, \mathbf{a}, \mathbf{b}|_{(i,j,k)} = 0$, respectively. It should also be said that in the latter done case study we always factor out the homogenizing factor $K$ if possible. Moreover we give the number $n$ of terms of not explicitly given polynomials $F$ in square brackets, i.e. $F[n]$.

## 2 Case (2a)

**Theorem 1.** *A non-architecturally singular planar SG platform, where the axis* $\mathsf{a}$ *is orthogonal to* $\varphi$ *and* $\Phi$*, is* $\mathsf{X}(\mathsf{a})$*-singular if and only if* $|\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{a}, \mathbf{b}, \mathbf{Ab} - \mathbf{Ba}|_1^6 = 0$ *and* $|\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{a}, \mathbf{b}, \mathbf{Aa} + \mathbf{Bb}|_1^6 = 0$ *are fulfilled.*

*Proof.* Without loss of generality (w.l.o.g.) we can choose Cartesian coordinate systems in $\Sigma$ and $\Sigma_0$ such that $A_1 = B_1 = B_2 = a_1 = b_1 = b_2 = 0$ hold. As both carrier planes are orthogonal to the axis we set $e_1 = e_2 = 0$ and compute the condition $Q := det(\mathbf{Q}) = 0$ as given in Section 1. $Q$ splits up into $z^3K^2[F_1(e_0^2 - e_3^2) + 2F_2e_0e_3]$ where $F_1$ and $F_2$ are the two conditions given in Theorem 1. □

*Remark 1.* A geometric interpretation of these 2 conditions is still missing. Moreover, it should be noted that the manipulators of the solution set of case (1a) also fulfill $F_1 = F_2 = 0$ due to their property $rk(\mathbf{1},\mathbf{A},\mathbf{B},\mathbf{a},\mathbf{b})_1^6 = 4$ (cf. [6]). ⋄

## 3 Case (2b)

**Theorem 2.** *A non-architecturally singular planar manipulator with* $0 \neq \alpha = \beta \neq \pi/2$ *is* $\mathsf{X}(\mathsf{a})$*-singular if and only if in a configuration with coinciding carrier planes the anchor points* $\{\mathsf{M}_i\}$ *and* $\{\mathsf{m}_i\}$ *are within an indirect similarity, which is the product of a dilation and the reflection on the orthogonal projection of* $\mathsf{a}$ *onto* $\Phi = \varphi$*.*

*Proof.* The proof of this theorem is given in the following two parts:

**Part [A]** *No four anchor points are collinear:*

Due to [2, 6] we can always choose coordinate systems in the platform and the base such that $a_2A_2B_3B_4B_5c_{(3,4,5)}(a_3 - a_4)(b_3 - b_4) \neq 0$ hold. Now we must distinguish again two cases, depending on whether $\gamma > \alpha$ or $\gamma = \alpha$ holds with $\gamma := \angle([\mathsf{M}_1, \mathsf{M}_2], \mathsf{a}) \in [0, \pi/2]$.

<u>Case $\gamma > \alpha$:</u>

Under this assumption we can rotate the platform about $\mathsf{a}$ such that the common line $\mathsf{s}$ of $\Phi$ and $\varphi$ is parallel to $[\mathsf{M}_1, \mathsf{M}_2]$. Therefore we can use the same coordinatisation as in the proof of Theorem 1 of [7], namely: $\mathbf{M}_i = (A_i, B_i, 0)$ and $\mathbf{m}_i = (a_i, b_i\cos\delta, b_i\sin\delta)$ with $A_1 = B_1 = B_2 = a_1 = b_1 = 0$ and $\sin\delta \neq 0$. We set $e_1 = e_4\cos\mu$, $e_3 = e_4\sin\mu$ and $e_2 = e_4n$ where $e_4$ is the homogenizing factor.

Therefore we only have to consider those cases in the proof of Theorem 1 of [7] which yield the contradiction $\alpha = \beta$. There is exactly one such case which will be discussed here in more detail. As given in [7] this case is characterized by $K_1 = K_2 = K_4 = 0$, $b_2 = 0$, $e_2 \neq 0$ and $b_i = b_3B_i/B_3$ for $i = 4, 5$ with

$$K_1 = |\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{a}|_2^6, \quad K_2 = |\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{b}|_2^6, \quad K_4 = |\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{Ab}|_2^6. \quad (4)$$

We proceed by computing $Q_{020}^{51} = 0$ where $Q_{ijk}^{uv}$ denotes the coefficient of $t_1^it_2^jt_3^ke_0^ue_4^v$ of $Q$. Its only non-contradicting factor can be solved for $A_4$ w.l.o.g.. Then $Q_{020}^{42} = 0$ implies an expression for $A_5$. Due to $Q_{010}^{71} = 0$ we must distinguish two cases:

1. $A_2 = a_2$: Solving the only non-contradicting factor of $Q_{100}^{62} = 0$ for $n$ yields $b_3\sin\mu\sin\delta/(B_3 - b_3\cos\delta)$. Note that for $B_3 = b_3\cos\delta$ the coefficient $Q_{100}^{62}$ cannot vanish without contradiction (w.c.). We proceed by expressing $A_3$ from

$Q^{44}_{100} = 0$, which can also be done w.l.o.g.. Now $Q^{35}_{100}$ and $Q^{26}_{100}$ can only vanish w.c. for:

a. $B_3 = b_3$: This yields a solution and the platform and the base are congruent.
b. $B_3 = -b_3$: We get again a solution; but now we have an indirect congruence.
c. $\cos\mu = 0$: In this case $Q^{35}_{010} = 0$ yields the contradiction.

2. $A_2 b_3(n\cos\delta + \sin\delta\sin\mu) - n a_2 B_3 = 0, A_2 \neq a_2$: As $Q^{71}_{010} = 0$ cannot vanish w.c. for $B_3 = b_3 A_2 \cos\delta / a_2$ we can solve the above condition for $n$ w.l.o.g.. Moreover we can express $A_3$ w.l.o.g. from the only non-contradicting factor of $Q^{62}_{010} = 0$. Now $Q^{44}_{100}$ and $Q^{35}_{100}$ can only vanish w.c. for:

   a. $b_3 = B_3 a_2 / A_2$: This yields a solution and the platform and the base are similar.
   b. $b_3 = -B_3 a_2 / A_2$: We get a solution; but now we have an indirect similarity.

Note that the solutions (1a) and (1b) as well as (2a) and (2b) are identical if one takes the relative position of $\varphi$, $\Phi$ and a into consideration.

<u>Case $\gamma = \alpha$:</u>

For this case we can use the same coordinatisation as in the case $\gamma > \alpha$ but now we have $e_2 = \delta = 0$. In the first step we show that $K_1 = K_2 = 0$ (cf. Eq. (4)) must hold. Therefore we compute $Q[30768]$ in its general form. Computation of the following two linear-combinations already yields the result:

$$K_1 = \frac{Q^{51}_{002}}{2\omega_{(0,0,1)}} + \frac{Q^{42}_{011}}{4\omega_{(0,0,2)}} + \frac{Q^{33}_{020}}{8\omega_{(0,0,3)}}, \quad K_2 = \frac{Q^{24}_{200}}{8\omega_{(0,1,3)}} - \frac{Q^{42}_{101}}{4\omega_{(0,0,2)}},$$

with $\omega_{(i,j,k)} := \sin\delta^i \sin\mu^j \cos\mu^k$. Moreover, due to $8b_2(K_4 + K_2) = Q^{53}_{100}/\omega_{(0,0,3)}$ we must distinguish the following two cases:

1. $b_2 \neq 0$ ($\Rightarrow K_4 = 0$): We compute $Q$ in dependency of $K_1, \ldots, K_4$ as given in the proof of Theorem 1 of [7] with $K_3 := |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Aa}|^6_2$. By setting $K_1 = K_2 = K_4 = 0$ we end up with $Q = K_3 e_4 \sin\mu A_2 F[2646]$. As $K_1 = K_2 = K_3 = K_4 = 0$ indicate the architecturally singularity (cf. [2]) we prove that $F$ cannot vanish w.c.. The resultant of $F^{05}_{200}$ and $F^{14}_{200}$ with respect to $a_2$ implies $|\mathbf{a}, \mathbf{b}, \mathbf{B}|^5_3 = 0$.

   a. $c_{(1,3,4)} \neq 0$: W.l.o.g. we can express $B_5$ from $|\mathbf{a}, \mathbf{b}, \mathbf{B}|^5_3 = 0$. Back-substitution yields $F^{05}_{200} = H[4]G_1[8]$ and $F^{14}_{200} = H[4]G_2[8]$, respectively. It is not difficult to verify that $G_1 = G_2 = 0$ yields a contradiction. Therefore we express $b_4$ from $H := B_3(a_2 b_4 - b_2 a_4) - B_4(a_2 b_3 - b_2 a_3)$.
      i. Assuming $b_3 \neq b_5$ we can compute $A_4$ from $F^{14}_{001} = 0$ and $A_5$ form $F^{14}_{020}$.
      ii. For $b_3 = b_5$ we can compute $A_5$ from $F^{14}_{001} = 0$ and $A_4$ form $F^{14}_{020}$.
   In both cases we proceed by expressing $A_6$ and $b_6$ from $K_1 = K_2 = 0$. Now we get $b_2 K_3 = a_2 K_4$. This is a contradiction as $K_4 = 0$ yields $K_3 = 0$.
   b. $c_{(1,3,4)} = 0$: W.l.o.g. we can express $a_3$ from $c_{(1,3,4)} = 0$. Then $|\mathbf{a}, \mathbf{b}, \mathbf{B}|^5_3 = 0$ yields $B_3 = b_3 B_4 / b_4$ and $F^{14}_{200} = 0$ implies an expression for $b_5$. Moreover, we can compute $A_5$ from $F^{14}_{110} = 0$. Now $F^{14}_{020}$ can only vanish w.c. for $A_3 = b_3 A_4 / b_4$. After computing $A_6$ and $b_6$ from $K_1 = K_2 = 0$, we get again the contradiction $b_2 K_3 = a_2 K_4$.

2. $b_2 = 0$: Here we distinguish again two cases:
   a. $K_4 = 0$: This assumption yields $Q = K_3 e_4 \sin\mu A_2 F[1338]$. The resultant of $F_{200}^{05}$ and $F_{200}^{14}$ with respect to $B_5$ can only vanish w.c. for $b_5|\mathbf{b},\mathbf{B}|_3^4 = 0$.
   For $b_5 = 0$ we get $B_3 = B_4$ from $F_{200}^{14} = 0$ and $F_{200}^{05} = 0$ yields the contradiction. Therefore we can set $B_3 = b_3B_4/b_4$ because not both $b_i$ $(i = 3,4)$ can be equal zero. Moreover we can assume $b_5 \neq 0$. Then from $F_{200}^{14} = 0$ we get $B_5 = b_5B_4/b_4$. We proceed by computing $A_5$ from $F_{110}^{14} = 0$. Now $F_{020}^{14} = 0$ implies an expression for $A_4$. Due to $F_{100}^{34} = 0$ we must distinguish two cases:
      i. $B_4 = -b_4A_2/a_2$, $A_2 \neq a_2$: In this case the 3 equations $F_{100}^{25} = 0$, $K_1 = 0$, $K_2 = 0$ imply an indirect similarity.
      ii. For $B_4 = -b_4$ the 4 equations $F_{010}^{43} = 0$, $F_{010}^{25} = 0$, $K_1 = 0$, $K_2 = 0$ imply an indirect congruence.
   b. $K_4 \neq 0$: We start by considering $Q_{200}^{24} = 0$ which implies $|\mathbf{b},\mathbf{B},\mathbf{Bb}|_3^5 = 0$. Now $|\mathbf{b},\mathbf{B},\mathbf{Bb}|_3^5 = 0$ cannot be solved for any $b_i$ if and only if $B_3 = B_4 = B_5$ hold. But for this special case we get the contradiction from $Q_{200}^{15} = 0$.
   Therefore we can assume w.l.o.g. that we can express $b_3$ from $|\mathbf{b},\mathbf{B},\mathbf{Bb}|_3^5 = 0$. Now $Q_{020}^{33}$ can only vanish w.c. for $(B_4 - B_5)G[16] = 0$. As for $B_4 = B_5$ we get the contradiction from $Q_{200}^{15}$ we set $G[16] = 0$. From this condition we can compute $a_3$ w.l.o.g.. Then $Q_{200}^{15}$ and $Q_{020}^{24}$ can only vanish w.c. for $|\mathbf{b},\mathbf{B}|_4^5 = 0$ and $|\mathbf{a},\mathbf{b},\mathbf{A}|_{(2,4,5)} = 0$, respectively. From these conditions we can express $A_5$ and $B_5$ w.l.o.g.. Moreover we can compute $B_6$ from $K_2 = 0$ which yields $b_6K_1 = K_4$, a contradiction.

**Part [B]** *Four anchor points are collinear:*

Similar considerations as for part [A] show that possible solutions of this problem must yield the contradiction $\alpha = \beta$ in the proof of Theorem 2 of [7]. But there do not exist such a contradiction in the mentioned proof.

There is only one case which is not covered by the proof of Theorem 2 of [7], namely the following one: $\mathsf{M}_1,\dots,\mathsf{M}_4$ collinear and $\mathsf{m}_1,\dots,\mathsf{m}_4$ collinear with $\alpha = \angle([\mathsf{M}_1,\dots,\mathsf{M}_4],\mathsf{a}) = \angle([\mathsf{m}_1,\dots,\mathsf{m}_4],\mathsf{a}) = \beta$. For this case we can use the same coordinatisation as in the case $\gamma > \alpha$ by setting $B_3 = B_4 = b_2 = b_3 = b_4 = e_2 = \delta = 0$. Now $Q$ splits up into $e_4 \sin\mu(ye_4\cos\mu - ze_0)H[6]F[56]$ where $H = 0$ indicates item 8 of Karger's list of architecturally singular manipulators given in Theorem 3 of [3]. From $F_{002}^{20} = 0$ we get $b_5 = b_6B_5/B_6$. Then $F_{100}^{13} = 0$ and $F_{100}^{04} = 0$ imply $B_5 = B_6$ and $a_5 = a_6$, respectively. Finally, $F_{001}^{22} = 0$ yields the contradiction. □

*Remark 2.* The manipulators of Theorem 2 are so-called equiform platforms. The singularities and self-motions of these manipulators were extensively studied by Karger [1]. For the special case of congruent platforms see also Karger [4].

In part [B] of the discussion we get no solution because an equiform manipulator with four collinear anchor points is already architecturally singular (cf. [3]). ⋄

## 4 Case (2c)

**Theorem 3.** *A non-architecturally singular planar SG platform is* $\mathsf{X}(\mathsf{a})$*-singular, where* $\mathsf{a}$ *is parallel to the x-axes of the fixed and moving system, if and only if one of the following cases hold:* $(1)\, rk(\mathbf{1},\mathbf{b},\mathbf{B},\mathbf{Bb})_1^6 = 2$*,* $(2)\, rk(\mathbf{1},\mathbf{b},\mathbf{B},\mathbf{Bb},\mathbf{A}-\mathbf{a})_1^6 = 3$ *or* $(3)\ rk(1,\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{a},\mathbf{b},\mathbf{Ab})_1^6 = 5$.

*Proof.* We can choose coordinate systems such that $\mathbf{M}_i = (A_i, B_i, 0)$ and $\mathbf{m}_i = (a_i, b_i, 0)$ with $A_1 = B_1 = a_1 = b_1 = 0$ hold. Now we can compute $Q$ in its general form according to Section 1 under consideration of $e_2 = e_3 = 0$. The necessity of $K_1$ and $K_2$ given in Eq. (4) follows directly from $Q_{002}^{51} + Q_{002}^{15} + Q_{002}^{33}$ and $Q_{101}^{42} + Q_{101}^{24}$, respectively, where $Q_{ijk}^{uv}$ denotes the coefficient of $t_1^i t_2^j t_3^k e_0^u e_1^v$ of $Q$. In the following we split up the proof of the necessity into two parts:

**Part [A]** $rk(\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb})_2^5 = 4$

Under this assumption we can perform the generalized version of the matrix manipulation given by Karger in [2]. The 5 steps of the generalization are given by:

(a) $\underline{\mathbf{l}}_i := \underline{\mathbf{l}}_i - \underline{\mathbf{l}}_1 \quad i = 2,\ldots,6$

(b) $\underline{\mathbf{l}}_i := \underline{\mathbf{l}}_i - \underline{\mathbf{l}}_2\, A_i/A_2 \quad i = 3,\ldots,6$

(c) $\underline{\mathbf{l}}_i := \underline{\mathbf{l}}_i - \underline{\mathbf{l}}_3 \dfrac{|\mathbf{A},\mathbf{B}|_{(2,i)}}{|\mathbf{A},\mathbf{B}|_2^3} \quad i = 4,5,6$

(d) $\underline{\mathbf{l}}_i := \underline{\mathbf{l}}_i - \underline{\mathbf{l}}_4 \dfrac{|\mathbf{A},\mathbf{B},\mathbf{Ba}|_{(2,3,i)}}{|\mathbf{A},\mathbf{B},\mathbf{Ba}|_2^4} \quad i = 5,6$

(e) $\underline{\mathbf{l}}_6 := \underline{\mathbf{l}}_6 - \underline{\mathbf{l}}_5 \dfrac{|\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb}|_{(2,3,4,6)}}{|\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb}|_2^5}$.

Then $\underline{\mathbf{l}}_6$ has the from $(v_1, v_2, v_3, 0, -w_3, w_2)$ with $v_i := r_{i1}K_1 + r_{i2}K_2$ and $w_j := r_{j1}K_3 + r_{j2}K_4$ with $K_3 := |\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{Aa}|_2^6$ and $K_4$ of Eq. (4).

Due to the above shown necessity of $K_1 = K_2 = 0$ we can set them equal to zero and compute $Q = K_4 F[1032]$ where $F$ do not depend on $K_3$ and $K_4$. Therefore there a two possibilities. For $K_4 = 0$ we get solution (3).

In the second case we have to consider the conditions under which $F$ is fulfilled identically. It can easily be seen that there are only 7 such conditions, namely:

$$P_1 := Q_{100}^{53} = |\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{b}|_2^5 = 0 \qquad P_2 := Q_{003}^{40} = |\mathbf{A},\mathbf{B},\mathbf{a},\mathbf{b}|_2^5 = 0$$
$$P_3 := Q_{001}^{62} - Q_{001}^{26} = |\mathbf{Ba},\mathbf{Bb},\mathbf{b},\mathbf{a}-\mathbf{A}|_2^5 = 0 \qquad P_4 := Q_{002}^{33} = |\mathbf{A},\mathbf{Bb},\mathbf{a},\mathbf{b}|_2^5 = 0$$
$$P_5 := Q_{001}^{62} + Q_{001}^{26} = |\mathbf{Ba},\mathbf{Bb},\mathbf{B},\mathbf{a}-\mathbf{A}|_2^5 = 0 \qquad P_6 := Q_{101}^{42} = |\mathbf{B},\mathbf{Bb},\mathbf{a},\mathbf{b}|_2^5 = 0$$
$$P_7 := Q_{011}^{42} = |\mathbf{B},\mathbf{Ba},\mathbf{b},\mathbf{a}-\mathbf{A}|_2^5 - |\mathbf{B},\mathbf{Bb},\mathbf{A},\mathbf{a}|_2^5 = 0$$

For the discussion of this system of equations we distinguish two cases:

1. $rk(\mathbf{b},\mathbf{B},\mathbf{Bb})_2^5 = 3$: We get $\mathbf{a}_2^5 = \lambda_a \mathbf{b}_2^5 + \mu_a \mathbf{B}_2^5 + \nu_a \mathbf{Bb}_2^5$ with $(\mu_a, \nu_a) \neq (0,0)$ from $P_6$. As a consequence $P_2$ and/or $P_4$ equal/s $|\mathbf{b},\mathbf{Bb},\mathbf{B},\mathbf{A}|_2^5 = 0$. This yields together with $P_1$ the relation $rk(\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{a})_2^5 = 3$, a contradiction.
2. $rk(\mathbf{b},\mathbf{B},\mathbf{Bb})_2^5 < 3$: Now $rk(\mathbf{b},\mathbf{B},\mathbf{Bb})_2^5 = 2$ must hold due to $rk(\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb})_2^5 = 4$. Therefore the vectors $\mathbf{B}$ and $\mathbf{Bb}$ are linearly independent and we can set $\mathbf{b}_2^5 =$

$\lambda_b\mathbf{B}_2^5+\mu_b\mathbf{Bb}_2^5$. It can easily be seen that for any linear combination of $\mathbf{b}$ only the following four conditions remain, namely:

$$|\mathbf{A},\mathbf{B},\mathbf{a},\mathbf{Bb}|_2^5=0, \quad |\mathbf{Ba},\mathbf{Bb},\mathbf{B},\mathbf{a}-\mathbf{A}|_2^5=0 \quad \text{and} \quad K_1=K_2=0. \tag{5}$$

As the vectors $\mathbf{A}_2^5$, $\mathbf{B}_2^5$ and $\mathbf{Bb}_2^5$ are linearly independent due to the assumption $rk(\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb})_2^5=4$ we can set $\mathbf{a}_2^5=\lambda_a\mathbf{B}_2^5+\mu_a\mathbf{Bb}_2^5+\nu_a\mathbf{A}_2^5$ without loss of generality. Now the remaining three equations can only vanish for:

a. $|\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb}|_2^5=0$ which is a contradiction or for
b. $\nu_a=1$, $b_6=\lambda_bB_6+\mu_bB_6b_6$ and $a_6=\lambda_aB_6+\mu_aB_6b_6+\nu_aA_6$, which yield solution (2).

This finishes this part. For the remaining one we can assume that there do not exist any $i,j,k,l\in\{2,\dots,6\}$ with $|\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb}|_{(i,j,k,l)}\neq 0$.

**Part [B]** $rk(\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb})_2^6<4$

Computation of $Q$ shows that it vanishes independently of $t_1,t_2,t_3,e_0,e_1$ under consideration of $rk(\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb})_2^6<4$ if and only if the following 9 conditions are fulfilled:

$$R_1:=Q_{003}^{40}-Q_{003}^{04}=|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Ab}|_2^6 \qquad R_2:=Q_{102}^{31}=|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Bb}|_2^6$$
$$R_3:=Q_{003}^{04}+Q_{003}^{04}=|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Ba}|_2^6 \qquad R_4:=Q_{020}^{33}=|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{Ab},\mathbf{Bb}|_2^6$$
$$R_5:=Q_{101}^{62}+Q_{101}^{26}=|\mathbf{a},\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{Ab}|_2^6 \qquad R_6:=Q_{110}^{33}=|\mathbf{a},\mathbf{b},\mathbf{B},\mathbf{Ab},\mathbf{Bb}|_2^6$$
$$R_7:=Q_{110}^{33}=|\mathbf{b},\mathbf{Ba},\mathbf{Bb},\mathbf{Ab},\mathbf{A}-\mathbf{a}|_2^6 \qquad R_8:=Q_{020}^{33}=|\mathbf{b},\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{Ab}|_2^6$$
$$R_9:=Q_{002}^{40}+Q_{002}^{04}=|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{Ba},\mathbf{Bb}|_2^6+|\mathbf{a},\mathbf{A},\mathbf{B},\mathbf{Ab},\mathbf{Bb}|_2^6+|\mathbf{b},\mathbf{B},\mathbf{Ab},\mathbf{Ba},\mathbf{A}-\mathbf{a}|_2^6$$

In this part we distinguish the following three subcases:

1. $rk(\mathbf{B},\mathbf{Ba},\mathbf{Bb})_2^6=3$: As $rk(\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb})_2^6=3$ must hold we can set $\mathbf{A}_2^6=\lambda_A\mathbf{B}_2^6+\mu_A\mathbf{Ba}_2^6+\nu_A\mathbf{Bb}_2^6$ with $(\mu_A,\nu_A)\neq(0,0)$. Then $R_2$ and/or $R_3$ equals the determinant of $(\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{a},\mathbf{b})_2^6$. If the rank of this matrix is 3 we get solution (3). Therefore we can assume rank 4. For $rk(\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{b})_2^6=4$ we get solution (3) from $R_8$. If $rk(\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{a})_2^6=4$ holds we get solution (3) from $R_5$.
2. $rk(\mathbf{B},\mathbf{Ba},\mathbf{Bb})_2^6=2$: We already get solution (3) if $rk(\mathbf{U},\mathbf{V},\mathbf{a},\mathbf{b})_2^6=2$ with $\mathbf{U},\mathbf{V}\in\{\mathbf{B},\mathbf{Ba},\mathbf{Bb}\}$ and $rk(\mathbf{U},\mathbf{V})_2^6=2$ holds. Therefore we assume $rk(\mathbf{U},\mathbf{V},\mathbf{a},\mathbf{b})_2^6=4$:

   a. $(\mathbf{U},\mathbf{V})=(\mathbf{B},\mathbf{Bb})$: Solution (3) is implied by $R_2$ and $R_6$.
   b. $(\mathbf{U},\mathbf{V})=(\mathbf{B},\mathbf{Ba})$: Now $R_3$ implies $rk(\mathbf{A},\mathbf{B},\mathbf{Ba},\mathbf{Bb},\mathbf{a},\mathbf{b})_2^6=4$. For $\mathbf{Bb}_2^6=\lambda_{Bb}\mathbf{B}_2^6+\mu_{Bb}\mathbf{Ba}_2^6$ with $\mu_{Bb}\neq0$ we get solution (3) from $R_6$. For $\mu_{Bb}=0$ and $\lambda_{Bb}\neq0$ we get it from $R_7$. For $\mu_{Bb}=\lambda_{Bb}=0$ we get solution (3) from $R_9$.

   In the remaining case of $rk(\mathbf{U},\mathbf{V},\mathbf{a},\mathbf{b})=3$ we can set $\mathbf{x}_2^6=\lambda_x\mathbf{U}_2^6+\mu_x\mathbf{V}_2^6+\nu_x\mathbf{y}_2^6$ with $(\lambda_x,\mu_x)\neq(0,0)$, $\mathbf{x},\mathbf{y}\in\{\mathbf{a},\mathbf{b}\}$ and $\mathbf{x}\neq\mathbf{y}$:

   a. $(\mathbf{U},\mathbf{V})=(\mathbf{B},\mathbf{Bb})$: $R_1$ resp. $R_4$ implies solution (3) for $\mu_x\neq0$ resp. $\lambda_x\neq0$.
   b. $(\mathbf{U},\mathbf{V})=(\mathbf{B},\mathbf{Ba})$: For $\mu_x\neq0$ we get solution (3) from $R_1$. For the case $\mu_x=0$, $\nu_x\neq0$ solution (3) is implied by $R_9$ for $\mathbf{Bb}=\mathbf{0}$, by $R_7$ for $\mathbf{Bb}=\mathbb{R}\mathbf{B}$ and

by $R_4$ in all other cases. The case $\mu_x = \nu_x = 0$ is the same as the last one for $\mathbf{x} = \mathbf{a}$ and $\mathbf{y} = \mathbf{b}$. But for $\mathbf{x} = \mathbf{b}$, $\mathbf{y} = \mathbf{a}$ and $\mu_x = \nu_x = 0$ we get solution (1) for $\mathbf{Bb} = \mathbf{0}$ or $\mathbf{Bb} = \mathbb{R}\mathbf{B}$. In all other cases $R_4$ implies solution (3).

3. $rk(\mathbf{B}, \mathbf{Ba}, \mathbf{Bb})_2^6 = 1$: In this case $R_1$ implies solution (3).

As $\mathbf{B}_2^6 \neq \mathbf{0}$ must hold the case $(\mathbf{U}, \mathbf{V}) = (\mathbf{Ba}, \mathbf{Bb})$ need not be discussed in item 2. This finishes the proof of the necessity of the conditions of solution (1), (2) and (3). The sufficiency of these conditions is proven in the next section.

## *4.1 Sufficiency of the conditions and their geometric meaning*

The sufficiency of the conditions of solution (3) is proven geometrically according to the method introduced by Röschel and Mick [9]. This method was also used in [6] to prove the sufficiency of the conditions characterizing the $\mathsf{X}(\mathsf{a})$-singular planar SG platforms where $\mathsf{a}$ is orthogonal to one of the carrier planes of the anchor points. For readers who are not familiar with line geometry we refer to [8].

All lines of a linear line complex $\mathscr{C}$ with homogeneous coordinates $(c_1 : \ldots : c_6)$ correspond with the null-lines of a null-polarity $\kappa$. This linear mapping $\kappa$ maps the point $\mathsf{P}$ with homogeneous coordinates $(p_0 : \ldots : p_3)$ onto the plane $\kappa(\mathsf{P})$ with homogeneous coordinates $[\xi_0 : \ldots : \xi_3]$ by

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & -c_4 & -c_5 & -c_6 \\ c_4 & 0 & -c_3 & c_2 \\ c_5 & c_3 & 0 & -c_1 \\ c_6 & -c_2 & c_1 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}. \tag{6}$$

If we restrict $\kappa$ to the points $\mathsf{M}_i$ of the base $\Phi$ and intersect $\kappa(\mathsf{M}_i)$ with the platform $\varphi$ we get a correlation $\gamma$ from points of $\Phi$ to lines of $\varphi$. Due to Lemma 2.1 of Mick and Röschel [5] we can assume that $\varphi$ is parallel to $\mathsf{a}$. Now the platform anchor points $\mathsf{M}_i$ with homogeneous coordinates $(1 : A_i : B_i : 0)$ and base anchor points $\mathsf{m}_i$ with $(1 : a_i : 0 : b_i)$ are conjugate points with respect to $\gamma$ if

$$(1, a_i, b_i) \begin{pmatrix} 0 & -c_4 & -c_5 \\ c_4 & 0 & -c_3 \\ c_6 & -c_2 & c_1 \end{pmatrix} \begin{pmatrix} 1 \\ A_i \\ B_i \end{pmatrix} = 0 \tag{7}$$

holds. Moreover this condition must hold for the whole Schönflies group $\mathsf{X}(\mathsf{a})$ where $\mathsf{a}$ is orthogonal to $\Phi$ and parallel to $\varphi$. Therefore Eq. (7) must hold independently of translations of $\Phi$ in $x$ and $y$ direction and independently of translations of $\varphi$ in $z$ direction. This yields $(1, a_i, b_i)\mathbf{A}(1, A_i, B_i)^T = 0$ with

$$\mathbf{A} := (a_{ij}) = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -c_4 & -c_5 \\ c_4 & 0 & -c_3 \\ c_6 & -c_2 & c_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{pmatrix}. \tag{8}$$

The $a_{ij}$ are homogeneous linear functions of the coordinates $c_i$. Therefore the set of linear line complexes spanned by Schönflies-singular manipulators determine a 3-parametric manifold (parameters $x,y,z$) of correlations. Moreover the three equation $a_{00}=0$, $a_{11}=0$ and $a_{01}+a_{10}=0$ must hold, where $a_{11}=0$ is fulfilled identically.

The remaining two conditions $a_{00}=0$ and $a_{01}+a_{10}=0$ can be written as

$$\begin{pmatrix} 1 & 0 & -z & -x & -y & xz & 0 & yz \\ 0 & 1 & 0 & 1 & 0 & -z & -y & 0 \end{pmatrix}(a_{00},a_{10},a_{20},a_{01},a_{02},a_{21},a_{12},a_{22})^T = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (9)$$

Due to the first two columns this $2\times 8$ matrix has rank 2 independently of the parameters $x,y,z$. Moreover we can also rewrite the 6 equations $(1,a_i,b_i)\mathbf{A}(1,A_i,B_i)^T=0$ $(i=1,\ldots,6)$ in an analogous form as

$$\begin{pmatrix} 1 & a_1 & b_1 & A_1 & B_1 & A_1b_1 & B_1a_1 & B_1b_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_6 & b_6 & A_6 & B_6 & A_6b_6 & B_6a_6 & B_6b_6 \end{pmatrix}(a_{00},a_{10},a_{20},a_{01},a_{02},a_{21},a_{12},a_{22})^T = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \qquad (10)$$

If this $6\times 8$ matrix has rank 5 the system of linear equations given in Eq. (9) and (10) has at least a 1-dimensional solution. As a consequence the 7-parametric linear manifold of correlations described by (3,3)-matrices $a_{ij}$ with $a_{11}=0$ contains at least one correlation $\gamma$ and therefore the manipulator is Schönflies-singular.

This proof also provides us the following geometric characterization of solution (3):

**Theorem 4.** *Given are two sets of points* $\{\mathsf{M}_i\}$ *and* $\{\mathsf{m}_i\}$ *(*$i=1,\ldots,6$*) in two non-parallel planes* $\Phi$ *and* $\varphi$*, respectively. Then the non-architecturally singular planar parallel manipulator of Stewart Gough type is* $\mathsf{X}(\mathsf{a})$*-singular with* $\mathsf{a}:=(\Phi,\varphi)$ *if* $\{\mathsf{M}_i,\mathsf{m}_i\}$ *are three-fold conjugate pairs of points with respect to a 2-dimensional linear manifold of correlations, whereas the ideal point of* $\mathsf{a}$ *is self-conjugate.*

The proof of the sufficiency of the conditions of solution (1) and (2) can be done analytically as follows: Due to the symmetry of the conditions with respect to the interchange of the platform and the base, the linear dependency of $\mathbf{b},\mathbf{B},\mathbf{Bb}$ only imply the following three cases:

i. $\mathbf{Bb}_1^6=\lambda_{Bb}\mathbf{b}_1^6+\mu_{Bb}\mathbf{B}_1^6$ and $\mathbf{A}_1^6=\lambda_A\mathbf{b}_1^6+\mu_A\mathbf{B}_1^6+\mathbf{a}_1^6$,
ii. $\mathbf{b}_1^6=\lambda_b\mathbf{B}_1^6+\mu_b\mathbf{Bb}_1^6$ and $\mathbf{A}_1^6=\lambda_A\mathbf{B}_1^6+\mu_A\mathbf{Bb}_1^6+\mathbf{a}_1^6$,
iii. or the special case $\mathbf{Bb}_1^6=\lambda_{Bb}\mathbf{b}_1^6$ and $\mathbf{B}_1^6=\mu_B\mathbf{b}_1^6$.

Computation shows that $Q$ vanishes in all three cases for $e_2=e_3=0$. This finishes the proof of the sufficiency and therefore Theorem 3 is proven. □

Now we also want to give a geometric interpretation of the condition of solution (1): W.l.o.g. we can choose special coordinate systems in the platform and the base, such that $a_1 = b_1 = A_1 = B_1 = 0$ holds. Moreover we can assume w.l.o.g. $\mathbf{b}_2^6 = \lambda_b \mathbf{B}_2^6$ and $\mathbf{Bb}_2^6 = \mu_{Bb}\mathbf{B}_2^6$. As $\mathbf{B}_2^6$ or $\mathbf{b}_2^6$ must not equal the zero vector (otherwise we get an architecturally singular manipulator) we can assume w.l.o.g. $\lambda_b B_6 \neq 0$. Therefore we get $\mu_{Bb} = \lambda_b B_6$. As a consequence the remaining equations can only hold for $b_i = B_i = 0$ or $B_i = B_6$ and $b_i = b_6$ for $i = 2,3,4,5$. There only exist two non-architecturally singular combinatorial cases:

1. $[\mathsf{M}_i, \mathsf{M}_j, \mathsf{M}_k, \mathsf{M}_l] \parallel [\mathsf{m}_i, \mathsf{m}_j, \mathsf{m}_k, \mathsf{m}_l] \parallel [\mathsf{M}_m, \mathsf{M}_n] \parallel [\mathsf{m}_m, \mathsf{m}_n] \parallel \mathsf{a}$,
2. $[\mathsf{M}_i, \mathsf{M}_j, \mathsf{M}_k] \parallel [\mathsf{m}_i, \mathsf{m}_j, \mathsf{m}_k] \parallel [\mathsf{M}_l, \mathsf{M}_m, \mathsf{M}_n] \parallel [\mathsf{m}_l, \mathsf{m}_m, \mathsf{m}_n] \parallel \mathsf{a}$,

with $(i,j,k,l,m,n)$ consisting of all indices from 1 to 6.

*Remark 3.* The geometric meaning of the condition $rk(\mathbf{1},\mathbf{b},\mathbf{B},\mathbf{Bb},\mathbf{A}-\mathbf{a})_1^6 = 3$ is still missing. Until now we are only able to identify a geometric meaning with $|\mathbf{1},\mathbf{b},\mathbf{B},\mathbf{Bb}|_{(i,j,k,l)} = 0$. This algebraic conditions equals $DV(\mathsf{G}_i,\mathsf{G}_j,\mathsf{G}_k,\mathsf{G}_l) = DV(\mathsf{g}_i,\mathsf{g}_j,\mathsf{g}_k,\mathsf{g}_l)$ with $\mathsf{G}_i := [\mathsf{M}_i,\mathsf{U}]$ and $\mathsf{g}_i := [\mathsf{m}_i,\mathsf{U}]$ where $DV$ denotes the cross-ratio and $\mathsf{U}$ the ideal point of the axis $\mathsf{a}$. ◇

### 4.2 Cartesian-singular planar parallel manipulators

**Definition 1.** Parallel manipulators which are singular with respect to the translational group $\mathsf{T}(3)$ are called Cartesian-singular or $\mathsf{T}(3)$-singular, respectively.

Due to the Lemma 2.1 of of Mick and Röschel [5] the solution set of case (2c) equals the set of Cartesian-singular planar SG platforms where $\Phi$ and $\varphi$ are not parallel. Therefore only the case with parallel platform and base is missing which follows from the proof of Theorem 1 by setting $e_0 = 1$ and $e_3 = 0$:

**Theorem 5.** *A non-architecturally singular planar SG platform, where $\varphi$ and $\Phi$ are parallel, is $\mathsf{T}(3)$-singular if and only if $|\mathbf{1},\mathbf{A},\mathbf{B},\mathbf{a},\mathbf{b},\mathbf{Ab}-\mathbf{Ba}|_1^6 = 0$ holds.*

## 5 Conclusion

In this paper we discussed the special cases of Schönflies-singular planar Stewart Gough platforms, where the angle $\alpha \in [0,\pi/2]$ between the rotation axis $\mathsf{a}$ of the Schönflies group and the carrier plane of the base anchor points equals the angle between $\mathsf{a}$ and the carrier plane of the platform anchor points.

We distinguished the three cases $\alpha = \pi/2$ (cf. Theorem 1), $\alpha \in ]0,\pi/2[$ (cf. Theorem 2) and $\alpha = 0$ (cf. Theorem 3). As side product we also characterized all Cartesian-singular planar parallel manipulators (cf. Section 4.2).

Moreover, the problem of determining all non-planar Schönflies-singular manipulators of Stewart Gough type remains open.

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