On the spin surface of RSSR mechanisms with parallel rotary axes

Georg Nawratil*
Vienna University of Technology, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstrasse 8-10/104, Vienna, A-1040, Austria

(Dated: May 30, 2009)

Due to Cayley's theorem the line $s \in \Sigma$ (= moving system) spanned by the centers of the spherical joints of an RSSR linkage generates a surface of degree 8. In the special case of parallel rotary axes of the R-joints the corresponding ruled surface is only of degree 6. Now the point locus of any point $X \in \Sigma \setminus \{s\}$ is a surface of order 16 (general case) or of order 12 (special case). Hunt suggested that the circularity of this so called spin-surface for the general case is 8 and this was later proved. We demonstrate that the circularity of the spin-surface for the special case is 4 instead of 6 as given in the literature. As a consequence generalized TSSM manipulators (the three rotary axes need not be coplanar) with two parallel rotary joints can have up to 16 solutions instead of 12. We show that this upper bound cannot be improved by constructing an example for which the maximal number of assembly modes is reached. Moreover we list all parallel manipulators of this type where more than $4 \times 2 = 8$ points are located on the imaginary spherical circle.

PACS numbers: Valid PACS appear here

I. INTRODUCTION

An RSSR mechanism is a closed kinematic chain, where two points B_i $(i = 1, 2)$ of the end-effector Σ are connected via spherical joints S_i centered in B_i with the systems Σ_i, which themselves are coupled via rotary joints R_i with the fixed system Σ_0. The RSSR linkage is illustrated in FIG. 1, where A_i denotes the footpoint on the rotary axis a_i of the joint R_i with respect to B_i.

Due to Grüblers formula the RSSR mechanism has two degrees of freedom (dof) where one degree is the rotation of Σ about the line s spanned by B_1 and B_2. Therefore we get a constrained motion for all points of s in contrast to those $X \in \Sigma \setminus \{s\}$ which are located on a so called spin surface Φ. It should be noted that Φ is also known as qeeroid (cf. [15]).

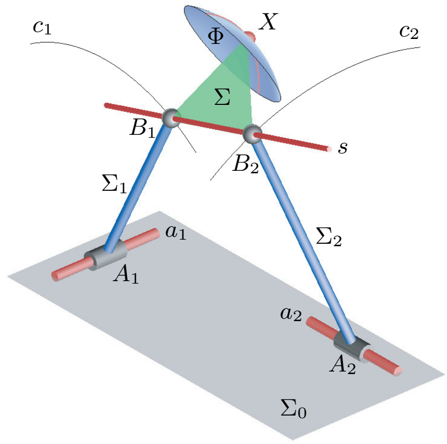

FIG. 1: Sketch of an RSSR mechanism and the spin surface Φ traced by a point $X \in \Sigma \setminus \{s\}$.

A. Related Work

The order of the ruled surface generated by the line s can be determined by applying the two so called Cayley theorems given by Fichter and Hunt [3]. The first Cayley theorem stats that if two points B_1 and B_2 a fixed distance apart on a line s are constrained to lie respectively on two curves c_1 and c_2 (either spatial curves or planar curves lying in non-parallel planes) then s generates a ruled surface whose degree is give by $2n_1(n_2-p_2)+2n_2(n_1-p_1)$, where n_i denotes the order of c_i and p_i the circularity of c_i $(i = 1, 2)$. In the special case that c_1 and c_2 are planar curves lying in parallel planes the order of the ruled surface can be computed by the second Cayley theorem as $2n_1(n_2-p_2)+2n_2(n_1-p_1)-2p_1p_2$.

For the RSSR mechanism c_i are circles ($n_i = 2$ and $p_i = 1$) and therefore the ruled surface is of order 8 (general case) and 6 (special case), respectively. Hunt [4] proved by means of algebraic connection theory that the spin surface Φ generated by the rotation of a point $X \in \Sigma \setminus \{s\}$ about s is of order $2 \times 8 = 16$ (general case) and $2 \times 6 = 12$ (special case), respectively. Moreover Hunt [5] suggested that the circularity of the spin surface is 8 which was later proved by Merlet [11].

For the special case Lazard and Merlet [10] stated that the circularity is 6. As a consequence the number of assembly modes for a parallel manipulator of TSSM type with two parallel rotary axes was given by 12 (cf. [13]). Moreover the circularity of 6 was used in the first approach of [10] to reason the number of assembly modes for the Stewart platform.

*Electronic address: nawratil@geometrie.tuwien.ac.at

B. Overview

In Section II we prove that the circularity of the spin surface of the special case is only 4. As a consequence the generalized TSSM manipulator (the three rotary axes need not be coplanar) with two rotary axes cannot have more than 16 real solutions (see Section III). Moreover we list all parallel manipulators of this type where more than $4 \times 2 = 8$ points located on the imaginary spherical circle. As one of these special cases the Stewart platform appears.

In Section IV we demonstrate that the TSSM with two parallel rotary axes can also have 16 solutions (as the general case; cf. [12]) by constructing an example for which the maximal number of assembly modes is reached.

II. CIRCULARITY OF THE SPIN-SURFACE

Theorem 1. *The spin surface of an RSSR mechanism with parallel rotary axes contains the imaginary spherical circle four times, i.e. the circularity equals four.*

Proof: Without loss of generality (w.l.o.g.) we can choose a Cartesian coordinate system in the fixed system Σ_0 such that the circles c_i traced by the points B_i are located in planes parallel to the xy-plane. Moreover we can assume that A_1 equals the origin and that A_2 is located in the positive quadrant of the xz-plane. Thus we get:

$$\mathbf{A}_1 = (0,0,0)^T \quad \text{and} \quad \mathbf{A}_2 = (t,0,u)^T, \tag{1}$$

with $t, u \geq 0$. The paths of B_i parametrized by

$$\mathbf{B}_i(\mu_i) = \mathbf{A}_i + (r_i \cos\mu_i, r_i \sin\mu_i, 0)^T \tag{2}$$

with $r_i := \overline{A_i B_i}$ are coupled by the constraint

$$P_0 := \|\mathbf{B}_1(\mu_1) - \mathbf{B}_2(\mu_2)\|^2 - b^2 = 0, \tag{3}$$

where $b > 0$ denotes the fixed distance $\overline{B_1 B_2}$. Moreover $b \geq u$ must hold because otherwise the RSSR mechanism can not be assembled. The locus of a point X with coordinates $\mathbf{X} = (x, y, z)$ of $\Sigma \setminus \{s\}$ is determined by the conditions

$$P_i := \|\mathbf{B}_i(\mu_i) - \mathbf{X}\|^2 - d_i^2 = 0 \quad \text{for} \quad i = 1, 2 \tag{4}$$

with $d_i := \overline{B_i X}$. After applying the half angle substitution

$$\cos\mu_i = \frac{1 - m_i^2}{1 + m_i^2} \quad \text{and} \quad \sin\mu_i = \frac{2m_i}{1 + m_i^2} \tag{5}$$

in Equ. 3 and 4 to get algebraic expressions we can start the elimination process. We compute the resultant of the numerator of P_0 and the numerator of P_1 with respect to m_1 which yields a polynomial P_{01}. Now we eliminate m_2 from P_{01} and P_2 again with the resultant method. This yields already the polynomial Q of the spin surface Φ which is indeed of degree 12 in the unknowns x, y, z.

For the determination of the circularity of this surface we introduce homogeneous coordinates by

$$x := x_1/x_0, \quad y := x_2/x_0, \quad z := x_3/x_0. \tag{6}$$

and multiply Q by x_0^{12}. Now we intersect the Φ with the plane at infinity ω determined by $x_0 = 0$ which yields $2^{16} r_1^4 r_2^4 F_1 F_2 (x_1^2 + x_2^2 + x_3^2)^4$ with

$$\begin{aligned} F_1 &:= (x_1^2 + x_2^2)(u - b)^2 + x_3 t\left[x_3 t - 2x_1(u - b)\right], \\ F_2 &:= (x_1^2 + x_2^2)(u + b)^2 + x_3 t\left[x_3 t - 2x_1(u + b)\right]. \end{aligned} \tag{7}$$

Up to this point we have only proven that the intersection multiplicity of ω and Φ along k_0 is 4. It is not clear that k_0 is a 4-fold curve of ϕ because it could for example also be the case that k_0 is a 3-fold curve and that ω touches Φ along k_0.

The proof can be done by intersecting Φ with an arbitrary circle c_3. By using the abbreviation $C := \frac{1-h^2}{1+h^2}$, $S := \frac{2h}{1+h^2}$ and $D := 1 - C$ the circle can be parametrized as $\mathbf{M} \cdot (\alpha, \beta, \gamma)^T + (f_1, f_2, f_3)^T$ with

$$\mathbf{M} = \begin{bmatrix} (1 - v_1^2)C + v_1^2 & v_1 v_2 D - v_3 S & v_1 v_3 D + v_2 S \\ v_1 v_2 D + v_3 S & (1 - v_2^2)C + v_2^2 & v_2 v_3 D + v_1 S \\ v_1 v_3 D - v_2 S & v_2 v_3 D - v_1 S & (1 - v_3^2)C + v_3^2 \end{bmatrix} \tag{8}$$

and where $(v_1, v_2, v_3)^T$ denotes the unit vector in direction of the rotary axis, $(\alpha, \beta, \gamma)^T$ the coordinates of any point of $\mathbb{R}^3$ and $(f_1, f_2, f_3)^t$ the translation vector. Plugging these expressions into the implicit representation of Φ yields a polynomial of degree 16 in the unknown h.

As the circle c_3 and Φ have $2 \times 12 = 24$ intersection points, the $24 - 16 = 8$ common points of c_3 and Φ must be located on the plane at infinity ω. Therefore the two cyclic points of c_3 (which are the intersection points of c_3 and ω located on the imaginary spherical circle) must be 4-fold points of the spin surface Φ. ☐

III. CONSEQUENCES FOR PARALLEL MANIPULATORS

A parallel manipulator of TSSM type consists of a platform Σ which is connected via three Spherical-Prismatical-Rotational legs with the base Σ_0 (see FIG. 2), where the axes a_i for $i = 1, 2, 3$ of the rotational joints are coplanar. If we skip the assumption of coplanarity we get a more generalized class of parallel manipulators which we will call GTSSM for the rest of this article.

Theorem 1 has the following consequences for this class of parallel manipulators:

Theorem 2. *GTSSM manipulators with two parallel rotary axes cannot have more than* 16 *assembly modes except the degenerated cases with infinitely many solutions.*

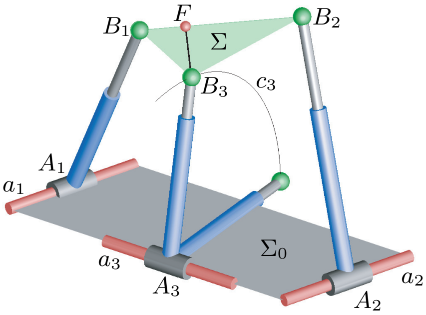

FIG. 2: TSSM with the assembled and disconnected third leg, respectively, and parallel rotary axes a_1, a_2.

Proof: For solving the direct kinematics of a given manipulator (geometry of the platform and base as well as the length of the three legs are known) we disconnect the third leg from the platform and consider the resulting RSSR mechanism with parallel rotary axes. Now the platform-anchor-point of the disconnected leg is located on the spin surface of order 12. The circle traced by the corresponding point of the third leg intersects this surface $12 \times 2 = 24$ times where $4 \times 2 = 8$ points are located on the imaginary spherical circle. Therefore such a mechanism cannot have more than 16 real solutions. □

A. Degenerated Cases

The classification of the self-motions of GTSSM manipulators is still an open problem because the computation of these degenerated cases with infinitely many solutions is a very complicated task (cf. [6–9]).

But the self-motions of the special class of TSSM manipulators are almost completely known:

- In the case of the TSSM parallel manipulator with intersecting axes the degenerated cases correspond to the three types of Bricard's flexible octahedra (cf. [1, 16]).

- The self-motions of TSSM manipulators with two parallel axes correspond to flexible octahedra with one vertex at infinity. Clearly, it is impossible to construct such flexible octahedra of type 1, because all three pairs of opposite vertices must be symmetric with respect to a common line. But there exist flexible octahedra of type 2 and 3 with one vertex at infinity.

  In type 2 two pairs of opposite vertices are symmetric with respect to a common plane which passes through the vertex at infinity and its opposite vertex. The construction of flexible octahedra of type 3 with one vertex at infinity was given by Stachel in [17].

  As the known proofs for Bricard's flexible octahedra assume all six vertices to be Euclidean points, it is not entirely clear if these are all cases of flexible octahedra with one vertex at infinity. But there are good reasons to conjecture that no other cases exist.

- For TSSM manipulators with 3 parallel axes the problem reduces to a planar one, as these manipulators possess a cylindrical singularity surface (cf. [14]). In this case the degenerated cases correspond to the well known self-motions of 3-dof RPR manipulators.

B. Special Cases

In the following we are interested in those cases where more than 8 intersection points are located on the imaginary spherical circle. A necessary condition for this is that the cyclic points of the circle c_3 traced by the end point of the third leg also belong to at least one of the conic sections $k_i \in \omega$ determined by $F_i = 0$ of Equ. (7) for $i = 1, 2$. Therefore we compute the intersection points of k_i and k_0 by eliminating x_3 from the corresponding equations.

The resulting two expressions L_i split up into $(x_1^2 + x_2^2)N_i$ where N_i is of degree 2 in the unknows x_1, x_2. Solving these factors for x_1 we get the corresponding x_3 values as common solution of k_0 and k_i after back-substitution of x_1. By setting $q_1 := u - b$ and $q_2 := u + b$ the homogeneous coordinates of the intersection points $J_0, \overline{J}_0, J_i, \overline{J}_i$ of k_0 and k_i can be written as

$$\mathbf{J}_0 = [1, I, 0]^T, \quad \mathbf{J}_i = \left[(q_i^2 - t^2)I, q_i^2 + t^2, -2tq_iI\right]^T, \tag{9}$$

for $i = 1, 2$. Moreover it should be noted that I denotes the imaginary unit and $\overline{J}_j$ the conjugate complex point of J_j for $j = 0, 1, 2$. A sketch of the situation at the plane at infinity is given in FIG. 3.

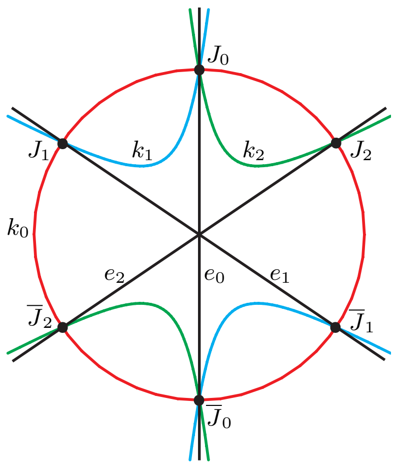

FIG. 3: Sketch of the situation at the plane at infinity ω.

As the carrier plane ε of c_3 has to intersect k_0 in conjugate complex points there are the following possibilities left for choosing $e := \varepsilon \cap \omega$:

a) $e_i := \left[J_i, \overline{J}_i\right]$ for $i = 1, 2$: Now e_i with projective line coordinates

$$\mathbf{J}_i \times \overline{\mathbf{J}}_i = [2q_it : 0 : q_i^2 - t^2]^T \tag{10}$$

intersects Φ in the point J_i and $\overline{J}_i$ with multiplicity 5, but these points are not 5-fold points of Φ which can be shown as follows: We intersect Φ with the line l spanned by the origin and the point J_i and $\overline{J}_i$, respectively, by inserting its parametric representation into Φ. Then the resulting equation splits up into $2^{16}r_1^4r_2^4F$ where F is a polynomial of degree 8 in the parameter of l. Therefore e_i touches Φ in J_i and $\overline{J}_i$ which are still 4-fold points of Φ.

b) $e_0 := \left[J_0, \overline{J}_0\right]$: As J_0 and $\overline{J}_0$ are located on k_1 and k_2 the line e_0 with projective line coordinates

$$\mathbf{J}_0 \times \overline{\mathbf{J}}_0 = [0:0:1]^T \tag{11}$$

intersects Φ in these points with a multiplicity of 6. Moreover J_0 and $\overline{J}_0$ are 6-fold points of Φ which can be proven by setting $v_1 = v_2 = 0$ and $v_3 = 1$ in Equ. (8). If we now plug the parametric representation of the circle c_3 into the equation of Φ we end up with a polynomial of degree 12 in the unknown h which finishes the proof.

In the following we have to discuss the special cases $e_1 = e_2$, $e_0 = e_1$, $e_0 = e_2$ and $e_0 = e_1 = e_2$, respectively:

- If $e_1 = e_2$ holds this line intersects Φ in $J_1 = J_2$ and $\overline{J}_1 = \overline{J}_2$ with multiplicity 6. This happens if

$$det \begin{bmatrix} 2tq_1 & q_i^2 - t^2 \\ 2tq_2 & q_2^2 - t^2 \end{bmatrix} = 4tb(u^2 - b^2 + t^2) = 0. \tag{12}$$

  As b must be greater than zero we only get as solution $t = 0$ and $b^2 = u^2 + 1$, respectively. For the latter it can be shown as in *a*) that $J_1 = J_2$ and $\overline{J}_1 = \overline{J}_2$ are only 4-fold points of Φ. $t = 0$ will be discussed as last point of this case study.

- e_0 equals e_1 for $q_1 t = 0$. As $t = 0$ will be treated latter on we set $u = b$ and assume $t \neq 0$. In this case $J_0 = J_1$ and $\overline{J}_0 = \overline{J}_1$ are 7-fold points of Φ which can be proven analogously to *b*). Therefore such a manipulator cannot have more than 10 assembly modes.

  But in this case the given threshold can be refined, because for $u = b$ and $t \neq 0$ the ruled surface generated by s can only have two real generators. Therefore the spin surface Φ degenerates into two coplanar circles which can be intersected by a further circle c_3 in a maximum of 4 real points. Moreover it should be noted that such a manipulator has only 3 dofs, namely the translations in x and y direction as well as the rotation about s.

- e_0 equals e_2 for $q_2 t = 0$. As we assumed (w.l.o.g.) $u \geq 0$ and $b > 0$ this can only happen for $t = 0$. But for $t = 0$ we get $e_0 = e_1 = e_2$ and the points $J_0 = J_1 = J_2$ and $\overline{J}_0 = \overline{J}_1 = \overline{J}_2$ are 8-fold points of Φ which can again be proven as in case *b*).

  If additionally $u = b$ holds the manipulator can only be assembled for $r_1 = r_2$. Now the ruled surface traced by s is a cylinder of rotation and the point X can only be located in an annulus. Trivially such a manipulator has again only 3 dofs and a maximum of 4 real solutions.

The results of this case study are summed up in the following theorem:

Theorem 3. *GTSSM manipulators with two parallel rotary axes (a_1 and a_2) cannot have more than*

i) 12 *assembly modes if the axis a_3 is parallel to a_1, a_2*

ii) 8 *assembly modes if the axis a_3 is parallel to the coinciding axes $a_1 = a_2$*

iii) 4 *assembly modes if $u = b$ holds*

except in the degenerated cases with infinitely many solutions.

In order to show that the upper bounds of assembly modes given in Theorem 2 and 3 cannot be improved we must present examples which possess the maximal number of real solutions for the direct kinematics.

For the manipulator of Theorem 3 *i*) which corresponds to the Stewart platform an example with 12 real solutions was given by Lazard and Merlet [10]. An example with 8 real solutions for the manipulator of Theorem 3 *ii*) can easily be constructed by applying the same approach used in [10] because this manipulator is nothing but a special case of the Stewart platform. The construction of a manipulator of Theorem 3 *iii*) with 4 real solutions is trivial anyway.

Therefore the only open problem in this regard is to demonstrate that there exists a parallel manipulator of GTSSM type with two parallel rotary axes with 16 assembly modes. How such an example can be constructed is outlined in the next section.

IV. MAXIMUM NUMBER OF ASSEMBLY MODES

One possibility to generate an example with a maximal number of assembly modes is the iterative algorithm presented by Dietmaier [2] which was used for the construction of a Stewart Gough Platform with 40 real solutions.

We will solve this problem with the help of a graphical tool after reducing it to a planar one. As we only have to find *one* example with 16 real solutions for the direct kinematics we make the following assumptions: As $u = 0$ does not reduce the maximal number of assembly modes (cf. Theorem 2 and 3) we set u equal to zero. As a consequence the RSSR mechanism with parallel rotary axes degenerates to an ordinary 4-bar mechanism in the xy-plane. Moreover we can assume that the rotary axis a_3 is also located in this plane.

Now the circles c_3 as well as the circle c_F traced by the point B_3 about $s := [B_1, B_2]$ lie in z-parallel planes where F denotes the footpoint on s with respect to B_3 (see FIG. 2 and 4). For the following considerations we also need the spheres Γ_3 and Γ_F determined by their centers A_3 and F, respectively, and by $c_3 \in \Gamma_3$ and $c_F \in \Gamma_F$. Trivially the carrier plane of the intersection circle c_Γ of these two spheres is also parallel to the z axis.

These three planes are mapped by the horizontal projection onto the straight lines l_3, l_F and l_Γ, respectively. Now the intersection point $G'_{1,2}$ of l_F and l_Γ corresponds to the horizontal projection of the intersection point G_1, G_2 of c_3 and c_F if and only if $G'_{1,2}$ is also located on l_3. For the case $l_F = l_\Gamma$ ($\Rightarrow l_F = l_\Gamma = l_3$) all points $G'_{1,2} \in l_3$ with $\overline{G'_{1,2}A'_3} \leq r_3$ correspond to points G_1, G_2 of $c_3 = c_F$.

In the following we have to find parameters such that the curve g' generated by $G'_{1,2}$ during the motion of the the 4-bar mechanism intersects l_3 in four real points L'_i. These points correspond in each case with two real intersection points of c_3 and c_F if $\overline{L'_iA'_3} < r_3$ holds. It should be noted that $\overline{L'_iA'_3} = r_3$ would yield a double solution ($G_1 = G_2$).

Based on these considerations it is not difficult to find such a configuration set with the help of a graphical tool (e.g. Euklid DynaGeo, http://www.dynageo.com). For the example illustrated in FIG. 4 the parameters are as follows:

$$\begin{aligned} &t = 100, \quad r_1 = 58, \quad r_2 = 119, \quad r_3 = \overline{B_3F} = 45 \\ &\mathbf{A}_3 = (25, 0), \quad b = 85, \quad \overline{B_1F} = 15, \quad \overline{B_2F} = 70 \end{aligned} \tag{13}$$

and a_3 equals the x-axis. Due to the symmetry of the chosen example the curve $\overline{g}'$ for the second assembly mode of the 4-bar mechanism can be obtained by reflecting g' on a_3. This yields the points $\overline{L}'_i$ for $i = 1, \ldots, 4$.

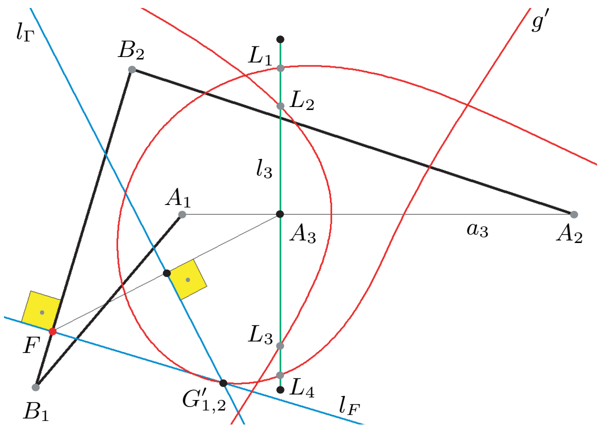

FIG. 4: Screenshot of the software Euklid DynaGeo, which was used to generate an example with 16 assembly modes.

We only illustrate 4 solutions (see FIG. 5–8) of the direct kinematics problem because the remaining 12 can be obtained by reflecting the given 4 solutions on the xy-plane and by applying a further reflection to the resulting 8 configurations on the xz-plane.

Moreover it should be noted that the given example not only belongs to the class of GTSSM manipulators but also to its subclass of TSSM type manipulators.

The numerical data of the platform anchor points B_i of the illustrated 4 solutions are given in the following table:

Solution	Point	x	y	z
L_1	B_1	-10.19	57.10	0
	B_2	55.91	110.53	0
	B_3	25	37.42	24.98
L_2	B_1	55.02	18.36	0
	B_2	38.16	101.67	0
	B_3	25	27.59	35.55
$\overline{L}_3$	B_1	56.25	14.14	0
	B_2	-14.22	-33.39	0
	B_3	25	33.65	29.88
$\overline{L}_4$	B_1	-16.30	55.66	0
	B_2	-15.33	-29.33	0
	B_3	25	41.13	18.25

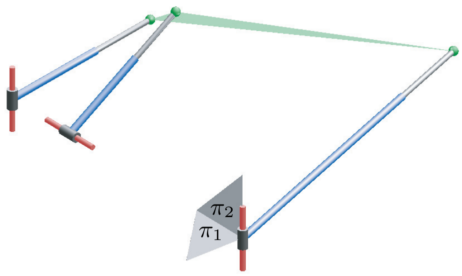

FIG. 5: Assembly mode which corresponds with L_1.

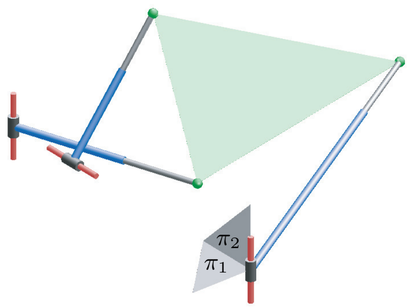

FIG. 6: Assembly mode which corresponds with L_2.

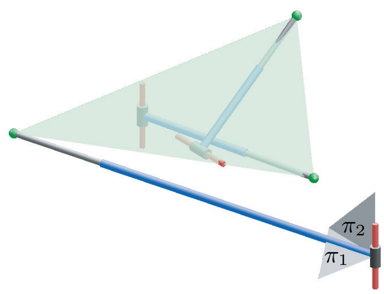

FIG. 7: Assembly mode which corresponds with $\overline{L}_3$.

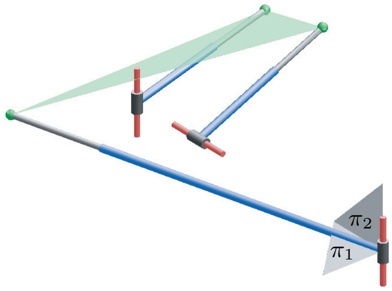

FIG. 8: Assembly mode which corresponds with $\overline{L}_4$.

V. CONCLUSION

We demonstrated that the circularity of the spin-surface for an RSSR mechanism with parallel rotary axes is 4 instead of 6 as given in the literature [10]. As a consequence a parallel manipulator of GTSSM type with two parallel rotary joints can have up to 16 solutions instead of 12 (cf. [13]). We showed that this upper bound cannot be improved by constructing an example for which the maximal number of assembly modes is reached.

Moreover we analyzed all parallel manipulators of GTSSM type with two parallel rotary axes where more than $4 \times 2 = 8$ points are located on the imaginary spherical circle. As one of these special cases the Stewart platform appears.

Acknowledgments

This research was carried out as part of the project S9206-N12 which was supported by the Austrian Science Fund (FWF).

The author would also like to thank the reviewers for their useful comments and suggestions, which have helped to improve the quality of this article.

[1] Bricard, R.: Mémoire sur la théorie de l'octaèdre articulé, Journal de Mathématiques pures et appliquées, Liouville **3** 113–148 (1897).

[2] Dietmaier, P.: The Stewart-Gough Platform of General Geometry can have 40 real Postures, Advances in Robot Kinematics: Analysis and Control (J. Lenarcic and M.L. Husty eds.), 7–16, Kluwer (1998).

[3] Fichter, E.F., and Hunt, K.H.: Mechanical Couplings - A General Geometrical Theory, Trans. ASME B, Journal of Engineering for Industry **99** 77–81 (1977).

[4] Hunt, K.H.: Kinematic Geometry of Mechanisms, Clarendon Press, Oxford (1978).

[5] Hunt, K.H.: Structural Kinematics of In-Parallel-Actuated Robot-Arms, Journal of Mechanisms, Transmissions, and Automation in Design **105** 705–712 (1983).

[6] Husty, M.: E. Borel's and R. Bricard's Papers on Displacements with Spherical Paths and their Relevance to Self-Motions of Parallel Manipulators, In Proc. of International Symposium on History of Machines and Mechanisms (M. Ceccarelli ed.), 163–172, Kluwer (2000).

[7] Husty, M., and Karger, A.: Self-Motions of Griffis-Duffy Type Platforms, In Proc. of IEEE International Conference on Robotics and Automation, 7–12 (2000).

[8] Karger, A.: New Self-Motions of Parallel Manipulators, Advances in Robot Kinematics - Analysis and Design (J. Lenarcic, P. Wenger eds.), 275-282, Springer (2008).

[9] Karger, A., and Husty, M.: Classification of all self-motions of the original Stewart-Gough platform, Computer-Aided Design **30** 205–215 (1998).

[10] Lazard, D., and Merlet, J-P.: The (true) Stewart Platform has 12 configurations, In Proc. of IEEE International Conference on Robotics and Automation, 2160–2165 (1994).

[11] Merlet, J-P.: Manipulateurs parallèles, 4eme partie : mode d'assemblage et cinématique directe sous forme polynomiale, Technical Report 1135, INRIA (1989).

[12] Merlet, J-P.: Direct Kinematics and Assembly modes of parallel manipulators, International Journal of Robotics Research **11** (2) 150–162 (1992).

[13] Merlet, J-P.: Parallel Robots, 2nd Edition, Springer (2006).

[14] Nawratil, G.: All planar parallel manipulators with cylindrical singularity surface. Mechanism and Machine Theory, submitted.

[15] Robertson, G.D., and Torfason, L.E.: The Qeeroid - a new kinematic surface, In Proc. of 4th World Congress on the Theory of Machines and Mechanisms, 717–719 (1975).

[16] Stachel, H.: Zur Einzigkeit der Bricardschen Oktaeder, Journal of Geometry **28** 41–56 (1987).

[17] Stachel, H.: Remarks on Bricard's flexible octahedra of type 3. In Proc. of 10th International Conference on Geometry and Graphics, 8–12 (2002).