Main theorem on Schönflies-singular planar Stewart Gough platforms

G. Nawratil

Institute of Discrete Mathematics and Geometry,
Vienna University of Technology, Austria,
e-mail: nawratil@geometrie.tuwien.ac.at

Abstract. Parallel manipulators which are singular with respect to the Schönflies motion group $\mathsf{X}(\mathsf{a})$ are called Schönflies-singular, or more precisely $\mathsf{X}(\mathsf{a})$-singular, where a denotes the direction of the rotary axis. A special class of such manipulators are architecturally singular ones because they are singular with respect to any Schönflies group. Another remarkable set of Schönflies-singular planar parallel manipulators of Stewart Gough type was already presented by the author. In this paper we give the main theorem on $\mathsf{X}(\mathsf{a})$-singular planar parallel manipulators and discuss the consequences of this result.

Key words: Schönflies-singular, Schönflies motion group, Stewart Gough platform, planar parallel manipulators, Architecture singularity

1 Introduction

The Schönflies motion group $\mathsf{X}(\mathsf{a})$ is the largest subgroup of the Special Euclidean motion group $\mathsf{SE}(3)$ and consists of three linearly independent translations and all the rotations about the infinity of axes with direction a. This 4-dimensional group, which is named after the German geometer Arthur Moritz Schönflies (cf. [1, 2]), is of importance in practice because it is well adapted for pick-and-place operations.

The geometry of a planar parallel manipulator of Stewart Gough type (SG type) is given by the six base anchor points $\mathsf{M}_i \in \Sigma_0$ with coordinates $\mathbf{M}_i := (A_i, B_i, 0)^T$ and by the six platform anchor points $\mathsf{m}_i \in \Sigma$ with coordinates $\mathbf{m}_i := (a_i, b_i, 0)^T$. By using Euler Parameters (e_0, e_1, e_2, e_3) for the parametrization of the spherical motion group $\mathsf{SO}(3)$ the coordinates $\mathbf{m}'_i$ of the platform anchor points with respect to the fixed space can be written as $\mathbf{m}'_i = K^{-1}\mathbf{R}\,\mathbf{m}_i + \mathbf{t}$ with

$$\mathbf{R} := (r_{ij}) = \begin{pmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2-e_0e_3) & 2(e_1e_3+e_0e_2) \\ 2(e_1e_2+e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3-e_0e_1) \\ 2(e_1e_3-e_0e_2) & 2(e_2e_3+e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{pmatrix}, \qquad (1)$$

the translation vector $\mathbf{t} := (t_1, t_2, t_3)^T$ and $K := e_0^2+e_1^2+e_2^2+e_3^2$.

It is well known (cf. Merlet [3]) that a SG platform is singular if and only if the carrier lines of the prismatic legs belong to a linear line complex, or analytically seen, if $Q := det(\mathbf{Q}) = 0$ holds, where the i^{th} row of the 6×6 matrix $\mathbf{Q}$ equals the Plücker coordinates $\underline{\mathbf{l}}_i := (\mathbf{l}_i, \widehat{\mathbf{l}}_i) := (\mathbf{m}_i' - \mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$ of the i^{th} carrier line.

1.1 Notation

Definition 1. Parallel manipulators which are singular with respect to the Schönflies motion group $\mathsf{X}(\mathsf{a})$ are called Schönflies-singular, or more precisely $\mathsf{X}(\mathsf{a})$-singular.

For proving the so-called main theorem on Schönflies-singular planar Stewart Gough platforms we use the notation introduced in [4]. We denote the determinant of certain $j\times j$ matrices as follows:

$$|\mathbf{X}, \mathbf{y}, \ldots, \mathbf{Xy}|_{(i_1,i_2,\ldots,i_j)} := det(\mathbf{X}_{(i_1,i_2,\ldots,i_j)}, \mathbf{y}_{(i_1,i_2,\ldots,i_j)}, \ldots, \mathbf{Xy}_{(i_1,i_2,\ldots,i_j)}) \tag{2}$$

$$\text{with} \quad \mathbf{X}_{(i_1,i_2,\ldots,i_j)} = \begin{bmatrix} X_{i_1} \\ X_{i_2} \\ \vdots \\ X_{i_j} \end{bmatrix}, \ \mathbf{y}_{(i_1,i_2,\ldots,i_j)} = \begin{bmatrix} y_{i_1} \\ y_{i_2} \\ \vdots \\ y_{i_j} \end{bmatrix}, \ \mathbf{Xy}_{(i_1,i_2,\ldots,i_j)} = \begin{bmatrix} X_{i_1}y_{i_1} \\ X_{i_2}y_{i_2} \\ \vdots \\ X_{i_j}y_{i_j} \end{bmatrix} \tag{3}$$

and $(i_1, i_2, \ldots, i_j) \in \{1, \ldots, 6\}$ and pairwise distinct. Moreover it should be noted that we write $|\mathbf{X}, \mathbf{y}, \ldots, \mathbf{Xy}|_{i_1}^{i_j}$ if $i_1 < i_2 < \ldots < i_j$ with $i_{k+1} = i_k + 1$ for $k = 1, \ldots, j-1$ hold. Moreover the algebraic condition that $\mathsf{M}_i, \mathsf{M}_j, \mathsf{M}_k$ or $\mathsf{m}_i, \mathsf{m}_j, \mathsf{m}_k$ are collinear is denoted by $C_{(i,j,k)} := |\mathbf{1}, \mathbf{A}, \mathbf{B}|_{(i,j,k)} = 0$ and $c_{(i,j,k)} := |\mathbf{1}, \mathbf{a}, \mathbf{b}|_{(i,j,k)} = 0$, respectively.

It should also be said that in the later done case study we always factor out the homogenizing factor K if possible. Moreover we give the number n of terms of not explicitly given polynomials F in square brackets, i.e. $F[n]$.

1.2 Related work

Special Schönflies-singular manipulators are the architecturally singular ones (cf. [5]) because they are singular with respect to any Schönflies group. As architecturally singular manipulators are already classified we are only interested in Schönflies-singular manipulators which are not architecturally singular.

For the characterization of architecturally singular planar SG platforms we refer to Karger [6, 7], Nawratil [8], Röschel and Mick [9] as well as Wohlhart [10]. For the non-planar case we refer to Karger [11] and Nawratil [12].

For the determination of $\mathsf{X}(\mathsf{a})$-singular planar parallel manipulators we distinguish the following cases depending on the angle $\alpha \in [0, \pi/2]$ enclosed by a and the carrier plane Φ of the base anchor points and the angle $\beta \in [0, \pi/2]$ between a and the carrier plane φ of the platform anchor points. Every $\mathsf{X}(\mathsf{a})$-singular manipulator belongs to one of the following 5 cases (after exchanging platform and base):

1. $\alpha \neq \beta$: (a) $\alpha = \pi/2, \beta \in [0, \pi/2[$ (b) $\alpha, \beta \in [0, \pi/2[$
2. $\alpha = \beta$: (a) $\alpha = \pi/2$ (b) $\alpha =]0, \pi/2[$ (c) $\alpha = 0$

According to [4] the solution set of case (1a) can be characterized as follows:

Theorem 1. *A non-architecturally singular planar manipulator is* $\mathsf{X}(\mathsf{a})$*-singular, where* a *is orthogonal to* Φ *and orthogonal to the x-axis of the moving frame if and only if* $rk(\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{Bb}, \mathbf{a}, \mathbf{b}, \mathbf{Ab})_1^6 = 4$ *holds.*

It should be noted that this solution set does not depend on the angle β due to the following lemma given by Mick and Röschel [13]:

Lemma 1. *If the connecting lines of* $\mathsf{M}_i \in \Phi$ *and* $\mathsf{m}_i \in \varphi$ *of two intersecting planes* Φ *and* φ *belong to a linear line complex, then this property remains unchanged under rotations of the planes about their intersection line.*

For more details on the self-motional behavior of the solution set of case (1a) as well as a geometric interpretation of the given rank condition we refer to [4].

In the following Sections 2 and 3 we prove that the manipulators of Theorem 1 are the only $\mathsf{X}(\mathsf{a})$-singular ones with $\alpha \neq \beta$ which are not architecturally singular.

2 Main Theorem for the general case

Theorem 2. $\nexists$ *non-architecturally singular planar SG platforms with no 4 collinear anchor points which are* $\mathsf{X}(\mathsf{a})$*-singular if* $\alpha \neq \beta$ *and* a *not orthogonal to* Φ *or* φ*.*

Proof. Without loss of generality (w.l.o.g.) we can assume that $\alpha > \beta$ and therefore Φ cannot be parallel to a. Then we can choose coordinate systems such that $a_2 A_2 B_3 B_4 B_5 c_{(3,4,5)} (a_3 - a_4)(b_3 - b_4) \neq 0$ hold (cf. [6, 4]). Moreover, due to $\alpha > \beta$ we can always rotate the platform about a such that the common line of Φ and φ is parallel to $[\mathsf{M}_1, \mathsf{M}_2]$. [1] This yields the following coordinatization: $\mathbf{M}_i = (A_i, B_i, 0)$ and $\mathbf{m}_i = (a_i, b_i \cos\delta, b_i \sin\delta)$ with $A_1 = B_1 = B_2 = a_1 = b_1 = 0$. As $\sin\delta = 0$ yields $\alpha = \beta$ we can assume $\sin\delta \neq 0$.

As no four anchor points are collinear we can apply the elementary matrix manipulations given by Karger [6] to the Jacobian $\mathbf{Q}$. We end up with $\underline{\mathbf{l}}_6 := (v_1, v_2, v_3, 0, -w_3, w_2)$ with

$$v_i := r_{i1} K_1 + (r_{i3} \sin\delta + r_{i2} \cos\delta) K_2, \quad w_j := r_{j1} K_3 + (r_{j3} \sin\delta + r_{j2} \cos\delta) K_4$$

and

$$\begin{aligned} K_1 &:= |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{a}|_2^6, \qquad & K_3 &:= |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Aa}|_2^6, \\ K_2 &:= |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{b}|_2^6, \qquad & K_4 &:= |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Ab}|_2^6. \end{aligned} \tag{4}$$

[1] Note that the common line of Φ and φ is no ideal line due to $\alpha \neq \beta$.

Due to Lemma 1 this manipulator must also be $\mathsf{X}(\mathsf{s})$-singular where s denotes the direction of the common line of Φ and φ.

In the first step we will use this property to show that $K_1 = K_2 = 0$ must hold. Therefore we can set $e_2 = e_3 = \delta = 0$ and compute $Q[4224]$ in its general form. The necessity of $K_1 = K_2 = 0$ follows immediately from $Q^{42}_{101} + Q^{24}_{101} = K_2$ and $Q^{51}_{002} + Q^{33}_{002} + Q^{15}_{002} = K_1$, where Q^{uv}_{ijk} denotes the coefficient of $t_1^i t_2^j t_3^k e_0^u e_1^v$ of Q.

Now we go back to the general case. We replace the sixth line of the Jacobian $\mathbf{Q}$ by $(v_1, v_2, v_3, 0, -w_3, w_2)$ under consideration of $K_1 = K_2 = 0$. In the following we prove by contradiction that also $K_3 = K_4 = 0$ must hold. This finishes the proof because $K_1 = K_2 = K_3 = K_4 = 0$ are the four necessary and sufficient conditions for a planar manipulators with no four points on a line to be architecturally singular (cf. Karger [6]).

Part [A] $e_2 = 0$

We set $e_1 = e_4 \cos\mu$ and $e_3 = e_4 \sin\mu$, where e_4 is the homogenizing factor. Moreover $\sin\mu\cos\mu \neq 0$ must hold. Then we compute $Q[35346]$ in dependency of K_3 and K_4 and denote the coefficients of $t_1^i t_2^j t_3^k e_0^u e_4^v$ of Q by Q^{uv}_{ijk}.

First we prove by contradiction that K_4 must also vanish. Assuming $K_4 \neq 0$ we get $b_2 = 0$ from $Q^{80}_{100} = 0$. Then the resultant of Q^{71}_{100} and Q^{51}_{200} with respect to B_3 can only vanish without contradiction (w.c.) for:

1. $b_i = 0$: Then $Q^{51}_{200} = 0$ implies $B_j = B_k$ (with $i, j, k \in \{3,4,5\}$ and pairwise distinct) and $Q^{33}_{200} = 0$ yields the contradiction.
2. $B_4 = B_5$, $b_3 b_4 b_5 \neq 0$: Then $Q^{51}_{200} = 0$ can only vanish w.c. for $B_3 = B_5$ or $b_4 = b_5$.
   a. $B_3 = B_5$: We get the contradiction from $Q^{33}_{200} = 0$.
   b. $b_4 = b_5$, $B_3 \neq B_5$: In this case $Q^{71}_{100} = 0$ yields the contradiction.

Now we can set $K_4 = 0$ and compute $Q = A_2 e_4 K_3 F[15090]$. We distinguish between the following two cases for proving that F cannot vanish w.c.:

1. $b_2 \neq 0$: W.l.o.g. we can compute a_5 from $F^{41}_{110} = 0$ and A_5 from $F^{50}_{101} = 0$.
   a. Assuming $b_3 \neq b_5 \neq b_4$ we can express A_4 from $F^{70}_{100} = 0$. Then $F^{61}_{100} = 0$ yields the contradiction.
   b. W.l.o.g. we set $b_4 = b_5$. Now F^{70}_{100} can only vanish w.c. for $b_5(b_2 - b_5) = 0$. In both cases $F^{61}_{100} = 0$ yields the contradiction.
2. $b_2 = 0$: Now F^{61}_{100} can only vanish w.c. for $b_3 b_4 b_5 C_{(3,4,5)} = 0$:
   a. $b_i = 0$: Then $F^{32}_{200} = 0$ implies $B_j = B_k$ and $F^{52}_{100} = 0$ yields $A_j = A_k$ (with $i, j, k \in \{3,4,5\}$ and pairwise distinct). Finally F^{43}_{100} cannot vanish w.c..
   b. $C_{(3,4,5)} = 0$, $b_3 b_4 b_5 \neq 0$: Assuming $B_3 \neq B_4$ we can compute A_5 from the collinearity condition and a_5 from $F^{41}_{020} = 0$. Now F^{32}_{200} can only vanish w.c. for $|\mathbf{B}, \mathbf{b}, \mathbf{Bb}|_3^5 = 0$. W.l.o.g. we can compute b_4 from this condition. Then $F^{52}_{100} = 0$ yields the contradiction.
      In the special case $B_3 = B_4 = B_5$ we can compute A_5 from $F^{52}_{100} = 0$ w.l.o.g.. Then $F^{14}_{200} = 0$ already yields the contradiction.

Part [B] $e_2 \neq 0$

We set $e_1 = e_4 \cos\mu$, $e_3 = e_4 \sin\mu$ and $e_2 = e_4 n$, where $n \sin\mu \neq 0$ holds. Moreover for $n\cos\delta + \sin\mu \sin\delta = 0$ we can assume $\cos\mu \neq 0$ because otherwise a is orthogonal to the platform. Again we prove by contradiction that K_4 must vanish.

Assuming $K_4 \neq 0$ we get $b_2 = 0$ from $Q^{80}_{100} = 0$. Then the resultant of Q^{60}_{110} and Q^{80}_{020} with respect to B_3 can only vanish w.c. in the following cases:

1. $A_2 = a_2$: In this case $Q^{60}_{110} = 0$ implies $|\mathbf{b}, \mathbf{B}, \mathbf{Bb}|^5_3 = 0$:
   a. For the special case $B_3 = B_4 = B_5$ we get $\mu = \zeta$ with $\zeta := -\arcsin(n\cot\delta)$ from $Q^{42}_{200} = 0$. Then $Q^{33}_{200} = 0$ yields the contradiction.
   b. W.l.o.g. we can solve $|\mathbf{b}, \mathbf{B}, \mathbf{Bb}|^5_3 = 0$ for b_5. Due to $Q^{42}_{200} = 0$ we must distinguish the following two cases:
      i. $b_4 = b_3 B_4/B_3$: W.l.o.g. we can express a_5 from the only non-contradicting factor of $Q^{60}_{020} = 0$. Then $Q^{51}_{020} = 0$ implies $a_4 = A_4 + B_4(a_3 - A_3)/B_3$. Now we can solve $K_1 = K_2 = 0$ for A_6 and b_6 w.l.o.g.. Moreover, substitution of these expressions into K_4 shows that it is fulfilled identically and this contradicts the assumption.
      ii. $\mu = \zeta$, $b_4 \neq b_3 B_4/B_3$: Then $Q^{33}_{200} = 0$ already implies the contradiction.
2. $b_3 b_4 b_5 = 0$, $A_2 \neq a_2$: W.l.o.g. we set $b_3 = 0$. Now $Q^{51}_{200} = 0$ implies two cases:
   a. $B_4 = B_5$: Then $Q^{42}_{200} = 0$ yields $\mu = \zeta$. $Q^{33}_{200} = 0$ yields the contradiction.
   b. $\mu = \zeta, B_4 \neq B_5$: $Q^{60}_{020} = 0$ yields $A_3 = a_3 A_2/a_2$ and $Q^{42}_{200} = 0$ the contradiction.
3. $B_4 = B_5$, $b_3 b_4 b_5 (A_2 - a_2) \neq 0$: Due to $Q^{51}_{200} = 0$ we must distinguish two cases:
   a. $B_3 = B_5$: Now $Q^{42}_{200} = 0$ implies $\mu = \zeta$. $Q^{33}_{200} = 0$ yields the contradiction.
   b. $b_4 = b_5$, $B_3 \neq B_5$: Then $Q^{80}_{010} = 0$ cannot vanish w.c..
   c. $\mu = \zeta$, $(b_4 - b_5)(B_3 - B_5) \neq 0$: $Q^{42}_{200} = 0$ yields the contradiction.

Now we can set $K_4 = 0$ and compute $Q = A_2 e_4 K_3 F[57528]$. We prove by contradiction that $K_3 = 0$ must hold, i.e. we assume $K_3 \neq 0$. Then we distinguish again between the following two cases for proving that F cannot vanish w.c.:

1. $b_2 \neq 0$: Now we can solve $F^{50}_{110} = 0$ for a_5. From $F^{32}_{200} = 0$ we can express a_4. From $F^{50}_{020} = 0$ we get A_5. $F^{41}_{020} = 0$ yields an expression for A_4. W.l.o.g. we can solve $K_1 = K_2 = 0$ for A_6 and b_6. Then $b_2 K_3 - a_2 K_4 = 0$ holds. This is a contradiction as $K_4 = 0$ implies $K_3 = 0$.
2. $b_2 = 0$: Now F^{50}_{200} implies $|\mathbf{b}, \mathbf{B}, \mathbf{Bb}|^5_3 = 0$. Again we start with the special case:
   a. $B_3 = B_4 = B_5$: $F^{41}_{110} = 0$ already yields the contradiction.
   b. W.l.o.g. we can compute b_5 from $|\mathbf{b}, \mathbf{B}, \mathbf{Bb}|^5_3 = 0$. Now F^{32}_{200} can only vanish w.c. in the following 2 cases:
      i. $b_4 = b_3 B_4/B_3$: An accurate case study shows that we only end up with contradictions. For the detailed discussion we refer to [14]. Moreover it should be noted that this case implies solutions for the the special case $\alpha = \beta \in]0, \pi/2[$.
      ii. $\mu = \zeta$, $b_4 \neq b_3 B_4/B_3$: Then $F^{23}_{200} = 0$ implies the contradiction. □

3 Main Theorem for the special case

Theorem 3. $\nexists$ *non-architecturally singular planar SG platforms with 4 collinear anchor points which are* $\mathsf{X}(\mathsf{a})$*-singular if* $\alpha \neq \beta$ *and* a *not orthogonal to* Φ *or* φ.

Proof. In order to prove this theorem efficiently we need a good choice for the coordinate systems in Σ and Σ_0. Based on some geometric considerations such a coordinatization can be done as follows: W.l.o.g. we can assume that the four collinear points are on the platform, i.e. $\mathsf{m}_1,\ldots,\mathsf{m}_4$ are situated on the line g. Now we must distinguish again two cases, depending on the property if $\gamma \geq \alpha$ or $\gamma < \alpha$ holds with $\gamma := \angle(\mathsf{g},\mathsf{a}) \in [0,\pi/2]$.

3.1 $\gamma \geq \alpha$

In this case we translate φ and Φ such that $\mathsf{M}_1 = \mathsf{m}_1$ holds. As $\gamma \geq \alpha$ there exist at least one position by rotating of φ about a such that $\mathsf{g} \in \Phi$ holds. This is the starting configuration of the following coordinatization: $\mathbf{M}_i = (A_i, B_i, 0)$ and $\mathbf{m}_i = (a_i, b_i\cos\delta, b_i\sin\delta)$ with $A_1 = B_1 = a_1 = b_1 = b_2 = b_3 = b_4 = 0$ and $\sin\delta \neq 0$.

Moreover we set $e_1 = e_4\cos\mu$, $e_3 = e_4\sin\mu$ and $e_2 = e_4 n$, where $n = \cos\mu = 0$, $n = \sin\mu = 0$ or $\cos\mu = n\cos\delta + \sin\mu\sin\delta = 0$ yield contradictions.

Part [A] $\sin\mu \neq 0$

Firstly, we show that we can assume $\mathsf{M}_5 \neq \mathsf{M}_6$ and that no 5 platform anchor points are collinear because these two cases yield a contradiction:

1. $b_5 = 0$: We give those 5 coefficients which imply $rk(\mathbf{A},\mathbf{a},\mathbf{B},\mathbf{Aa},\mathbf{Ba})_2^5 \leq 3$. This yields a contradiction due to [9]. We distinguish 3 cases:
   a. $n = 0$: Four conditions are given by $Q^{13}_{201} = Q^{15}_{200} = Q^{22}_{021} = Q^{24}_{020} = 0$. For $B_6 \neq 0$ we get the fifth condition from $Q^{62}_{001} = 0$. For $B_6 = 0$ and $A_6 \neq 0$ we get it from $Q^{53}_{001} = 0$. For the case $\mathsf{M}_1 = \mathsf{M}_6$ it is given by $Q^{33}_{101} = 0$.
   b. $n = \nu := -\sin\mu\tan\delta$: Four conditions are given by $Q^{13}_{201} = Q^{24}_{200} = Q^{31}_{021} = Q^{42}_{020} = 0$. For $B_6 \neq 0$ we get the fifth condition from $Q^{71}_{001} = 0$. For $B_6 = 0$ and $A_6 \neq 0$ we get it from $Q^{53}_{001} = 0$. For the case $\mathsf{M}_1 = \mathsf{M}_6$ it is given by $Q^{51}_{002} = 0$.
   c. $\nu \neq n \neq 0$: Four conditions are given by $Q^{22}_{201} = Q^{33}_{200} = Q^{31}_{021} = Q^{42}_{020} = 0$. For $B_6 \neq 0$ we get the fifth condition from $Q^{71}_{001} = 0$. For $B_6 = 0$ and $A_6 \neq 0$ we get it from $Q^{62}_{001} = 0$. For the case $\mathsf{M}_1 = \mathsf{M}_6$ and $\cos\delta \neq 0$ it is given by $Q^{51}_{002} = 0$. If additionally $\cos\delta = 0$ hold we get the last condition from $Q^{42}_{101} = 0$.
2. $\mathsf{M}_5 = \mathsf{M}_6$: We give the 4 necessary and sufficient conditions indicating the degenerated cases of architecturally singular planar parallel manipulators (cf. [8]):
   a. $n = 0$: $Q^{22}_{021} = Q^{24}_{020} = Q^{13}_{201} = Q^{15}_{200} = 0$.
   b. $n = \nu$: $Q^{31}_{021} = Q^{42}_{020} = Q^{13}_{201} = Q^{24}_{200} = 0$.
   c. $\nu \neq n \neq 0$: $Q^{31}_{021} = Q^{42}_{020} = Q^{22}_{201} = Q^{33}_{200} = 0$.

Moreover, w.l.o.g. we can assume that if 3 points of $\mathsf{M}_1,\ldots,\mathsf{M}_4$ are collinear and pairwise distinct they are $\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3$. We can also assume that if 2 points of $\mathsf{M}_1,\ldots,\mathsf{M}_4$ coincide, they are M_2 and M_3.

Now $Q_{111}^{40}=0$ and $Q_{021}^{40}=0$ imply $|\mathbf{a},\mathbf{A},\mathbf{B}|_2^4=0$. W.l.o.g. we can express a_2 from this condition. In the next step we prove by contradiction that W must vanish with

$$W := a_3(A_2B_4-A_4B_2)(B_2-B_3)+a_4(A_3B_2-A_2B_3)(B_2-B_4).$$

From $Q_{101}^{60}=0$ we get $B_5=B_6$. Now Q_{200}^{42} can only vanish w.c. under consideration of $Q_{011}^{60}=0$ for $n=0$ or $n=\nu$. In both cases $Q_{200}^{33}=0$ yields the contradiction.

Part [B] $(B_2-B_3)\sin\mu\neq 0$

Under this assumption we can express a_3 from $W=0$. Then $Q_{102}^{22}=0$ together with $Q_{021}^{31}=0$ imply an expression for a_5. Now Q_{100}^{71} can only vanish w.c. for:

1. $n=0$: Now $Q_{100}^{62}=0$ implies $B_5=B_6$ or $B_2B_3B_4=0$.
   a. $B_5=B_6$: Assuming $B_2B_3\neq 0$ we can express A_4 from $Q_{101}^{42}=0$. From $Q_{101}^{33}=0$ we get A_6 and $Q_{100}^{53}=0$ yields the contradiction. For the special case $B_2B_3=0$ we can set $B_2=0$ w.l.o.g.. Then $Q_{101}^{42}=0$ implies $B_3=B_4$. From $Q_{101}^{33}=0$ we get $b_5=b_6$ and $Q_{010}^{53}=0$ yields the contradiction.
   b. $B_2B_3B_4=0$, $B_5\neq B_6$: In all 3 cases we get the contradiction from $Q_{101}^{51}=0$.
2. $B_5\neq B_6$, $n\neq 0$: Now $Q_{020}^{42}=0$ and $Q_{110}^{42}=0$ can only hold if the common factor $G[48]$ vanishes or for $H_1[6]=H_2[6]=0$. As the latter case yield easy contradictions we set $G=0$ and introduce the following notation:

$$R := A_2B_3B_4(B_4-B_3)(B_2-B_6)-A_3B_2B_4(B_4-B_2)(B_3-B_6)+A_4B_2B_3(B_3-B_2)(B_4-B_6).$$

   a. $R\neq 0$: Now we can compute A_6 from $G=0$. Then Q_{100}^{62} can only vanish w.c. for $n=\mu$, but in this case $Q_{100}^{53}=0$ yields the contradiction.
   b. $R=0$, $B_2B_3(B_6-B_4)\neq 0$: Under this assumption we can compute A_4 from $R=0$. Now $G=0$ can only vanish w.c. for $b_5=b_6$. Then $Q_{101}^{42}=0$ implies $n=\nu$ and $Q_{100}^{44}=0$ yields the contradiction.
   c. $R=0$, $B_2B_3=0$: W.l.o.g. we set $B_2=0$. Then $R=0$ can only vanish w.c. for:
      i. $B_6=0$: Due to $Q_{100}^{53}=0$ we must distinguish two cases: For $B_3=B_4$ we get $n=\nu$ from $Q_{010}^{62}=0$ and $Q_{010}^{53}=0$ yields the contradiction. For the second case $n=\nu$, $B_3\neq B_4$ we get the contradiction from $Q_{100}^{35}=0$.
      ii. $B_3=B_4$, $B_6\neq 0$: Due to $Q_{110}^{33}=0$ we must distinguish 3 cases: For the cases $b_5=b_6$ and $B_4=B_6$ we get $n=\nu$ from $Q_{011}^{51}=0$ and the contradiction from $Q_{011}^{42}=0$. For the third case $n=\nu$, $(B_4-B_6)(b_5-b_6)\neq 0$ we get the contradiction from $Q_{110}^{24}=0$.
   d. $R=0$, $B_4=B_6$, $B_2B_3\neq 0$: Now R can only vanish w.c. for:
      i. $B_6=0$: $Q_{100}^{53}=0$ implies $n=\nu$ and $Q_{100}^{35}=0$ yields the contradiction.
      ii. $B_2=B_6\neq 0$: $Q_{101}^{42}=0$ yields $n=\nu$ and $Q_{101}^{24}=0$ the contradiction.
3. $B_4=0$, $n(B_5-B_6)\neq 0$: We get the contradiction from $Q_{110}^{51}=0$.
4. $B_2B_3=0$, $nB_4(B_5-B_6)\neq 0$: W.l.o.g. we set $B_2=0$. Now $Q_{110}^{51}=0$ implies $B_3=B_4$ and then $Q_{010}^{71}=0$ yields the contradiction.

Part [C] $B_2 = B_3$, $\sin\mu \neq 0$

Now W can only vanish w.c. in the following 2 cases:

1. $a_4 = 0$: Now $Q^{22}_{102} = 0$ and $Q^{31}_{021} = 0$ imply $|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B}|^6_3 = 0$. W.l.o.g. we can solve this condition for a_5. Due to $Q^{71}_{100} = 0$ we must distinguish four cases:

   a. $n = 0$: Then $Q^{62}_{100} = 0$ can only vanish w.c. in the following 2 cases: For $B_5 = B_6$ we get $B_4 = 0$ from $Q^{42}_{101} = 0$ and $Q^{53}_{010} = 0$ yields the contradiction. For the 2^{nd} case $B_i = 0$, $B_5 \neq B_6$ for $i = 3,4$ we get the contradiction from $Q^{51}_{101} = 0$.
   b. $B_5 = B_6$, $n \neq 0$: Now Q^{42}_{020} and Q^{42}_{110} can only vanish w.c. for:
      i. $B_3 = B_4$: Due to $Q^{51}_{011} = 0$ we must distinguish 2 cases: For $A_4 = B_4(A_3 - a_3)/B_6$ we get $n = \nu$ from $Q^{42}_{101} = 0$ and the contradiction from $Q^{24}_{101} = 0$. In the second case $n = \mu$ we get the contradiction from $Q^{42}_{011} = 0$.
      ii. $b_6(A_4B_5 - B_4A_5) + b_5(A_6B_4 - A_4B_6) = 0$, $B_3 \neq B_4$: Assuming $B_4 \neq 0$ we can express A_6 from this condition. Then $Q^{62}_{100} = 0$ implies $n = \nu$ and $Q^{53}_{100} = 0$ yields the contradiction. For the special case $B_4 = 0$ the above condition can only vanish w.c. for $B_6(b_5 - b_6) = 0$. In both cases $Q^{51}_{011} = 0$ implies $n = \nu$ and $Q^{42}_{011} = 0$ yields the contradiction.
   c. $B_3 = 0$, $n(B_5 - B_6) \neq 0$: We get immediately the contradiction from $Q^{51}_{110} = 0$.
   d. $B_4 = 0$, $nB_3(B_5 - B_6) \neq 0$: In this case $Q^{42}_{110} = 0$ implies $n = \nu$ and finally $Q^{33}_{110} = 0$ yields the contradiction.

2. $B_3 = 0$, $a_4 \neq 0$: Now $Q^{22}_{102} = 0$ and $Q^{31}_{021} = 0$ imply again $|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B}|^6_3 = 0$. W.l.o.g. we can solve this condition for a_5. Then $Q^{51}_{110} = 0$ can only vanish w.c. for:

   a. $n = 0$: $Q^{51}_{101} = 0$ implies $B_5 = B_6$ and $Q^{42}_{101} = 0$ yields the contradiction.
   b. $B_5 = B_6$, $n \neq 0$: Now $Q^{42}_{110} = 0$ implies an expression for A_6. From $Q^{62}_{100} = 0$ we get $n = \nu$ and $Q^{53}_{100} = 0$ yields the contradiction.

Now only the discussion of the special case $\sin\mu = 0$ ($\Leftrightarrow \Phi \parallel \mathsf{a}$) is missing. This case study can exactly be done as the one for $\sin\mu \neq 0$. The only differences are that we always get $\cos\delta = 0$ instead of $n = \nu$ and that $n = 0$ yields a contradiction. This finishes the case study of $\gamma \geq \alpha$.

3.2 $\gamma < \alpha$

In this case we translate φ and Φ such that $\mathsf{M}_1 = \mathsf{m}_1$ holds. As $\gamma < \alpha$ there exist two positions by rotating of φ about a such that $[\mathsf{M}_1, \mathsf{M}_2] \in \varphi$ holds. This reasons the following coordinatization: $\mathbf{M}_i = (A_i, B_i, 0)$ and $\mathbf{m}_i = (a_i, b_i\cos\delta, b_i\sin\delta)$ with $A_1 = B_1 = B_2 = a_1 = b_1 = 0$, $a_i = b_ia_2/b_2$ for $i = 3,4$ and $b_2\sin\delta \neq 0$.

Again we set $e_1 = e_4\cos\mu$, $e_3 = e_4\sin\mu$ and $e_2 = e_4n$. As $\beta \leq \gamma < \alpha$ holds, $\sin\mu = 0$ yields a contradiction as well as $n = \cos\mu = 0$ or $\cos\mu = n\cos\delta + \sin\mu\sin\delta = 0$.

Moreover, due to the result of Sec. 3.1 we can stop the case study if 4 base anchor points are collinear or if $b_5 = b_6 = b_i = b_j$ holds with $i, j \in \{1, \ldots, 4\}$ and $i \neq j$.

Part [A]

We show that $\mathsf{M}_5 = \mathsf{M}_6$ or $a_i = b_i a_2/b_2$ for $i \in \{5,6\}$ yields a contradiction:

1. $a_5 = b_5 a_2/b_2$: We distinguish the following three subcases:
   a. If $\mathsf{m}_1, \ldots, \mathsf{m}_5$ are pairwise distinct $Q_{110}^{60} = 0$ and $Q_{111}^{40} = 0$ indicate item 10 of Karger's list of architecturally singular manipulators (cf. [11]).
   b. If 3 of the 5 collinear platform points coincide (w.l.o.g. $\mathsf{m}_1 = \mathsf{m}_4 = \mathsf{m}_5$) $Q_{111}^{40} = 0$ and $Q_{021}^{40} = 0$ yield the contradiction.
   c. Only 2 of the 5 collinear platform points coincide (w.l.o.g. $b_3 = 0$). Now $Q_{110}^{60} = 0$ and $Q_{020}^{60} = 0$ imply $C_{(2,4,5)} = 0$ and $Q_{111}^{40} = 0$ indicates the special case of item 10 of Karger's list.
2. $\mathsf{M}_5 = \mathsf{M}_6$: The four conditions $Q_{110}^{60} = Q_{020}^{60} = Q_{111}^{40} = Q_{021}^{40} = 0$ imply the degenerated cases of architecturally singular planar parallel manipulators (cf. [8]).

Therefore we can assume for the remaining discussion that no 5 platform anchor points are collinear and that $\mathsf{M}_5 \neq \mathsf{M}_6$ holds. Now we compute the resultant of Q_{100}^{80} and Q_{110}^{60} with respect to a_2 which yields $(B_5 - B_6)|\mathbf{a},\mathbf{b}|_5^6 I_5 I_6$ with

$$I_i := B_3 B_4 b_i (b_3 - b_4)(A_i - A_2) + B_3 B_i b_4 (b_3 - b_i)(A_2 - A_4) + B_4 B_i b_3 (b_4 - b_i)(A_3 - A_2).$$

As a consequence we must distinguish the following three parts:

Part [B] $B_5 = B_6$

1. Assuming $I_j \neq 0$ we can compute a_i from $Q_{110}^{60} = 0$ for $i, j \in \{5,6\}$ and $i \neq j$. W.l.o.g. we set $i = 5$. Then $Q_{200}^{42} = 0$ can only vanish w.c. for:
   a. $n = 0$: Assuming $B_6 B_j b_i (b_5 - b_j) \neq 0$ we solve $Q_{100}^{71} = 0$ for A_i with $i, j \in \{3,4\}$ and $i \neq j$. W.l.o.g. we set $i = 3$. Then $Q_{111}^{40} = 0$ cannot vanish w.c..
   It is an easy task to verify that all cases in which $Q_{100}^{71} = 0$ cannot be solved for A_3 and A_4 yield a contradiction.
   b. $B_3 B_4 = 0$, $n \neq 0$: W.l.o.g. we set $B_3 = 0$. Then Q_{100}^{71} can only vanish w.c. for $n = \nu$. If we assume $J_l := A_2 b_3 (b_2 - b_l) - A_3 b_2 (b_3 - b_l) \neq 0$ we can compute A_k from $Q_{021}^{40} = 0$ with $k, l \in \{5,6\}$ and $k \neq l$. W.l.o.g. we set $k = 5$. Then Q_{021}^{31} can only vanish w.c. for $A_3 = b_3 A_2/b_2$. Now $Q_{101}^{24} = 0$ yields the contradiction. The special case $J_5 = J_6 = 0$ implies $b_5 = b_6 = b_2 b_3 (A_2 - A_3)/|\mathbf{A},\mathbf{b}|_2^3$. But then $Q_{021}^{31} = 0$ yields the contradiction.
   c. $b_3 = b_4$, $B_3 B_4 n \neq 0$: Then Q_{100}^{71} can only vanish w.c. for $n = \nu$ and $Q_{100}^{62} = 0$ implies the contradiction.
   d. $n = \nu$, $B_3 B_4 n (b_3 - b_4) \neq 0$: Now $Q_{110}^{51} = 0$ already yields the contradiction.
2. We remain with the discussion of the special case $I_j = 0$. We express A_i from $I_j = 0$ with $j \in \{5,6\}$ and $i \in \{3,4\}$. W.l.o.g. we set $j = 6$ and $i = 3$. Then $Q_{110}^{60} = 0$ can only vanish w.c. in the following cases:
   a. $B_3 = 0$: Then $Q_{101}^{60} = 0$ implies an expression for a_5.
      i. $b_5 \neq 0$: Now we can compute A_5 from $Q_{021}^{40} = 0$. Then Q_{021}^{31} can only vanish w.c. for $n = 0$. Finally $Q_{021}^{22} = 0$ yields the contradiction.

ii. $b_5 = 0$: Now Q_{021}^{40} can only vanish w.c. for $b_6 = 0$. Then $Q_{021}^{31} = 0$ implies $n = 0$ and $Q_{021}^{22} = 0$ yields the contradiction.

b. $b_3 = b_4$, $B_3 \neq 0$: This case can exactly be done as item a.

c. $T := b_4 b_5 C_{(2,4,5)} - b_4 b_6 C_{(2,4,6)} + b_5 b_6 B_4 (A_5 - A_6) = 0$, $B_3(b_3 - b_4) \neq 0$:

i. $b_5 \neq 0$: Under this assumption we can compute A_5 from $T = 0$. Then Q_{100}^{71} can only vanish w.c. for $n = 0$. Finally $Q_{200}^{33} = 0$ yields the contradiction.

ii. $b_5 = 0$: Now T can only vanish w.c. for $b_6 = 0$. Then $Q_{100}^{71} = 0$ implies $n = 0$ and $Q_{200}^{33} = 0$ yields the contradiction.

3. It is impossible to solve $I_j = 0$ for A_i with $i \in \{5,6\}$ and $j \in \{3,4\}$ for:

a. $B_l = 0$, $b_k = b_5 = b_6$ with $l,k \in \{3,4\}$, $l \neq k$: W.l.o.g. we set $l = 3$.

i. $J_{5,6} \neq 0$: Under this assumption we can express a_2 from $Q_{021}^{40} = 0$. Then $Q_{021}^{31} = 0$ can only vanish w.c. for $n|\mathbf{A},\mathbf{b}|_2^3 = 0$. For $A_3 = b_3 A_2 / b_2$ we get $n = \mu$ from $Q_{101}^{42} = 0$ and finally $Q_{100}^{53} = 0$ yields the contradiction.
For $n = 0$, $|\mathbf{A},\mathbf{b}|_2^3 \neq 0$ we get the contradiction from $Q_{021}^{22} = 0$.

ii. $J_{5,6} = 0$: W.l.o.g. we can express A_3 from $J_{5,6} = 0$. Now $Q_{110}^{60} = 0$ can only vanish w.c. for $(a_5 - a_6)(B_4 - B_6) = 0$. In both cases $Q_{021}^{31} = 0$ yields $n = 0$ and $Q_{021}^{22} = 0$ the contradiction.

b. $b_3 = b_4 = 0$: Now $Q_{021}^{40} = 0$ implies $|\mathbf{a},\mathbf{b},\mathbf{Ab}|_{(2,5,6)} = 0$ which can be solved for A_6 w.l.o.g.. Then $Q_{021}^{31} = 0$ yields $n = 0$ and $Q_{021}^{22} = 0$ the contradiction.

Part [C] $I_5 I_6 = 0$, $B_5 \neq B_6$

We express A_i from $I_j = 0$ with $j \in \{5,6\}$ and $i \in \{3,4\}$. W.l.o.g. we set $j = 6$ and $i = 3$. Then $Q_{110}^{60} = 0$ can only vanish w.c. for $B_3(b_3 - b_4)L = 0$ with

$$L := B_4 B_5 b_6 (b_4 - b_5)(A_6 - A_2) + B_4 B_6 b_5 (b_4 - b_6)(A_2 - A_5) + B_5 B_6 b_4 (b_5 - b_6)(A_4 - A_2).$$

1. $B_3 = 0$: Then $Q_{101}^{60} = 0$ implies an expression for a_5.

a. $b_5 \neq 0$: Under this assumption we can compute A_5 from $Q_{021}^{40} = 0$. Then Q_{021}^{31} can only vanish w.c. for $n b_4 G[14] = 0$.

i. $b_4 = 0$: We get $n = \nu$ from $Q_{101}^{51} = 0$ and the contradiction from $Q_{101}^{42} = 0$.

ii. $G = 0$, $b_4 \neq 0$: Assuming $b_5 \neq b_6$ we can express A_6 from $G = 0$. Then $Q_{110}^{51} = 0$ implies $n = \nu$ and $Q_{110}^{42} = 0$ yields the contradiction.
For the remaining case $b_5 = b_6$ we get $A_2 = A_4$ from $G = 0$. Then $Q_{021}^{31} = 0$ implies $n = \nu$ and $Q_{021}^{22} = 0$ yields the contradiction.

iii. $n = 0$, $b_4 G \neq 0$: We get the contradiction from $Q_{021}^{22} = 0$.

b. $b_5 = 0$: We distinguish again two cases:

i. $b_6 \neq 0$: Now we can express A_6 from $Q_{021}^{40} = 0$. Then Q_{021}^{31} can only vanish w.c. for $(A_5 - A_6)n = 0$. For $A_5 = A_6$ we get the contradiction from $Q_{111}^{40} = 0$. For the remaining case $n = 0$, $A_5 \neq A_6$ we get the contradiction from $Q_{021}^{22} = 0$.

ii. $b_6 = 0$: $Q_{021}^{40} = 0$ implies $A_2 = A_4$. Now $Q_{021}^{31} = 0$ can only vanish w.c. for $(A_5 - A_6)n = 0$. We can construct the same contradiction as in case i.

2. $b_3 = b_4, B_3 \neq 0$: Now Q_{101}^{60} can only vanish w.c. for $B_5[a_2b_4(b_6-b_5)+a_5b_2(b_4-b_6)+a_6b_2(b_5-b_4)] = 0$. In both cases $Q_{201}^{31} = 0$ implies $n = \nu$ and $Q_{201}^{22} = 0$ yields the contradiction.
3. $L = 0, B_3(b_3-b_4) \neq 0$: We distinguish the following two cases:
   a. $b_5 \neq 0$: Under this assumption we can express A_5 from $L = 0$. Then we get $Q_{010}^{80} = b_4b_6C_{(2,4,6)}R[162]$. As all 3 cases $b_4b_6C_{(2,4,6)} = 0$ yield easy contradictions we compute $R+Q_{101}^{60}$ which cannot vanish w.c..
   b. $b_5 = 0$: In this case L can only vanish w.c. in the following 3 cases:
      i. $b_4 \neq 0$: Now $Q_{200}^{42} = 0$ implies $n = 0$ or $n = \nu$. In both cases $Q_{200}^{33} = 0$ yields the contradiction.
      ii. $B_5 = 0$, $b_4 \neq 0$: In this case the conditions $Q_{200}^{42} = 0$ and $Q_{200}^{33} = 0$ show that $b_4(A_4B_6 - A_6B_4 + A_5B_4 - A_2B_6) + b_6B_4(A_2 - A_5) = 0$ must hold. W.l.o.g. we can express A_5 from this condition. Then $Q_{101}^{60} = 0$. implies an expression for A_6 and Q_{010}^{80} can only vanish w.c. for $b_6 = 0$. Finally $Q_{001}^{80} = 0$ yields the contradiction.
      iii. $b_6 = 0$, $b_4B_5 \neq 0$: Now $Q_{100}^{71} = 0$ can only vanish w.c. for $nC_{(2,5,6)} = 0$. Firstly, we express A_6 from the collinearity condition. Then $Q_{100}^{62} = 0$ can only vanish w.c. for $n = 0$ or $n = \nu$. In both cases $Q_{100}^{53} = 0$ yields the contradiction. In the remaining case $n = 0$, $C_{(2,5,6)} \neq 0$ we get the contradiction from $Q_{100}^{62} = 0$.

It is impossible to solve $I_j = 0$ for A_i with $i \in \{5,6\}$ and $j \in \{3,4\}$ for:

1. $B_l = 0, b_k = b_5 = b_6$ with $l,k \in \{3,4\}$, $l \neq k$: W.l.o.g. we set $l = 3$.
   a. $J_{5,6} \neq 0$: Under this assumption we can express A_5 from $Q_{021}^{40} = 0$. W.l.o.g. we can solve $Q_{010}^{80} = 0$ for A_6. Then Q_{100}^{71} can only vanish w.c. for $n = \nu$. Finally $Q_{100}^{62} = 0$ yields the contradiction.
   b. $J_{5,6} = 0$: As $|\mathbf{A},\mathbf{b}|_2^3 = 0$ yields together with $J_{5,6} = 0$ a contradiction we can solve $J_{5,6} = 0$ for b_6 w.l.o.g.. Then we can express a_2 from $Q_{021}^{40} = 0$. Now we get a_5 from the only non-contradicting factor of $Q_{010}^{80} = 0$. Then Q_{100}^{71} can only vanish w.c. for $n = \nu$. Finally $Q_{100}^{62} = 0$ yields the contradiction.

2. $b_3 = b_4 = 0$: W.l.o.g. we can be solved for A_5 and B_5 from $Q_{021}^{40} = 0$ and $Q_{111}^{40} = 0$. Then $Q_{021}^{31} = 0$ can only vanish w.c. for $(A_2 - A_6)n = 0$. For $A_2 = A_6$ we get $n = 0$ from $Q_{111}^{31} = 0$ and $Q_{111}^{22} = 0$ yields the contradiction.
   For the remaining case $n = 0$, $A_2 \neq A_6$ we get the contradiction from $Q_{021}^{22} = 0$.

Part [D] $|\mathbf{a},\mathbf{b}|_5^6 = 0$, $(B_5 - B_6)I_5I_6 \neq 0$

1. We start with the special case $b_5 = b_6 = 0$. In this case $Q_{100}^{80} = 0$ implies $a_5 = a_6$. Then $Q_{110}^{60} = 0$ already yields the contradiction.
2. Therefore we can assume w.l.o.g. that $b_6 \neq 0$. We set $a_5 = b_5a_6/b_6$. Then the resultant of Q_{100}^{80} and Q_{110}^{60} with respect to A_6 can only vanish w.c. for $B_3B_4(b_3 - b_4) = 0$. For all cases we get the contradiction from $Q_{100}^{80} = 0$ and $Q_{110}^{60} = 0$. □

4 Conclusion

In this article we proved the following main theorem (cf. Theorem 2 and 3):

Main Theorem. $\mathsf{X}(\mathsf{a})$-*singular planar Stewart Gough platforms with* $\alpha \neq \beta$ *and where* a *is not orthogonal to* Φ *or* φ *are necessarily architecturally singular.*

Consequences of this main theorem are the following:

- The manipulators given in Theorem 1 are the only non-architecturally singular planar SG platforms with $\alpha \neq \beta$ which are Schönflies-singular.
  Moreover it should be noted, that the missing special cases (i.e. $\alpha = \beta$) of Schönflies-singular planar Stewart Gough platforms are given in [14]. Therefore paper [14] also finishes the discussion of Schönflies-singular planar parallel manipulators which was started by Wohlhart [16] by giving an example for a $\mathsf{X}(\mathsf{a})$-singular planar SG platform of case (2a).
  The presented example was the so-called *polygon platform*, i.e. a manipulator where the platform and base anchor points are related by an inversion. This manipulator even possesses a Schönflies self-motion because it is a special case of a parallel manipulator with Schönflies Borel-Bricard motions (cf. Husty and Zsombor-Murray [17]) listed by Borel [18]. That Borel's list is complete was proven by Husty and Karger in [19].
  Therefore the only open problem in this context is the determination of all non-planar Schönflies-singular Stewart Gough platforms.
- Mick and Röschel proved in Theorem 4.1 of [13] that a planar SG platform is architecturally singular if and only if it is singular with respect to a special 5-parametric set of displacements. Due to the given main theorem for Schönflies-singular manipulators we can improve this statement even to 4-parametric sets of displacements, namely the Schönflies motion groups for which Theorem 2 and 3, respectively, hold.
  Note that this is a new characterization of architecturally singular planar SG platforms beside the already existing ones (cf. Karger [6, 7], Nawratil [8], Röschel and Mick [9] as well as Wohlhart [10]).
  The question remains open, if this statement can further be improved to an even 3-dimensional Lie subgroup of $\mathsf{SE}(3)$, which are $\mathsf{SO}(3)$ and $\mathsf{H}(\mathsf{d}) \rtimes \mathbb{R}^2$ (cf. [20]). The latter is composed of translations on a plane and a helical motion (with pitch p) along the normal direction d of the plane. $\mathsf{H}(\mathsf{d}) \rtimes \mathbb{R}^2$ also includes the Cartesian motion group $\mathsf{T}(3)$ ($p = \infty$) and the planar motion group $\mathsf{SE}(2)$ ($p = 0$) as special cases. Due to the presented main theorem and the results given in [4, 14] we can restrict $\mathsf{H}(\mathsf{d}) \rtimes \mathbb{R}^2$ to $p \in [0, \infty[$ with $\angle(\Phi, \mathsf{d}) \neq \angle(\varphi, \mathsf{d})$ and d not orthogonal to Φ or φ.

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