Reducible compositions of spherical four-bar linkages without a spherical coupler component

Georg Nawratil

Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstrasse 8-10/104, Vienna, A-1040, Austria

Abstract

We use the output angle of a spherical four-bar linkage $\mathcal{C}$ as the input angle of a second four-bar linkage $\mathcal{D}$, where the two frame links are assumed in aligned position as well as the follower of $\mathcal{C}$ and the input link of $\mathcal{D}$. We determine all cases, where the relation between the input angle of the input link of $\mathcal{C}$ and the output angle of the follower of $\mathcal{D}$ is reducible and where none of the components produces a transmission, which equals that of a single spherical coupler. The problem under consideration is of importance for the classification of flexible 3×3 complexes.

Key words: Spherical four-bar linkage, reducible composition, 3×3 complex

1 Introduction

Let a spherical four-bar linkage $\mathcal{C}$ be given by the quadrangle $I_{10}A_1B_1I_{20}$ (see Fig. 1a) with the frame link $I_{10}I_{20}$, the coupler A_1B_1 and the driving arm $I_{10}A_1$. We use the output angle φ_2 of this linkage as the input angle of a second coupler motion $\mathcal{D}$ with vertices $I_{20}A_2B_2I_{30}$. The two frame links are assumed in aligned position as well as the driven arm $I_{20}B_1$ of $\mathcal{C}$ and the driving arm $I_{20}A_2$ of $\mathcal{D}$.

The goal of this paper is to complete the determination of all cases, where the relation between the input angle φ_1 of the arm $I_{10}A_1$ and the output angle φ_3

Email addresses: `nawratil@geometrie.tuwien.ac.at`, `Tel: (+43)-(0)1-58801/11322`, `Fax: (+43)-(0)1-58801/11399` (Georg Nawratil).
URL: `http://www.geometrie.tuwien.ac.at/nawratil` (Georg Nawratil).

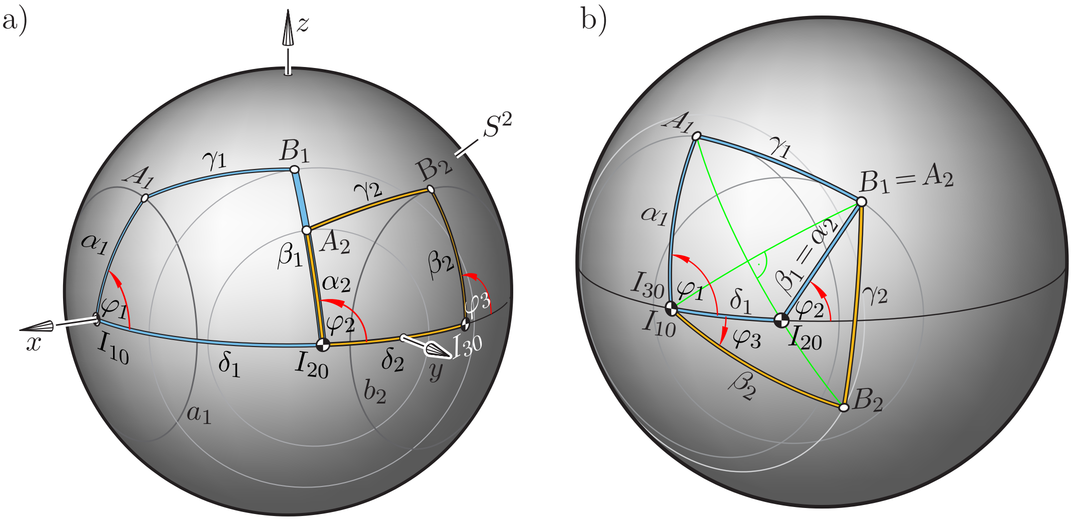

Fig. 1. a) Composition of the two spherical four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ with spherical side lengths $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1, 2$. b) Composition of two orthogonal four-bar linkages with $I_{30} = I_{10}$. (Courtesy H. Stachel)

of $I_{30}B_2$ is reducible. As all reducible compositions of spherical coupler components with a so-called spherical coupler component were already computed by the author in [1], we are only interested in cases without a spherical coupler component, i.e. no component produces a transmission which equals that of a single spherical coupler.

The problem under consideration is of importance for the classification of flexible Kokotsakis meshes (cf. [2], [3] and [4]) with a 4-sided planar central polygon, which are compounds of 3×3 planar quadrangular plates with hinges between neighboring plates.[1] This results from the fact that the spherical image of a 3×3 complex consists of two compositions of spherical four-bars sharing the transmission $\varphi_1 \mapsto \varphi_3$ (see Fig. 1a).

Based on this article and [1] it should be possible to give a complete list of flexible 3×3 complexes if *Stachel's conjecture* holds true that all multiply decomposable compounds of spherical four-bars are reducible (with exception of the translatory type and planar-symmetric type). The work towards this goal is in progress. Such a listing is of great interest, because Bobenko et al. [3] showed that a polyhedral mesh (in general position), where four planar quadrilaterals meet at each vertex, is flexible if and only if all 3×3 complexes are flexible. One possible application is the architectural design of flexible claddings composed of planar quads [5]. Further applications are e.g. the foldings of the roof at cabriolets [6] or new unfoldings of solar panels beside the Miura-ori technique [7]. Moreover, a listing of all flexible 3×3 complexes would also have an impact on the art of Origami [8].

[1] Such a structure is also known as 3×3 complex or Neunflach in German.

Remark: Based on the reducible compositions with a spherical coupler component (cf. [1]) the author was able to determine all flexible octahedra in the projective extension of the Euclidean 3-space (cf. [9] and [10]). $\diamond$

1.1 *Transmission by one spherical four-bar linkage*

We start with the analysis of the first spherical four-bar linkage $\mathcal{C}$ with the frame link $I_{10}I_{20}$ and the coupler A_1B_1 (Fig. 1a). We set $\alpha_1 := \overline{I_{10}A_1}$ for the spherical length (= arc length) of the driving arm, $\beta_1 := \overline{I_{20}B_1}$ for the output arm, $\gamma_1 := \overline{A_1B_1}$, and $\delta_1 := \overline{I_{10}I_{20}}$. We may suppose $0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < \pi$.

The movement of the coupler remains unchanged when A_1 is replaced by its antipode $\overline{A}_1$ and at the same time α_1 and γ_1 are substituted by $\pi - \alpha_1$ and $\pi - \gamma_1$, respectively. The same holds for the other vertices (cf. supplementary spherical four-bar linkages in [11]). When I_{10} is replaced by $\overline{I}_{10}$, also the sense of orientation changes, when the rotation of the driving bar $I_{10}A_1$ is inspected from outside of the unit sphere S^2 either at I_{10} or at $\overline{I}_{10}$.

We use a Cartesian coordinate frame with its origin at the spherical center, I_{10} on the positive x-axis and $I_{10}I_{20}$ in the xy-plane that I_{20} has a positive y-coordinate (see Fig. 1a). The input angle φ_1 is measured between $I_{10}I_{20}$ and the driving arm $I_{10}A_1$ in mathematically positive sense. The output angle $\varphi_2 = \sphericalangle\, \overline{I}_{10}I_{20}B_1$ is the oriented exterior angle at vertex I_{20}.

As given in [4], the constant spherical length γ_1 of the coupler implies the following equation:

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0, \tag{1}$$

with $t_i := \tan(\varphi_i/2)$, $c_{11} = 4\,\mathrm{s}\alpha_1\,\mathrm{s}\beta_1 \neq 0$,

$$\begin{aligned} c_{22} &= N_1 + K_1 - L_1 + M_1, \quad & c_{02} &= N_1 + K_1 + L_1 - M_1, \\ c_{20} &= N_1 - K_1 - L_1 - M_1, \quad & c_{00} &= N_1 - K_1 + L_1 + M_1, \end{aligned} \tag{2}$$

and

$$K_1 = \mathrm{c}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{s}\delta_1, \quad L_1 = \mathrm{s}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{s}\delta_1, \quad M_1 = \mathrm{s}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{c}\delta_1, \quad N_1 = \mathrm{c}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{c}\delta_1 - \mathrm{c}\gamma_1. \tag{3}$$

Herein s and c are abbreviations for the sine and cosine function. In these equations the spherical lengths α_1, β_1 and δ_1 are signed. For a more detailed explanation and alternative expressions of Eq. (1) see [4].

Remark: Note that the 2-2-correspondence of Eq. (1) depends only on the ratio of the coefficients $c_{22} : \cdots : c_{00}$ (cf. Lemma 1 of [12]). $\diamond$

Now we use the output angle φ_2 of the first four-bar linkage $\mathcal{C}$ as input angle of a second four-bar linkage $\mathcal{D}$ with vertices $I_{20}A_2B_2I_{30}$ and consecutive spherical side lengths α_2, γ_2, β_2, δ_2 (Fig. 1a). The two frame links are assumed in aligned position. In the case $\sphericalangle I_{10}I_{20}I_{30} = \pi$ the spherical length δ_2 is positive, otherwise negative. Analogously, a negative α_2 expresses the fact that the aligned bars $I_{20}B_1$ and $I_{20}A_2$ are pointing to opposite sides. Changing the sign of β_2 means replacing the output angle φ_3 by $\varphi_3 - \pi$. The sign of γ_2 has no influence on the transmission.

Due to Eq. (1), the transmission between the angles φ_1, φ_2 and the output angle φ_3 of the second four-bar with $t_3 := \tan(\varphi_3/2)$ can be expressed by the two biquadratic equations

$$\begin{aligned} \mathsf{C} &:= c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0\,, \\ \mathsf{D} &:= d_{22}t_2^2t_3^2 + d_{20}t_2^2 + d_{02}t_3^2 + d_{11}t_2t_3 + d_{00} = 0\,. \end{aligned} \tag{4}$$

The d_{ik} are defined by equations analogously to Eqs. (2) and (3). We eliminate t_2 by computing the *resultant* X (cf. [13]) of the two polynomials with respect to t_2. The resulting biquartic equation $X = 0$, where each monomial is of even degree, expresses a *4-4-correspondence* between points A_1 and B_2 on the circles a_1 and b_2, respectively (Fig. 1a). We call the composition reducible if the bivariate polynomial X is non-trivially reducible.

1.2.1 Known example of reducible compositions

Due to [1], the reducible compositions with a spherical coupler component can be summarized as follows:

Corollary 1 *If a reducible composition of two spherical four-bar linkages $\mathcal{C}$ and $\mathcal{D}$ with a spherical coupler component is given, then it is one of the following cases or a special case of them, respectively:*

a. One of the following four cases holds:

$$c_{00} = c_{22} = 0, \quad d_{00} = d_{22} = 0, \quad c_{20} = c_{02} = 0, \quad d_{20} = d_{02} = 0,$$

b. The following conditions hold for $\lambda \in \mathbb{R} \setminus \{0\}$:

$$\begin{gathered} c_{00}c_{20} = \lambda d_{00}d_{02}, \quad c_{22}c_{02} = \lambda d_{22}d_{20}, \\ c_{11}^2 - 4(c_{00}c_{22} + c_{20}c_{02}) = \lambda[d_{11}^2 - 4(d_{00}d_{22} + d_{20}d_{02})], \end{gathered}$$

c. One of the following two cases holds:

$$c_{22} = c_{02} = d_{00} = d_{02} = 0, \quad d_{22} = d_{20} = c_{00} = c_{20} = 0,$$

d. One of the following two cases holds for $A \in \mathbb{R} \setminus \{0\}$ and $B \in \mathbb{R}$:

- $c_{20} = Ad_{02},\ c_{22} = Ad_{22},\ c_{02} = Bd_{22},\ c_{00} = Bd_{02}, d_{00} = d_{20} = 0,\ d_{02}d_{22} \neq 0,$
- $d_{02} = Ac_{20},\ d_{22} = Ac_{22},\ d_{20} = Bc_{22},\ d_{00} = Bc_{20}, c_{00} = c_{02} = 0,\ c_{20}c_{22} \neq 0.$

Until now, to the author's best knowledge, there is only the following example of a reducible composition without a spherical coupler component known. This example was given by STACHEL [4] and reads as follows:

Two orthogonal four-bars [2] are combined that they have one diagonal in common (see Fig. 1b) with $\alpha_2 = \beta_1$ and $\delta_2 = -\delta_1$, hence $I_{30} = I_{10}$. Then the 4-4-correspondence between A_1 and B_2 is the square of a 2-2-correspondence of the form

$$s_{21}t_1^2t_3 + s_{12}t_1t_3^2 + s_{10}t_1 + s_{01}t_3 = 0$$

and therefore this component cannot produce a transmission, which equals that of a single spherical coupler.

Moreover, in the conclusion of [1] it was noted that this so-called *orthogonal* type can be generalized as follows: $d_{00} = Ac_{20}$, $d_{20} = Ac_{22}$, $d_{22} = Bc_{02}$, $d_{02} = Bc_{00}$ with $A, B \in \mathbb{R}$ and $\mathcal{C}$ being an orthogonal coupler ($\Leftrightarrow c_{00}c_{22} = c_{02}c_{20}$). [3]

2 Possible cases of reducible compositions without spherical coupler component

Given are the two spherical couplers $\mathcal{C}$ and $\mathcal{D}$ and their corresponding transmission equations C and D, respectively (see Eq. (4)). In the following we are interested in the conditions the c_{ij}'s and d_{ij}'s have to fulfill that the resultant X splits up into the product FG of two factors F and G. Depending on the structure of the factors F and G, we have to distinguish eleven cases, which are grouped as follows:

- **Symmetric symmetric (SS) case:** The first "symmetric" means that the structure of each of the factors F and G do not change if we interchange the unknowns t_1 and t_3; i.e. with each monomial $f_{ij}t_1^it_3^j$ resp. $g_{rs}t_1^rt_3^s$ there is also a monomial $f_{ji}t_1^jt_3^i$ resp. $g_{sr}t_1^st_3^r$ included in F resp. G.

  The second "symmetric" means that with each monomial $f_{ij}t_1^it_3^j$ in F there is also a term $g_{ji}t_1^jt_3^i$ in G and vice versa. [4] We distinguish two types of (SS) cases:

[2] Four bar-mechanisms, where the diagonals of the corresponding spherical quadrangle are orthogonal, are called orthogonal spherical four-bars (cf. [4]).
[3] This case will correspond to item 3 of the later given Theorem 1.
[4] Clearly, this also implies the explanation for the other three cases abbreviated by (SA), (AS) and (AA), respectively.

I. The first case is given by

$$\begin{aligned} \mathsf{F} =& f_{21}t_1^2t_3 + f_{12}t_1t_3^2 + f_{10}t_1 + f_{01}t_3, \\ \mathsf{G} =& g_{21}t_1^2t_3 + g_{12}t_1t_3^2 + g_{10}t_1 + g_{01}t_3, \end{aligned} \tag{5}$$

with $f_{21} \neq 0 \neq f_{12}$.

II. The second case is given by

$$\begin{aligned} \mathsf{F} =& f_{22}t_1^2t_3^2 + f_{20}t_1^2 + f_{02}t_3^2 + f_{00}, \\ \mathsf{G} =& g_{22}t_1^2t_3^2 + g_{20}t_1^2 + g_{02}t_3^2 + g_{00}, \end{aligned} \tag{6}$$

with $f_{22} \neq 0$.

- **Symmetric asymmetric (SA) case:** Here we distinguish again two types:

I. The first case is given by

$$\begin{aligned} \mathsf{F} =& f_{10}t_1 + f_{01}t_3, \\ \mathsf{G} =& g_{32}t_1^3t_3^2 + g_{30}t_1^3 + g_{23}t_1^2t_3^3 + g_{21}t_1^2t_3 + g_{12}t_1t_3^2 + g_{10}t_1 + g_{03}t_3^3 + g_{01}t_3, \end{aligned} \tag{7}$$

with $f_{10} \neq 0 \neq f_{01}$.

II. The second case is given by

$$\begin{aligned} \mathsf{F} =& f_{20}t_1^2 + f_{02}t_3^2 + f_{00}, \\ \mathsf{G} =& g_{22}t_1^2t_3^2 + g_{20}t_1^2 + g_{02}t_3^2 + g_{00}, \end{aligned} \tag{8}$$

with $f_{20} \neq 0 \neq f_{02}$.

- **Asymmetric symmetric (AS) case:** Here we get three types:

I. The first case is given by

$$\begin{aligned} \mathsf{F} =& f_{30}t_1^3 + f_{21}t_1^2t_3 + f_{10}t_1 + f_{01}t_3, \\ \mathsf{G} =& g_{03}t_3^3 + g_{12}t_1t_3^2 + g_{01}t_3 + g_{10}t_1, \end{aligned} \tag{9}$$

with $f_{30} \neq 0$. Moreover, if $f_{01} = f_{21} = 0$ holds we get a special case of the later given case (AA,IV).

II. The second case is given by

$$\begin{aligned} \mathsf{F} =& f_{31}t_1^3t_3 + f_{20}t_1^2 + f_{11}t_1t_3 + f_{00}, \\ \mathsf{G} =& g_{13}t_1t_3^3 + g_{02}t_3^2 + g_{11}t_1t_3 + g_{00}, \end{aligned} \tag{10}$$

with $f_{31} \neq 0$. Moreover, if $g_{13} = 0$ holds we can also assume $g_{11} = 0$, as otherwise we get a component with a spherical coupler component.

III. The third case is given by

$$\begin{aligned} \mathsf{F} =& f_{40}t_1^4 + f_{20}t_1^2 + f_{00}, \\ \mathsf{G} =& g_{04}t_3^4 + g_{02}t_3^2 + g_{00}, \end{aligned} \tag{11}$$

with $f_{40} \neq 0$.

- **Asymmetric asymmetric (AA) case:** In this case we even get four types:

  I. The first case is given by

$$\begin{aligned}\mathsf{F} =& f_{10}t_1,\\ \mathsf{G} =& g_{34}t_1^3t_3^4 + g_{32}t_1^3t_3^2 + g_{30}t_1^3 + g_{10}t_1 + g_{23}t_1^2t_3^3 + g_{21}t_1^2t_3 + g_{03}t_3^3 + g_{01}t_3 + g_{14}t_1t_3^4 + g_{12}t_1t_3^2,\end{aligned} \tag{12}$$

  with $f_{10} \neq 0$.

  II. The second case is given by

$$\begin{aligned}\mathsf{F} =& f_{20}t_1^2 + f_{00},\\ \mathsf{G} =& g_{24}t_1^2t_3^4 + g_{22}t_1^2t_3^2 + g_{20}t_1^2 + g_{13}t_1t_3^3 + g_{11}t_1t_3 + g_{04}t_3^4 + g_{02}t_3^2 + g_{00},\end{aligned} \tag{13}$$

  with $f_{20} \neq 0$. Moreover, if $g_{24} = g_{13} = g_{04} = 0$ holds we can also assume $g_{11} = 0$, as otherwise we get a component with a spherical coupler component.

  III. The third case is given by

$$\begin{aligned}\mathsf{F} =& f_{21}t_1^2t_3 + f_{10}t_1 + f_{01}t_3,\\ \mathsf{G} =& g_{23}t_1^2t_3^3 + g_{21}t_1^2t_3 + g_{12}t_1t_3^2 + g_{10}t_1 + g_{03}t_3^3 + g_{01}t_3,\end{aligned} \tag{14}$$

  with $f_{21} \neq 0$.

  IV. The fourth case is given by

$$\begin{aligned}\mathsf{F} =& f_{30}t_1^3 + f_{10}t_1,\\ \mathsf{G} =& g_{14}t_1t_3^4 + g_{12}t_1t_3^2 + g_{10}t_1 + g_{03}t_3^3 + g_{01}t_3,\end{aligned} \tag{15}$$

  with $f_{30} \neq 0$.

Lemma 1 *The types of compositions given in Eqs. (5–15) are all possible compositions without a spherical coupler component with respect to the interchange of the unknowns t_1 and t_3.*

Proof: We only give a sketch for the proof of this lemma, because then it can easily be verified by the reader. As the case $\mathsf{F} = f_{00} = const.$ does not make sense, we start with $\mathsf{F} = f_{10}t_1$. Now G can only be of the form given in Eq. (12) so that the product FG is a biquartic polynomial in t_1 and t_3 with the same structure as the resultant X.

Then we further by considering the case $\mathsf{F} = f_{10}t_1 + f_{00}$. It can easily be seen that there does not exist a polynomial G (with exception of the zero-polynomial) so that the product FG is a polynomial with the same structure as the resultant X (even-degree terms only). Therefore this case does not imply an entry into our above given listing. The reason for this is that both factors, F and G, can either contain even-degree monomials or odd-degree monomials only.

As the cases $\mathsf{F} = f_{11}t_1t_3$ and $\mathsf{F} = f_{11}t_1t_3 + f_{00}$ yield spherical coupler components, we further the discussion by considering $\mathsf{F} = f_{20}t_1^2$ and so on. Finally, this procedure yields the above given eleven cases. □

In the following we denote the coefficients of $t_1^i t_3^j$ of $Y := \mathsf{FG}$ and X by Y_{ij} and X_{ij}, respectively. By the comparison of these coefficients, we get the following 13 equations $Q_{ij} = 0$ with $Q_{ij} := Y_{ij} - X_{ij}$ and

$$(i,j) \in \{(4,4),(4,2),(4,0),(3,3),(3,1),(2,4),(2,2),(2,0),(1,3),(1,1),(0,4),(0,2),(0,0)\},$$

which must be fulfilled. In the following we discuss the solution of this non-linear system of 13 equations for the above given eleven possible compositions. As the number of unknowns f_{ij}'s and g_{ij}'s is ranging between six and eleven (depending on the respective case), there have to be relations between the c_{ij}'s and d_{ij}'s to allow the solution of the whole system. The intention of the following discussion is to determine the subvarieties in the space of design variables c_{ij} and d_{ij}, that there exists a decomposition of the transmission function, where none of the resulting components corresponds to the transmission function of a spherical coupler. Therefore we first eliminate the unknowns f_{ij} and g_{ij} in order to get the equations, which only depend on the c_{ij}'s and d_{ij}'s. By solving the resulting equations, we obtain the desired relations between these unknowns.

As we compute the resultant X with respect to t_2, the coefficient of t_2^2 in C and D must not vanish. Therefore the two cases $c_{22} = c_{02} = 0$ and $d_{22} = d_{20} = 0$ are excluded and hence they have to be discussed separately.

In the next section we prove the main theorem on reducible compositions without a spherical coupler component, which reads as follows:

Theorem 1 *If a reducible composition without a spherical coupler component is given, then it is one of the following cases:*

1. *One of the cases given in item a of Corollary 1,*
2. *The case given in item b of Corollary 1,*
3. *The following conditions hold for $A, B \in \mathbb{R} \setminus \{0\}$:*

$$c_{00} = Ac_{02}, \quad c_{20} = Ac_{22}, \quad d_{00} = Bd_{20}, \quad d_{02} = Bd_{22}, \quad A = B, \tag{16}$$

4. *One of the following four cases holds:*

$$c_{00} = c_{02} = 0, \quad d_{00} = d_{02} = 0, \quad c_{00} = c_{20} = 0, \quad d_{00} = d_{20} = 0. \tag{17}$$

3 Proof of the main theorem

In the following we discuss the eleven possible cases. If the case study yields a solution, we write the corresponding item(s) of Theorem 1 in parentheses.

3.1 Discussing type (SS,I)

3.1.1 General case: $c_{02} \neq 0$

Assuming $c_{02} \neq 0$ we can solve the equations $Q_{44} = Q_{40} = Q_{04} = 0$ for $c_{20} = c_{00}c_{22}/c_{02}$, $d_{00} = c_{00}d_{20}/c_{02}$ and $d_{02} = c_{00}d_{22}/c_{02}$. Moreover, we can express g_{21} and g_{12} from $Q_{42} = 0$ and $Q_{24} = 0$, respectively. Now we distinguish the following two cases:

1. $f_{01} \neq 0$: In this case we can express g_{01} from $Q_{02} = 0$.
   a. $f_{10} \neq 0$: Now we can compute g_{10} from $Q_{20} = 0$. As for $c_{00} = 0$ all g_{ij}'s would vanish (which yields a contradiction), we can solve $Q_{13} = 0$ and $Q_{11} = 0$ for f_{12} and f_{10}, respectively. Now the remaining three equations $Q_{33} = Q_{31} = Q_{22} = 0$ can only vanish without contradiction (w.c.) for $c_{22} = 0$ (item 3) or $f_{01} = c_{02}f_{21}/c_{22}$ (general case of item 3).
   b. $f_{10} = 0$: We can express g_{10} from $Q_{11} = 0$. As for $c_{00} = 0$ all g_{ij}'s would vanish, we get $d_{20} = 0$ from $Q_{20} = 0$. Finally, we can compute f_{21} and f_{12} from $Q_{22} = 0$ and $Q_{13} = 0$, respectively (item 3 and 4).
2. $f_{01} = 0$: We distinguish two cases:
   a. $f_{10} \neq 0$: Under this assumption we can express g_{10} and g_{01} from $Q_{20} = 0$ and $Q_{11} = 0$, respectively. Then Q_{02} cannot vanish w.c..
   b. $f_{10} = 0$: Now $Q_{02} = 0$ implies $c_{00} = 0$ and $Q_{31} = 0$ yields $g_{10} = 0$. Finally, Q_{13} cannot vanish w.c..

3.1.2 Special case: $c_{02} = 0$

Now we can assume without loss of generality (w.l.o.g.) that $c_{22} \neq 0$ holds, as otherwise we get the excluded case. Therefore we can express $d_{02} = c_{20}d_{22}/c_{22}$ and $d_{00} = c_{20}d_{20}/c_{22}$ from $Q_{44} = 0$ and $Q_{40} = 0$, respectively. Then Q_{04} and Q_{00} can only vanish w.c. for $c_{00} = 0$. Moreover, we can compute g_{21} and g_{12} from $Q_{42} = 0$ and $Q_{24} = 0$. Now we distinguish two cases:

1. $f_{10} \neq 0$: In this case we can solve $Q_{20} = 0$ for g_{10}. Now it can easily be seen that Q_{02}, Q_{13} and Q_{11} can only vanish w.c. for $f_{01} = g_{01} = 0$. Finally, we can express f_{12} and f_{10} from $Q_{33} = 0$ and $Q_{31} = 0$ (item 3 and 4).
2. $f_{10} = 0$: We distinguish again two cases:
   a. $f_{01} = 0$: $Q_{13} = 0$ implies $g_{01} = 0$ and Q_{20} can only vanish w.c. for:

i. $c_{20} = 0$: $Q_{22} = 0$ yields the contradiction.
ii. $d_{20} = 0$, $c_{20} \neq 0$: Then $Q_{22} = 0$ implies $g_{10} = 0$. Finally, $Q_{33} = 0$ remains, which yields $f_{12} = -f_{21}d_{22}c_{11}/(c_{22}d_{11})$ (item 2, 3 and 4).

b. $f_{01} \neq 0$: Now $Q_{11} = 0$ and $Q_{02} = 0$ imply $g_{10} = 0$ and $g_{01} = 0$, respectively. As for $c_{20} = 0$ all g_{ij}'s vanish (a contradiction), we get $d_{20} = 0$ from $Q_{20} = 0$. Finally, Q_{22} cannot vanish w.c..

3.1.3 Excluded cases

We only discuss the case $c_{22} = c_{02} = 0$, under the assumption that $d_{22} = d_{20} = 0$ does not hold. [5] Due to $Q_{40} = Q_{00} = 0$, we have to distinguish two cases:

1. $d_{20} = 0$: As a consequence we can assume w.l.o.g. $d_{22} \neq 0$. Then we get $g_{12} = 0$ from $Q_{24} = 0$. Moreover, we can express $g_{21} = 0$ from $Q_{42} = 0$. Now $Q_{13} = 0$ implies $g_{01} = 0$. Then we get $c_{00} = 0$ and $c_{20} = 0$ from $Q_{02} = 0$ and $Q_{33} = 0$, respectively. Finally, Q_{31} cannot vanish w.c..
2. $d_{20} \neq 0$: Now $Q_{40} = 0$ and $Q_{00} = 0$ imply $c_{20} = 0$ and $c_{00} = 0$, respectively. Then we get $g_{21} = 0$ from $Q_{42} = 0$, $g_{12} = 0$ from $Q_{33} = 0$ and $g_{10} = 0$ from $Q_{31} = 0$. Finally, Q_{13} cannot vanish w.c..

Due to footnote 5, we are only left with the special case $c_{22} = c_{02} = d_{22} = d_{20} = 0$. We get $g_{21} = 0$ from $Q_{42} = 0$, $g_{12} = 0$ from $Q_{33} = 0$ and $g_{10} = 0$ from $Q_{31} = 0$. Finally, Q_{13} cannot vanish w.c..

3.2 Discussing type (SS,II)

3.2.1 General case: $c_{02} \neq 0$

Due to $c_{02} \neq 0$, we can solve $Q_{33} = Q_{31} = Q_{13} = 0$ for $c_{20} = c_{00}c_{22}/c_{02}$, $d_{00} = -c_{00}d_{20}/c_{02}$ and $d_{02} = -c_{00}d_{22}/c_{02}$, respectively. Moreover, we can express g_{22} from $Q_{44} = 0$ w.l.o.g.. Now we have to distinguish the following eight cases:

1. $f_{00}f_{02}f_{20} \neq 0$: Now we can compute g_{00}, g_{02} and g_{20} from $Q_{00} = 0$, $Q_{04} = 0$ and $Q_{40} = 0$, respectively. Moreover, we can assume $c_{00} \neq 0$, as otherwise all g_{ij}'s vanish. Then we distinguish further six cases:
   a. $d_{20}d_{22}c_{22} \neq 0$: In this case we compute the resultant R of Q_{20} and Q_{02} with respect to f_{00}. Moreover, we compute the resultant S of Q_{24} and Q_{42} with respect to f_{22}. The only non-contradicting factor of S equals the only non-contradicting factor of T. We denote this factor with $F_1[15]$, where the number in the square brackets gives the number of terms.

[5] For the discussion of the case $d_{22} = d_{20} = 0$, under the assumption that $c_{22} = c_{02} = 0$ does not hold, we refer to analogy.

Moreover, we eliminate f_{22} from Q_{22} by computing the resultant T of Q_{22} and Q_{42} with respect to f_{22}. Then we calculate the resultant U of T and Q_{20} with respect to f_{00}. Now F_1 and the only non-contradicting factor F_2[18] of U only depend on f_{20} and f_{02}. Therefore we compute the resultant of F_1 and F_2 with respect to one of these unknowns, which shows that the following condition has to hold (item 2):

$$c_{02}c_{22}d_{11}^2 + 16c_{00}c_{22}d_{22}d_{20} - d_{22}d_{20}c_{11}^2 = 0. \tag{18}$$

b. $c_{22} = 0$, $d_{20}d_{22} \neq 0$: In this case we can express f_{02} from $Q_{24} = 0$, f_{00} from $Q_{20} = 0$ and f_{20} from $Q_{22} = 0$. Then $Q_{02} = 0$ yields the contradiction.
c. $d_{22} = 0$, $d_{20}c_{22} \neq 0$: In this case we can express f_{20} from $Q_{42} = 0$, f_{00} from $Q_{02} = 0$ and f_{22} from $Q_{22} = 0$. Then $Q_{20} = 0$ yields the contradiction.
d. $d_{20} = 0$, $d_{22}c_{22} \neq 0$: In this case we can express f_{22} from $Q_{42} = 0$, f_{02} from $Q_{02} = 0$ and f_{20} from $Q_{22} = 0$. Then $Q_{24} = 0$ yields the contradiction.
e. $d_{20} = c_{22} = 0$, $d_{22} \neq 0$: Now $Q_{22} = 0$ already yields the contradiction.
f. $d_{22} = c_{22} = 0$, $d_{20} \neq 0$: Again $Q_{22} = 0$ yields the contradiction.

2. $f_{00} = 0$, $f_{02}f_{20} \neq 0$: W.l.o.g. we can express g_{02} from $Q_{04} = 0$, g_{20} from $Q_{40} = 0$ and g_{00} from $Q_{02} = 0$. Moreover, we can assume $c_{00} \neq 0$, as otherwise all g_{ij}'s vanish. Therefore $Q_{00} = 0$ implies $d_{20} = 0$. Finally, $Q_{20} = 0$ yields the contradiction.
3. $f_{20} = 0$, $f_{00}f_{02} \neq 0$: W.l.o.g. we can express g_{02} from $Q_{04} = 0$, g_{00} from $Q_{00} = 0$ and g_{20} from $Q_{20} = 0$. Moreover, we can assume $c_{00} \neq 0$, as otherwise all g_{ij}'s vanish. Then Q_{42} and Q_{40} can only vanish w.c. for $c_{22} = d_{20} = 0$. Now $Q_{24} = 0$ implies an expression for f_{02}. Finally, we remain with $Q_{02} = 0$, which can be solved w.l.o.g. for f_{22} (item 2 and 4).
4. $f_{02} = 0$, $f_{00}f_{20} \neq 0$: W.l.o.g. we can express g_{20} from $Q_{40} = 0$, g_{00} from $Q_{00} = 0$ and g_{02} from $Q_{24} = 0$. Moreover, we can assume $c_{00} \neq 0$, as otherwise all g_{ij}'s vanish. Therefore $Q_{04} = 0$ implies $d_{22} = 0$. Finally, $Q_{02} = 0$ yields the contradiction.
5. $f_{02} = f_{20} = 0$, $f_{00} \neq 0$: W.l.o.g. we can express g_{00} from $Q_{00} = 0$, g_{02} from $Q_{24} = 0$ and g_{20} from $Q_{20} = 0$. Moreover, we can assume $c_{00} \neq 0$, as otherwise all g_{ij}'s vanish. Therefore $Q_{04} = 0$ implies $d_{22} = 0$. Now Q_{42} and Q_{40} cannot vanish w.c..
6. $f_{00} = f_{02} = 0$, $f_{20} \neq 0$: W.l.o.g. we can express g_{20} from $Q_{40} = 0$, g_{02} from $Q_{24} = 0$ and g_{00} from $Q_{20} = 0$. Moreover, we can assume $c_{00} \neq 0$, as otherwise all g_{ij}'s vanish. Now Q_{42} and Q_{40} cannot vanish w.c..
7. $f_{00} = f_{20} = 0$, $f_{02} \neq 0$: W.l.o.g. we can express g_{02} from $Q_{04} = 0$, g_{20} from $Q_{42} = 0$ and g_{00} from $Q_{02} = 0$. Moreover, we can assume $c_{00} \neq 0$, as otherwise all g_{ij}'s vanish. Therefore $Q_{00} = 0$ implies $d_{20} = 0$. Then we can express f_{22} from $Q_{22} = 0$. Finally, $Q_{24} = 0$ yields the contradiction.
8. $f_{00} = f_{20} = f_{02} = 0$: W.l.o.g. we can express g_{20} from $Q_{42} = 0$, g_{02} from $Q_{24} = 0$ and g_{00} from $Q_{22} = 0$. We can assume $c_{00} \neq 0$, as otherwise all g_{ij}'s vanish. Then $Q_{04} = 0$ implies $d_{22} = 0$. Finally, Q_{00} cannot vanish w.c..

3.2.2 *Special case:* $c_{02} = 0$

W.l.o.g. we can express $d_{02} = -c_{20}d_{22}/c_{22}$ from $Q_{33} = 0$ and $d_{00} = -c_{20}d_{20}/c_{22}$ from $Q_{31} = 0$. Then $Q_{13} = 0$ and $Q_{11} = 0$ can only vanish w.c. for $c_{00} = 0$. Moreover, we can compute g_{22} from $Q_{44} = 0$, g_{02} from $Q_{24} = 0$ and g_{20} from $Q_{42} = 0$. Now $Q_{00} = -g_{00}f_{00}$ implies two cases:

1. $g_{00} = 0$: Now we can assume w.l.o.g. that $c_{20} \neq 0$ holds, as otherwise all g_{ij}'s vanish. Due to $Q_{04} = 0$ and $Q_{02} = 0$, we have to distinguish the following three cases:
   a. $d_{22} = 0$: Now $Q_{22} = 0$ implies $f_{02} = 0$. Then we can compute f_{20} from $Q_{40} = 0$ and f_{00} from $Q_{20} = 0$ (item 2 and 4).
   b. $f_{22} = -4f_{02}c_{20}c_{22}/c_{11}^2$, $d_{22} \neq 0$: In this case we can compute f_{02} from $Q_{22} = 0$. We distinguish two cases:
      i. $d_{20} \neq 0$: Now we can express f_{20} from $Q_{20} = 0$. Finally, Q_{40} cannot vanish w.c..
      ii. $d_{20} = 0$: Now $Q_{20} = 0$ implies $f_{00} = 0$ (item 2 and 4).
   c. $f_{00} = f_{02} = 0$, $d_{22}(f_{22}c_{11}^2 + 4f_{02}c_{20}c_{22}) \neq 0$: We get $d_{20} = 0$ from $Q_{20} = 0$ and $f_{20} = 0$ from $Q_{22} = 0$ (item 2 and 4).
2. $g_{00} \neq 0$: Now $Q_{00} = 0$ implies $f_{00} = 0$ and from $Q_{02} = 0$ we get $f_{02} = 0$. Moreover, we can express f_{20} from $Q_{20} = 0$ and f_{22} from $Q_{22} = 0$. Finally, Q_{40} cannot vanish w.c..

3.2.3 *Excluded cases*

Here we only discuss the case $c_{22} = c_{02} = 0$ assuming that $d_{22} = d_{20} = 0$ does not hold (cf. footnote 5): We get $c_{20} = 0$ and $c_{00} = 0$ from $Q_{31} = 0$ and $Q_{11} = 0$, respectively. Moreover, we get $g_{22} = 0$ from $Q_{44} = 0$, $g_{20} = 0$ from $Q_{42} = 0$ and $g_{02} = 0$ from $Q_{24} = 0$. Therefore we do not get a reducible composition as G is a constant function.

Due to footnote 5, we are only left with the special case $c_{22} = c_{02} = d_{22} = d_{20} = 0$. We get $g_{22} = 0$ from $Q_{44} = 0$ and $g_{20} = 0$ from $Q_{42} = 0$. Finally, Q_{24} cannot vanish w.c..

3.3 *Discussing type (SA,I)*

Due to the conditions

$$c_{20}d_{22} - c_{22}d_{02} = 0, \qquad c_{00}d_{20} - c_{02}d_{00} = 0. \tag{19}$$

implied by $Q_{44} = 0$ and $Q_{00} = 0$, we have to distinguish the following cases:

3.3.1 General case

We set $c_{00} = Ac_{02}$, $d_{00} = Ad_{20}$, $c_{20} = Bc_{22}$ and $d_{02} = Bd_{22}$ with $A, B \in \mathbb{R}$. Then we can express g_{32} from $Q_{42} = 0$, g_{23} from $Q_{24} = 0$, g_{10} from $Q_{20} = 0$, g_{01} from $Q_{02} = 0$, g_{30} from $Q_{40} = 0$, g_{03} from $Q_{04} = 0$, g_{21} from $Q_{31} = 0$ and g_{12} from $Q_{13} = 0$. Due to $Q_{33} = Q_{11} = 0$, we have to distinguish four cases:

1. $AB \neq 0$: We distinguish further two cases:
   a. $c_{22} \neq 0$: Now we can compute f_{01} from $Q_{33} = 0$. Then the remaining equations $Q_{11} = 0$ and $Q_{22} = 0$ can only vanish w.c. for the common factor (item 2):
   $$c_{02}c_{22}d_{11}^2 - d_{20}d_{22}c_{11}^2 = 0. \tag{20}$$
   b. $c_{22} = 0$: Now $Q_{33} = 0$ implies $d_{22} = 0$. Then only $Q_{11} = 0$ remains, which can be solved w.l.o.g. for f_{01} or f_{10}, respectively (item 2).
2. $A = 0$, $B \neq 0$: Again we distinguish two cases:
   a. $c_{22} \neq 0$: Now we can compute f_{01} from $Q_{33} = 0$. Then the remaining equation $Q_{22} = 0$ can only vanish w.c. for Eq. (20) (item 2).
   b. $c_{22} = 0$: Now $Q_{33} = 0$ implies $d_{22} = 0$, but this yields a contradiction as all g_{ij}'s equal zero.
3. $B = 0$, $A \neq 0$: We distinguish two cases:
   a. $d_{20} \neq 0$: Now we can compute f_{01} from $Q_{11} = 0$. Then the remaining equation $Q_{22} = 0$ can only vanish w.c. for Eq. (20) (item 2).
   b. $d_{20} = 0$: Now $Q_{11} = 0$ implies $c_{02} = 0$, but this yields a contradiction as all g_{ij}'s equal zero.
4. $A = B = 0$: This already yields a contradiction as all g_{ij}'s equal zero.

3.3.2 Special cases

Now we discuss the two cases, which are not covered by the general case:

1. $d_{20} = c_{02} = 0$, $c_{20} = Bc_{22}$, $d_{02} = Bd_{22}$ with $B \in \mathbb{R}$: In this case we can assume w.l.o.g. that $d_{22}c_{22} \neq 0$ holds. Then we can express g_{32} from $Q_{42} = 0$ and g_{23} from $Q_{24} = 0$. Moreover, $Q_{20} = 0$ and $Q_{02} = 0$ imply $g_{10} = 0$ and $g_{01} = 0$. We can also compute g_{30} from $Q_{40} = 0$, g_{03} from $Q_{04} = 0$, g_{21} from $Q_{31} = 0$ and g_{12} from $Q_{13} = 0$. We distinguish two cases:
   a. $B = 0$: Now only $Q_{22} = 0$ remains, which is a quartic homogeneous polynomial in f_{10} and f_{01} (item 1 and 2).
   b. $B \neq 0$: In this case Q_{33} can only vanish w.c. for $f_{10} = -d_{11}c_{22}f_{01}/(d_{22}c_{11})$. Now the last remaining equation $Q_{22} = 0$ can only vanish for (item 2):
   $$c_{00}c_{22}d_{11}^2 - d_{00}d_{22}c_{11}^2 = 0. \tag{21}$$
2. $d_{22} = c_{22} = 0$, $c_{00} = Ac_{02}$, $d_{00} = Ad_{20}$ with $A \in \mathbb{R}$: In this case we can assume w.l.o.g. that $d_{20}c_{02} \neq 0$ holds. Then we get $g_{32} = 0$ from $Q_{42} = 0$ and $g_{23} = 0$ from $Q_{24} = 0$. Moreover, we can express $g_{10} = 0$ from $Q_{20} = 0$

and $g_{01} = 0$ from $Q_{02} = 0$. We can also compute g_{30} from $Q_{40} = 0$, g_{03} from $Q_{04} = 0$, g_{21} from $Q_{31} = 0$ and g_{12} from $Q_{13} = 0$. We distinguish two cases:

a. $A = 0$: Now only $Q_{22} = 0$ remains, which is a quartic homogeneous polynomial in f_{10} and f_{01} (item 1 and 2).
b. $A \neq 0$: In this case Q_{33} can only vanish w.c. for $f_{10} = -c_{11}d_{20}f_{01}/(c_{02}d_{11})$. Now the last remaining equation $Q_{22} = 0$ can only vanish for (item 2):

$$c_{02}c_{20}d_{11}^2 - d_{02}d_{20}c_{11}^2 = 0. \tag{22}$$

3.3.3 Excluded cases

Here we only discuss the case $c_{22} = c_{02} = 0$ assuming that $d_{22} = d_{20} = 0$ does not hold (cf. footnote 5): In this case we get $g_{23} = 0$ from $Q_{24} = 0$ and $g_{03} = 0$ from $Q_{04} = 0$. Moreover, $Q_{33} = 0$ implies $g_{32} = 0$ and $Q_{13} = 0$ yields $g_{12} = 0$. Then we can express g_{30} from $Q_{40} = 0$, g_{10} from $Q_{20} = 0$, g_{21} from $Q_{22} = 0$ and g_{01} from $Q_{02} = 0$. Due to $Q_{42} = d_{22}c_{20}^2$ and $Q_{00} = d_{20}c_{00}^2$, we have to distinguish three cases:

1. $c_{20} = c_{00} = 0$: Then $Q_{31} = 0$ implies $d_{02} = 0$ and from $Q_{11} = 0$ we get $d_{00} = 0$, which already yields the contradiction as all g_{ij}'s equal zero.
2. $c_{20} = d_{20} = 0$: Now $Q_{31} = 0$ implies again $d_{02} = 0$. Finally, $Q_{11} = 0$ remains, which is homogeneous quadratic in f_{01} and f_{10} (item 1 and 2).
3. $d_{22} = c_{00} = 0$: Now $Q_{11} = 0$ implies $d_{00} = 0$. Finally, $Q_{31} = 0$ remains, which is again homogeneous quadratic in f_{01} and f_{10} (item 1, 2 and 4).

Due to footnote 5, we are only left with the special case $c_{22} = c_{02} = d_{22} = d_{20} = 0$. We get $g_{32} = 0$ from $Q_{42} = 0$, $g_{30} = 0$ from $Q_{40} = 0$, $g_{23} = 0$ from $Q_{33} = 0$, $g_{21} = 0$ from $Q_{31} = 0$, $g_{03} = 0$ from $Q_{04} = 0$, $g_{10} = 0$ from $Q_{20} = 0$, $g_{12} = 0$ from $Q_{22} = 0$ and $g_{01} = 0$ from $Q_{02} = 0$, which yields the contradiction.

3.4 Discussing type (SA,II)

In this case we solve $Q_{40} = 0$ for g_{20}, $Q_{04} = 0$ for g_{02}, $Q_{42} = 0$ for g_{22} and $Q_{02} = 0$ for g_{00}. Due to $Q_{44} = 0$ and $Q_{33} = 0$, the conditions $c_{22}c_{20} = 0$ and $d_{22}c_{20} = 0$ have to hold, which imply the following four cases:

1. $c_{22} = c_{20} = 0$: Therefore we can assume w.l.o.g. that $c_{02} \neq 0$ holds. Due to $Q_{24} = d_{22}d_{02}c_{11}^2$, we have to distinguish two cases:
   a. $d_{22} = 0$: Then $Q_{13} = 0$ implies $d_{02} = 0$. Now $g_{22} = g_{20} = g_{02} = 0$ holds and we do not get a reducible composition as G is a constant function.
   b. $d_{22} \neq 0$: Therefore $Q_{24} = 0$ implies $d_{02} = 0$. Then $Q_{13} = 0$ and $Q_{11} = 0$ imply $c_{00} = 0$ and $d_{00} = 0$, respectively. This already yields the contra-

diction as all g_{ij}'s equal zero.

2. $c_{22} = d_{22} = 0$, $c_{20} \neq 0$: Now Q_{24} cannot vanish w.c..
3. $d_{22} = d_{02} = 0$: W.l.o.g. we can assume that $d_{20} \neq 0$ holds. Due to $Q_{24} = 0$, we distinguish two cases:
   a. $c_{22} = 0$: Then $Q_{31} = 0$ implies $c_{20} = 0$. Now $g_{22} = g_{20} = g_{02} = 0$ holds and we do not get a reducible composition as G is a constant function.
   b. $c_{22} \neq 0$: Therefore $Q_{24} = 0$ implies $c_{20} = 0$. Then $Q_{31} = 0$ and $Q_{11} = 0$ imply $d_{00} = 0$ and $c_{00} = 0$, respectively. This already yields the contradiction as all g_{ij}'s equal zero.
4. $c_{20} = d_{02} = 0$, $d_{22} \neq 0$: Now $Q_{13} = 0$ implies $c_{00} = 0$. Moreover, Q_{31} and Q_{11} can only vanish w.c. for $d_{00} = 0$. Again all g_{ij}'s equal zero.

3.4.1 Excluded cases

Here we only discuss the case $c_{22} = c_{02} = 0$ assuming that $d_{22} = d_{20} = 0$ does not hold (cf. footnote 5): Now we get $g_{22} = 0$ and $g_{02} = 0$ from $Q_{24} = 0$ and $Q_{04} = 0$. Then $Q_{31} = 0$ and $Q_{11} = 0$ imply $c_{20} = 0$ and $c_{00} = 0$, respectively. Finally, $Q_{02} = 0$ and $Q_{40} = 0$ yield $g_{00} = g_{20} = 0$, a contradiction.

Due to footnote 5, we are only left with the special case $c_{22} = c_{02} = d_{22} = d_{20} = 0$. Now we get $g_{22} = 0$ and $g_{20} = 0$ from $Q_{42} = 0$ and $Q_{40} = 0$. Then $Q_{20} = 0$ and $Q_{04} = 0$ yield $g_{00} = g_{02} = 0$, a contradiction.

3.5 Discussing type (AS,I)

W.l.o.g. we can express g_{10} from $Q_{40} = 0$, g_{03} from $Q_{33} = 0$, g_{01} from $Q_{31} = 0$ and g_{12} from $Q_{42} = 0$. Due to the conditions implied by $Q_{44} = Q_{00} = 0$ (cf. Eq. (19)), we distinguish again following cases:

3.5.1 General case

We set $c_{00} = Ac_{02}$, $d_{00} = Ad_{20}$, $c_{20} = Bc_{22}$ and $d_{02} = Bd_{22}$ with $A, B \in \mathbb{R}$. Now we distinguish two cases:

1. $B \neq 0$: We distinguish further two cases:
   a. $c_{22} \neq 0$: Under this assumption we can compute f_{21} from $Q_{24} = 0$. Moreover, we can express f_{10} from $Q_{22} = 0$ and f_{01} from $Q_{13} = 0$. Now Q_{04} can only vanish w.c. for:
      i. $d_{22} = 0$: W.l.o.g. we can assume that $d_{20} \neq 0$ holds. Now Q_{02} can only vanish w.c. for:
         ⋆ $c_{02} = 0$: Then $Q_{20} = 0$ implies $A = 0$. Now all conditions are fulfilled, but we have $f_{01} = f_{21} = 0$ and therefore we get a special

case of (AA,IV).

  - $\star$ $A = 0$, $c_{02} \neq 0$: Now $Q_{20} = 0$ already yields the contradiction.

  ii. $A = B$, $d_{22} \neq 0$: Then $Q_{02} = 0$ implies $c_{02} = 0$. Now we get $d_{20} = 0$ from $Q_{20} = 0$ (item 2, 3 and 4).

  iii. $c_{02} = d_{22}d_{20}c_{11}^2/(c_{22}d_{11}^2)$, $(A-B)d_{22} \neq 0$: Now Q_{20} can only vanish w.c. in one of the following three cases:
  - $\star$ $d_{20} = 0$: All conditions are fulfilled identically (item 2 and 4).
  - $\star$ $d_{20} = d_{11}^2 A/[(A-B)^2 d_{22}]$, $d_{20} \neq 0$: Now Q_{11} cannot vanish w.c..
  - $\star$ $d_{20} = d_{11}^2 B/[(A-B)^2 d_{22}]$, $d_{20} \neq 0$: Again Q_{11} cannot vanish w.c..

 b. $c_{22} = 0$: Then $Q_{24} = 0$ implies $d_{22} = 0$. Finally, we get $A = 0$ from $Q_{20} = 0$, but this already yields a contradiction as all g_{ij}'s equal zero.

2. $B = 0$: Moreover, we can assume $A \neq 0$, as otherwise all g_{ij}'s would vanish. Therefore Q_{04} can only vanish w.c. for:

 a. $c_{02} = 0$: Therefore we can assume $c_{22} \neq 0$ w.l.o.g.. Then we can express f_{10} from $Q_{20} = 0$ and f_{21} from $Q_{02} = 0$. Moreover, Q_{22} can only vanish w.c. for $d_{22} = 0$. Then $Q_{11} = 0$ implies $f_{01} = 0$ (item 1, 2 and 4).

 b. $d_{22} = 0$, $c_{02} \neq 0$: We can assume $Ac_{22}d_{20} \neq 0$ w.l.o.g., as otherwise all g_{ij}'s would vanish. Therefore we can express f_{10} from $Q_{20} = 0$, f_{21} from $Q_{11} = 0$ and f_{01} from $Q_{02} = 0$. Finally, Q_{22} cannot vanish w.c..

3.5.2 Special cases

Now we discuss the two cases, which are not covered by the general case:

1. $d_{20} = c_{02} = 0$, $c_{20} = Bc_{22}$, $d_{02} = Bd_{22}$ with $B \in \mathbb{R}$: In this case we can assume w.l.o.g. that $d_{22}c_{22} \neq 0$ holds. Due to $Q_{20} = 0$, we have to distinguish the following two cases:

 a. $d_{00} = 0$: We can assume $B \neq 0$, as otherwise all g_{ij}'s would vanish. Therefore we can express f_{21} from $Q_{24} = 0$ and f_{10} from $Q_{22} = 0$. Then we get $f_{01} = 0$ from $Q_{13} = 0$ and $c_{00} = 0$ from $Q_{04} = 0$ (item 2, 3 and 4).

 b. $f_{10} = 0$, $d_{00} \neq 0$: Now $Q_{11} = 0$ implies $f_{01} = 0$ and from $Q_{13} = 0$ we get $c_{00} = 0$. We distinguish two cases:

  i. $B = 0$: Then only $Q_{22} = 0$ remains, which is a homogeneous quadratic equation in f_{30} and f_{21} (item 1, 2 and 4).

  ii. $B \neq 0$: Now $Q_{24} = 0$ implies an expression for f_{21}. Finally, Q_{22} cannot vanish w.c..

2. $d_{22} = c_{22} = 0$, $c_{00} = Ac_{02}$, $d_{00} = Ad_{20}$ with $A \in \mathbb{R}$: Now we can assume w.l.o.g. that $d_{20}c_{02} \neq 0$ holds. Then $Q_{13} = 0$ implies $d_{02} = 0$. Moreover, we can assume $c_{20} \neq 0$, as otherwise all g_{ij}'s would vanish. Then we can express f_{10} from $Q_{20} = 0$ and f_{01} from $Q_{11} = 0$. We distinguish two cases:

 a. $A = 0$: Then only $Q_{22} = 0$ remains, which is a homogeneous quadratic equation in f_{30} and f_{21} (item 1, 2 and 4).

 b. $A \neq 0$: Now $Q_{02} = 0$ implies an expression for f_{21}. Finally, Q_{22} cannot vanish w.c..

3.5.3 Excluded cases

We start by discussing the case $c_{22} = c_{02} = 0$, under the assumption that $d_{22} = d_{20} = 0$ does not hold. We express g_{12} from $Q_{42} = 0$, g_{10} from $Q_{40} = 0$, g_{03} from $Q_{33} = 0$ and g_{01} from $Q_{31} = 0$. Then we get $Q_{00} = d_{20}c_{00}^2$:

1. $d_{20} = 0$: Now $Q_{20} = 0$ implies $d_{00} = 0$. Moreover, we can assume $c_{20} \neq 0$, as otherwise all g_{ij}'s would vanish. Then we can express f_{01} from $Q_{13} = 0$ and f_{10} from $Q_{11} = 0$. Now we get $f_{21} = 0$ from $Q_{24} = 0$, $c_{00} = 0$ from $Q_{02} = 0$ and $d_{02} = 0$ from $Q_{22} = 0$. All conditions are fulfilled, but we have $f_{01} = f_{21} = 0$ and therefore we get a special case of (AA,IV).
2. $c_{00} = 0$, $d_{20} \neq 0$: Moreover, we can assume $c_{20} \neq 0$, as otherwise all g_{ij}'s would vanish. Then Q_{24} can only vanish w.c. for:
   a. $f_{21} = 0$: Then $Q_{02} = 0$ implies $f_{01} = 0$ and $Q_{11} = 0$ yields $f_{10} = 0$. Then $Q_{22} = Q_{20} = 0$ imply $d_{02} = d_{00} = 0$. Now all conditions are fulfilled, but we have $f_{01} = f_{21} = 0$ and therefore we get a special case of (AA,IV).
   b. $d_{22} = 0$, $f_{21} \neq 0$: Then we can express f_{10} from $Q_{20} = 0$ and f_{01} from $Q_{11} = 0$. Now Q_{02} can only vanish w.c. for:
      i. $d_{00} = 0$: Then only $Q_{22} = 0$ remains, which is a homogeneous quadratic equation in f_{30} and f_{21} (item 1, 2 and 4).
      ii. $f_{21} = -c_{11}d_{11}f_{30}/(c_{20}d_{20})$, $d_{00} \neq 0$: $Q_{22} = 0$ implies $d_{02} = 0$, which yields a solution (item 1, 2 and 4).

The discussion of the second excluded case $d_{22} = d_{20} = 0$, under the assumption that $c_{22} = c_{02} = 0$ does not hold, can be done as follows: We get $g_{12} = 0$ from $Q_{42} = 0$, $g_{10} = 0$ from $Q_{40} = 0$, $g_{03} = 0$ from $Q_{33} = 0$ and $g_{01} = 0$ from $Q_{31} = 0$. As all g_{ij}'s equal zero, we get a contradiction. The discussion of the special case $c_{22} = c_{02} = d_{22} = d_{20} = 0$ can be done analogously to the second excluded case.

3.6 Discussing type (AS,II)

W.l.o.g. we can express g_{13} from $Q_{44} = 0$, g_{02} from $Q_{33} = 0$, g_{00} from $Q_{31} = 0$ and g_{11} from $Q_{42} = 0$. Due to the conditions

$$d_{20}c_{20} - d_{00}c_{22} = 0, \qquad c_{00}d_{22} - d_{02}c_{02} = 0, \tag{23}$$

implied by $Q_{40} = 0$ and $Q_{04} = 0$, we have to discuss the following cases:

3.6.1 General case

We set $c_{00} = Ac_{02}$, $d_{02} = Ad_{22}$, $c_{20} = Bc_{22}$ and $d_{00} = Bd_{20}$ with $A, B \in \mathbb{R}$. Now we distinguish two cases:

1. $B \neq 0$: We have to distinguish further two cases:
   a. $c_{22} \neq 0$: Now $Q_{20} = 0$ implies an expression for f_{20}. Moreover, we can express f_{11} from $Q_{22} = 0$ and f_{00} from $Q_{11} = 0$. Then Q_{00} can only vanish w.c. in one of the following three cases:
      i. $d_{20} = 0$: Therefore we can assume w.l.o.g. that $d_{22} \neq 0$ holds. Now Q_{13} and Q_{02} can only vanish w.c. for $c_{02} = 0$. Finally, the last remaining equation $Q_{24} = 0$ implies $A = 0$ (item 1, 2 and 4).
      ii. $A = B$, $d_{20} \neq 0$: We get $d_{22} = 0$ from $Q_{24} = 0$ and $c_{02} = 0$ from $Q_{02} = 0$. All conditions are fulfilled, but we do not get a reducible composition without a spherical coupler component as $g_{13} = 0$ and $g_{11} \neq 0$ hold. Moreover, g_{11} cannot vanish w.c..
      iii. $d_{22} = c_{02}c_{22}d_{11}^2/(d_{20}c_{11}^2)$, $(A - B)d_{20} \neq 0$: Now Q_{02} can only vanish w.c. for one of the following three cases:
         - ⋆ $c_{02} = 0$: All conditions are fulfilled, but we do not get a solution as $g_{13} = 0$ and $g_{11} \neq 0$ hold. Moreover, g_{11} cannot vanish w.c..
         - ⋆ $c_{22} = Ac_{11}^2/(c_{02}(A - B)^2)$, $c_{02} \neq 0$: Then Q_{13} cannot vanish w.c..
         - ⋆ $c_{22} = Bc_{11}^2/(c_{02}(A - B)^2)$, $c_{02} \neq 0$: Then Q_{13} cannot vanish w.c..
   b. $c_{22} = 0$: Now $Q_{20} = 0$ implies $d_{20} = 0$. Therefore we can assume w.l.o.g. that $d_{22} \neq 0$ holds. Then we get $A = 0$ from $Q_{20} = 0$, which already yields the contradiction as all g_{ij}'s vanish.
2. $B = 0$: W.l.o.g. we can assume $Ac_{22}d_{22} \neq 0$, as otherwise all g_{ij}'s would vanish. Due to $Q_{00} = A^2c_{02}^2d_{20}^2$, we have to distinguish two cases:
   a. $c_{02} = 0$: Then we can express f_{11} from $Q_{24} = 0$, f_{00} from $Q_{13} = 0$ and f_{20} from $Q_{02} = 0$. Now $Q_{22} = 0$ implies $d_{20} = 0$ (item 1, 2 and 4).
   b. $d_{20} = 0$, $c_{02} \neq 0$: Then we can express f_{11} from $Q_{24} = 0$, f_{00} from $Q_{13} = 0$ and f_{20} from $Q_{02} = 0$. Now Q_{22} cannot vanish w.c..

3.6.2 Special cases

Now we discuss the two cases, which are not covered by the general case:

1. $c_{02} = d_{22} = 0$, $c_{20} = Bc_{22}$, $d_{00} = Bd_{20}$ with $B \in \mathbb{R}$: W.l.o.g. we can assume $c_{22}d_{20} \neq 0$. Due to $Q_{24} = 0$, we have to distinguish two cases:
   a. $d_{02} = 0$: W.l.o.g. we can assume $B \neq 0$, as otherwise all g_{ij}'s would vanish. Moreover, we can express f_{11} from $Q_{22} = 0$, f_{20} from $Q_{20} = 0$ and f_{00} from $Q_{00} = 0$. Finally, the last remaining equation $Q_{11} = 0$ implies $c_{00} = 0$. All conditions are fulfilled, but we do not end up with a solution as $g_{13} = 0$ and $g_{11} \neq 0$ hold. Moreover, g_{11} cannot vanish w.c..
   b. $f_{11} = 0$, $d_{02} \neq 0$: We get $f_{00} = 0$ from $Q_{13} = 0$ and $c_{00} = 0$ from $Q_{11} = 0$.

Then we distinguish two cases:

i. $B = 0$: Now only the equation $Q_{22} = 0$ remains, which is a homogeneous quadratic polynomial in f_{31} and f_{20} (item 1, 2 and 4).

ii. $B \neq 0$: In this case $Q_{20} = 0$ implies an expression for f_{20}. Then Q_{22} cannot vanish w.c..

2. $c_{22} = d_{20} = 0$, $c_{00} = Ac_{02}$, $d_{02} = Ad_{22}$ with $A \in \mathbb{R}$: W.l.o.g. we can assume $c_{02}d_{22} \neq 0$. As a consequence $Q_{11} = 0$ implies $d_{00} = 0$. Moreover, we can assume $c_{20} \neq 0$, as otherwise all g_{ij}'s would vanish. Then we can express f_{11} from $Q_{24} = 0$ and f_{00} from $Q_{13} = 0$. We distinguish two cases:
   a. $A = 0$: Now only the equation $Q_{22} = 0$ remains, which is a homogeneous quadratic polynomial in f_{31} and f_{20} (item 1, 2 and 4).
   b. $A \neq 0$: Under this assumption $Q_{02} = 0$ implies an expression for f_{20}. Then Q_{22} cannot vanish w.c..

3.6.3 Excluded cases

We start by discussing the case $c_{22} = c_{02} = 0$, under the assumption that $d_{22} = d_{20} = 0$ does not hold. In this case we get $g_{13} = 0$ from $Q_{44} = 0$ and $g_{02} = 0$ from $Q_{33} = 0$. Moreover, we can express g_{11} from $Q_{42} = 0$ and g_{00} from $Q_{31} = 0$. Trivially, this cannot yield a reducible composition without a spherical coupler component.

The discussion of the second excluded case $d_{22} = d_{20} = 0$, under the assumption that $c_{22} = c_{02} = 0$ does not hold, can be done as follows: We get $g_{13} = 0$ from $Q_{44} = 0$, $g_{02} = 0$ from $Q_{33} = 0$, $g_{11} = 0$ from $Q_{42} = 0$ and $g_{00} = 0$ from $Q_{31} = 0$. As all g_{ij}'s equal zero, we get a contradiction. The discussion of the special case $c_{22} = c_{02} = d_{22} = d_{20} = 0$ can be done analogously to the second excluded case.

3.7 Discussing type (AS,III)

W.l.o.g. we can express g_{04} from $Q_{44} = 0$, g_{02} from $Q_{42} = 0$ and g_{00} from $Q_{40} = 0$. Then Q_{33}, Q_{31}, Q_{13} and Q_{11} can only vanish w.c. for:

$$c_{22}d_{02}+c_{20}d_{22} = 0,\ c_{22}d_{00}+c_{20}d_{20} = 0,\ c_{02}d_{02}+c_{00}d_{22} = 0,\ c_{02}d_{00}+c_{00}d_{20} = 0.$$

Therefore we have to discuss the following three cases:

1. We set $c_{00} = Ac_{02}$, $d_{00} = -Ad_{20}$, $c_{20} = Ac_{22}$ and $d_{02} = -Ad_{22}$ with $A \in \mathbb{R}$. Then we can assume $Ac_{22} \neq 0$, as otherwise all g_{ij}'s would vanish. We distinguish two cases:
   a. $d_{22} \neq 0$: Now $Q_{24} = 0$ implies an expression for f_{20} and $Q_{04} = 0$ can be solved for f_{00}. Then Q_{22} cannot vanish w.c..

b. $d_{22} = 0$: Now $Q_{22} = 0$ implies an expression for f_{20} and $Q_{02} = 0$ can be solved for f_{00}. Then the only remaining equation $Q_{20} = 0$ implies $d_{20} = 0$ (item 1, 3 and 4).

2. $c_{02} = d_{20} = c_{00} = d_{00} = 0$, $c_{20} = Ac_{22}$ and $d_{02} = -Ad_{22}$ with $A \in \mathbb{R}$: Now $Q_{22} = 0$ implies $f_{20} = 0$ and from $Q_{02} = 0$ we get $f_{00} = 0$. Then $Q_{24} = 0$ implies $d_{22} = 0$ (item 1, 2, 3 and 4).
3. $c_{20} = d_{02} = c_{22} = d_{22} = 0$, $c_{00} = Ac_{02}$ and $d_{00} = -Ad_{20}$ with $A \in \mathbb{R}$: In this case we get a contradiction, because all g_{ij}'s vanish.

3.7.1 Excluded cases

We start by discussing the case $c_{22} = c_{02} = 0$, under the assumption that $d_{22} = d_{20} = 0$ does not hold. Now we get $g_{04} = 0$ from $Q_{44} = 0$. Moreover, we can express g_{02} from $Q_{42} = 0$ and g_{00} from $Q_{40} = 0$. Finally, $Q_{31} = 0$ implies $c_{20} = 0$, but this already yields a contradiction as all g_{ij}'s vanish.

The discussion of the second excluded case $d_{22} = d_{20} = 0$, under the assumption that $c_{22} = c_{02} = 0$ does not hold, can be done as follows: We get $g_{04} = 0$ from $Q_{44} = 0$, $g_{02} = 0$ from $Q_{42} = 0$ and $g_{00} = 0$ from $Q_{40} = 0$. As all g_{ij}'s equal zero, we get a contradiction. The special case $c_{22} = c_{02} = d_{22} = d_{20} = 0$ can be discussed analogously to the second excluded case.

3.8 Discussing type (AA,I)

W.l.o.g. we can express g_{34} from $Q_{44} = 0$, g_{30} from $Q_{40} = 0$, g_{23} from $Q_{33} = 0$, g_{32} from $Q_{42} = 0$, g_{21} from $Q_{31} = 0$, g_{14} from $Q_{24} = 0$, g_{10} from $Q_{20} = 0$, g_{03} from $Q_{13} = 0$, g_{01} from $Q_{11} = 0$ and g_{12} from $Q_{22} = 0$. Now Q_{04} and Q_{00} imply

$$c_{00}d_{22} - c_{02}d_{02} = 0, \qquad c_{00}d_{20} - c_{02}d_{00} = 0. \tag{24}$$

Therefore we have to distinguish the following three cases:

1. $c_{00} = Ac_{02}$, $d_{00} = Ad_{20}$ and $d_{02} = Ad_{22}$ with $A \in \mathbb{R}$: Then Q_{02} can only vanish w.c. for $Ac_{02} = 0$ (both cases yield item 4).
2. $c_{00} = c_{02} = d_{20} = 0$: This already yields a solution (item 4).
3. $c_{00} = c_{02} = d_{22} = 0$: This also yields a solution (item 4).

3.8.1 Excluded cases

We start by discussing the case $c_{22} = c_{02} = 0$, under the assumption that $d_{22} = d_{20} = 0$ does not hold. In this case we can compute all g_{ij}'s as in the general case. Then we remain with $Q_{02} = d_{22}c_{00}^2$ and $Q_{00} = d_{20}c_{00}^2$, which implies $c_{00} = 0$ (item 1 and 4).

The discussion of the second excluded case $d_{22} = d_{20} = 0$, under the assumption that $c_{22} = c_{02} = 0$ does not hold, can be done as follows: Again all g_{ij}'s can be computed as in the general case. Due to $Q_{04} = c_{02}d_{02}^2$ and $Q_{00} = c_{02}d_{00}^2$, we have to distinguish two cases:

1. $c_{02} = 0$: The last remaining equation $Q_{02} = 0$ implies $c_{00} = 0$ (item 4).
2. $d_{00} = d_{02} = 0$: Now $Q_{02} = 0$ implies $c_{00} = 0$ (item 1 and 4).

The discussion of the special case $c_{22} = c_{02} = d_{22} = d_{20} = 0$ shows that the computation of the g_{ij}'s as done in the general case already yields a contradiction (all g_{ij}'s equal zero).

3.9 Discussing type (AA,II)

W.l.o.g. we can express g_{24} from $Q_{44} = 0$, g_{22} from $Q_{42} = 0$, g_{20} from $Q_{40} = 0$, g_{13} from $Q_{33} = 0$, g_{11} from $Q_{31} = 0$, g_{04} from $Q_{24} = 0$, g_{02} from $Q_{22} = 0$ and g_{00} from $Q_{20} = 0$. Now we distinguish the following cases:

3.9.1 General case: $c_{22}d_{02} + c_{20}d_{22} \neq 0$

Under this assumption we can compute f_{00} from $Q_{13} = 0$. Then Q_{11} can only vanish w.c. if $\mathcal{C}$ or $\mathcal{D}$ are orthogonal; i.e. if one of the following equations holds:

$$c_{00}c_{22} - c_{02}c_{20} = 0, \qquad d_{00}d_{22} - d_{02}d_{20} = 0. \tag{25}$$

1. $\mathcal{C}$ is orthogonal: W.l.o.g. we can set $c_{00} = Ac_{02}$ and $c_{20} = Ac_{22}$ with $A \in \mathbb{R}$. Now it can easily be seen that the remaining equations $Q_{04} = Q_{02} = Q_{00} = 0$ can only vanish w.c. for the following two cases:
   a. $c_{02} = 0$: This yields a solution (general case of item 4).
   b. $d_{00} = d_{02} = 0$: This also yields a solution (item 4).
2. $\mathcal{D}$ is orthogonal: W.l.o.g. we can set $d_{00} = Ad_{20}$ and $d_{02} = Ad_{22}$ with $A \in \mathbb{R}$.
   a. $A = 0$: We get a solution (general case of item 4).
   b. $A \neq 0$: Under this assumption Q_{04} and Q_{00} can only vanish w.c. if their common factor $F[7]$ is fulfilled identically. Therefore we compute the resultant of F and Q_{02} with respect to A, which can only vanish w.c. for:
      i. $c_{02} = 0$: Now we distinguish again two cases:
         ⋆ $c_{00} = 0$: This already yields a solution (item 4).
         ⋆ $c_{00} \neq 0$: Recomputation of the resultant of F and Q_{02} with respect to A implies $c_{22} = c_{11}^2/(4c_{00})$. Then $F = 0$ yields $c_{20} = 0$. Finally, Q_{02} cannot vanish w.c..
      ii. $c_{22} = 0$, $c_{02} \neq 0$: Now the condition of the heading of section 3.9.1 yields $c_{20} \neq 0$. Recomputation of the resultant of F and Q_{02} with

respect to A implies $c_{02} = c_{11}^2/(4c_{20})$. Now $F = 0$ yields $c_{00} = 0$. Finally, Q_{02} cannot vanish w.c..
   iii. $c_{00} = Ac_{02}$, $c_{20} = Ac_{22}$ with $A \in \mathbb{R}$ and $c_{02}c_{22} \neq 0$: Now F cannot vanish w.c..

3.9.2 Special case: $c_{22}d_{02} + c_{20}d_{22} = 0$

As for $c_{22} = d_{22} = 0$ the expression Q_{13} cannot vanish w.c., we can set w.l.o.g. $c_{20} = Ac_{22}$ and $d_{02} = -Ad_{22}$ with $A \in \mathbb{R}$. Now Q_{13} can only vanish w.c. for:

1. $d_{22} = 0$: W.l.o.g. we can assume that $d_{20} \neq 0$ holds. Moreover, we have to distinguish the following cases:
   a. $c_{22}(d_{00} + Ad_{20}) \neq 0$: Under this assumption we can compute f_{00} from $Q_{11} = 0$. Now Q_{02} can only vanish w.c. for $d_{00} = 0$ or $c_{00} = Ac_{02}$. In both cases we do not get a reducible composition without a spherical coupler component as $g_{24} = g_{13} = g_{04} = 0$ and $g_{11} \neq 0$ hold. Moreover, g_{11} cannot vanish w.c..
   b. $c_{22} = 0$: Therefore we can assume w.l.o.g. that $c_{02} \neq 0$ holds. Now $Q_{02} = 0$ implies $c_{00} = 0$ and from $Q_{11} = 0$ we get $d_{00} = 0$. This already yields a contradiction as all g_{ij}'s equal zero.
   c. $d_{00} = -Ad_{20}$, $c_{22} \neq 0$: Then $Q_{11} = 0$ implies $c_{00} = Ac_{02}$. We can assume $A \neq 0$, as otherwise all g_{ij}'s would vanish. Therefore $Q_{02} = 0$ implies an expression for f_{00}. Finally, Q_{00} can only vanish w.c. for $c_{02} = 0$ (item 2 and 4).
2. $c_{00} = Ac_{02}$, $d_{22} \neq 0$: Q_{11} can only vanish w.c. for:
   a. $d_{00} = -Ad_{20}$: We can assume $A \neq 0$, as otherwise all g_{ij}'s would vanish. Under this assumption Q_{04} and Q_{00} can only vanish w.c. if their common factor $G[4]$ is fulfilled identically. Therefore we compute the resultant of G and Q_{02} with respect to A, which can only vanish w.c. for $c_{02}f_{20} - c_{22}f_{00} = 0$:
      i. $c_{22} \neq 0$: Now we can express f_{00} from the last condition. Then G can only vanish w.c. for $c_{02} = 0$ (item 2).
      ii. $c_{22} = c_{02} = 0$: In this case $G = 0$ implies $f_{00} = 0$ (item 1 and 4).
   b. $c_{02}f_{20} - c_{22}f_{00} = 0$, $d_{00} + Ad_{20} \neq 0$: We distinguish further two cases:
      i. $c_{22} \neq 0$: Now we can express f_{00} from the last condition. Then it can easily be seen that the remaining equations $Q_{04} = 0$, $Q_{02} = 0$ and $Q_{00} = 0$ can only vanish w.c. for $c_{02} = 0$ (item 4).
      ii. $c_{22} = c_{02} = 0$: Now the remaining equations $Q_{04} = 0$, $Q_{02} = 0$ and $Q_{00} = 0$ can only vanish w.c. for $f_{00} = 0$ (item 1 and 4).

3.9.3 Excluded cases

We start by discussing the case $c_{22} = c_{02} = 0$, under the assumption that $d_{22} = d_{20} = 0$ does not hold. In this case we can compute all g_{ij}'s as in the general case. Then we distinguish two cases:

1. $c_{20} \neq 0$: Under this assumption we can express f_{00} from $Q_{11} = 0$. Then the remaining equations $Q_{02} = 0$ and $Q_{00} = 0$ can only vanish w.c. for $c_{00} = 0$ or $d_{02} = d_{00} = 0$. In both cases we do not get a solution as $g_{24} = g_{13} = g_{04} = 0$ and $g_{11} \neq 0$ hold. Moreover, g_{11} cannot vanish w.c..
2. $c_{20} = 0$: Now $Q_{11} = 0$ implies $c_{00} = 0$. Then the remaining equations $Q_{02} = 0$ and $Q_{00} = 0$ can only vanish w.c. for:
   a. $f_{00} = 0$: This yields a solution (item 1 and 4).
   b. $d_{02} = d_{00} = 0$, $f_{00} \neq 0$: This yields a contradiction as all g_{ij}'s vanish.

The discussion of the second excluded case $d_{22} = d_{20} = 0$, under the assumption that $c_{22} = c_{02} = 0$ does not hold, can be done as follows: Again all g_{ij}'s can be computed as in the general case. Then $Q_{11} = 0$ implies $d_{00} = 0$ and from $Q_{13} = 0$ we get $d_{02} = 0$. W.l.o.g. we can assume that $c_{20} \neq 0$ holds, as otherwise all g_{ij}'s would vanish. Therefore we can solve $Q_{02} = 0$ for f_{00} w.l.o.g.. We get a further solution (item 1 and 4).

The discussion of the special case $c_{22} = c_{02} = d_{22} = d_{20} = 0$ shows that the computation of the g_{ij}'s, as done in the general case, already yields a contradiction (all g_{ij}'s equal zero).

3.10 Discussing type (AA,III)

W.l.o.g. we can express g_{23} from $Q_{44} = 0$, g_{21} from $Q_{42} = 0$, g_{12} from $Q_{33} = 0$, g_{10} from $Q_{31} = 0$, g_{03} from $Q_{24} = 0$ and g_{01} from $Q_{22} = 0$. Now Q_{40} and Q_{00} can only vanish for:

$$c_{20}d_{20} - d_{00}c_{22} = 0, \qquad c_{00}d_{20} - d_{00}c_{02} = 0. \tag{26}$$

3.10.1 General case

In this case we set $d_{00} = Ad_{20}$, $c_{20} = Ac_{22}$ and $c_{00} = Ac_{02}$ with $A \in \mathbb{R}$. Then we have to discuss two cases:

1. $A = 0$: We can assume $d_{02} \neq 0$, as otherwise all g_{ij}'s would vanish. We distinguish the following cases:
   a. $c_{22}(2f_{10}c_{22}d_{02} + f_{21}c_{11}d_{11}) \neq 0$: Now we can express f_{01} from $Q_{13} = 0$.

i. $f_{10} = 0$: The last remaining equations $Q_{04} = 0$ and $Q_{02} = 0$ can only vanish w.c. for $c_{02} = 0$ (item 1 and 4).

ii. $d_{22} = 0$, $f_{10} \neq 0$: Now $Q_{11} = 0$ can only vanish if a homogeneous quadratic equation in f_{21} and f_{10} is fulfilled (item 1 and 4).

iii. $d_{22}f_{10} \neq 0$: Now Q_{04} and Q_{02} can only vanish w.c. for $H[3] = 0$ and $I[5] = 0$, respectively. Both factors H and I are homogeneous quadratic in f_{21} and f_{10}. Therefore we compute the resultant of H and I with respect to f_{10}, which can only vanish w.c. for:

$$d_{20}d_{22}c_{11}^2 + 4d_{20}d_{02}c_{02}c_{22} - c_{02}c_{22}d_{11}^2 = 0.$$

If this condition is fulfilled, then the remaining equations can be solved w.c. (item 2 and 4).

b. $c_{22} = 0$: Therefore we can assume w.l.o.g. that $c_{02} \neq 0$ holds. Then we can compute f_{21} from $Q_{13} = 0$. Now $Q_{11} = 0$ implies $d_{20} = 0$. Then the last remaining equation $Q_{04} = 0$ can be solved for f_{01} (item 1, 2 and 4).

c. $2f_{10}c_{22}d_{02} + f_{21}c_{11}d_{11} = 0$ and $c_{22} \neq 0$: Therefore we can solve this equation w.l.o.g. for f_{10}. Then $Q_{13} = 0$ implies $d_{22} = 0$ and from $Q_{11} = 0$ we get $d_{20} = d_{11}^2/(4d_{02})$. Moreover, the last remaining equation $Q_{04} = 0$ can be solved for f_{01} (item 1 and 4).

2. $A \neq 0$: We distinguish two cases:

a. $c_{22} \neq 0$: In this case we can express f_{10} from $Q_{20} = 0$. Now $Q_{11} = 0$ can only vanish w.c. for:

i. $d_{20} = 0$: We can express f_{01} from $Q_{02} = 0$. Q_{04} can only vanish w.c. for $c_{02} = 0$ (item 2 and 4) or $d_{02} = 0$ (item 1 and 4).

ii. $d_{02} = Ad_{22}$, $d_{20} \neq 0$: Now we can express f_{01} from $Q_{02} = 0$. Moreover, Q_{13} and Q_{04} can only vanish w.c. for $d_{22} = 0$ (item 3).

b. $c_{22} = 0$: Now $Q_{20} = 0$ implies $d_{20} = 0$. Finally, Q_{02} cannot vanish w.c..

3.10.2 Special cases

Now we discuss the two cases, which are not covered by the general case:

1. $c_{22} = d_{00} = d_{20} = 0$: Therefore we can assume $c_{02}d_{22} \neq 0$. We distinguish further three cases:

a. $c_{20} = 0$: Then $Q_{02} = 0$ implies $c_{00} = 0$. Moreover, we can assume $d_{02} \neq 0$, as otherwise all g_{ij}'s would vanish. Then we can solve the remaining two equations $Q_{13} = 0$ and $Q_{04} = 0$ for f_{10} and f_{01} (item 1, 2 and 4).

b. $f_{10} = 0$, $c_{20} \neq 0$: Now we can express f_{01} from $Q_{02} = 0$. Moreover, $Q_{13} = 0$ implies $d_{02} = 0$ (item 1 and 4).

c. $f_{10}c_{20} \neq 0$: In this case we compute the resultant of Q_{11} and Q_{02}, which can only vanish w.c. for:

i. $d_{22} = 0$: Now $Q_{11} = 0$ implies $c_{02} = 0$, which yields a contradiction as all g_{ij}'s vanish.

ii. $c_{02} = 0$, $d_{22} \neq 0$: We can express $f_{10} \neq 0$ from $Q_{11} = 0$. Moreover, we can solve $Q_{13} = 0$ for f_{01}. Then Q_{04} can only vanish w.c. for $c_{00} = 0$ (item 1, 2 and 4) or $d_{02} = 0$ (item 1 and 4).

iii. $c_{00} = 0$, $c_{02}d_{22} \neq 0$: In this case Q_{11} is a factor of Q_{02}. Therefore we compute the resultant of Q_{11} and Q_{13} with respect to f_{10}, which can only vanish w.c. for $(4c_{02}c_{20} - c_{11}^2)K[4] = 0$, where K is a homogeneous quadratic expression in f_{21} and f_{01}. Moreover, K equals the numerator of the last remaining equation $Q_{04} = 0$ (beside $Q_{11} = 0$ and $Q_{13} = 0$). As $Q_{04} = 0$ has to be fulfilled, we do not have to discuss the case $4c_{02}c_{20} - c_{11}^2 = 0$. Clearly, the other case $K = 0$ implies a solution (item 1, 2 and 4).

2. $c_{02} = d_{00} = d_{20} = 0$: Therefore we can assume $c_{22}d_{22} \neq 0$. Then Q_{20} can only vanish w.c. for:
   a. $c_{20} = 0$: Now Q_{11} and Q_{02} can only vanish w.c. for:
      i. $f_{10} = f_{01} = 0$: Then $Q_{13} = 0$ implies $c_{00} = 0$ (item 1, 2 and 4).
      ii. The common factor $L[3]$ of Q_{11} and Q_{02} vanishes, where L is quadratic homogeneous in f_{21} and f_{10}. Now the resultant of L and Q_{13} can only vanish w.c. for $(4c_{00}c_{22} - c_{11}^2)M[4] = 0$, where M is a homogeneous quadratic expression in f_{21} and f_{01}. Moreover, M equals the numerator of the last remaining equation $Q_{04} = 0$ (beside $L = 0$ and $Q_{13} = 0$). As $Q_{04} = 0$ has to be fulfilled, we do not have to discuss the case $4c_{00}c_{22} - c_{11}^2 = 0$. Clearly, the other case $M = 0$ implies a solution (item 1, 2 and 4).
   b. $f_{10} = 0$, $c_{20} \neq 0$: Q_{02} can only vanish w.c. for:
      i. $f_{01} = 0$: Now $Q_{13} = 0$ implies $c_{00} = 0$ (item 2 and 4).
      ii. $f_{01} = c_{00}f_{21}/c_{20}$ with $c_{00} \neq 0$: Now $Q_{13} = 0$ implies $d_{02} = 0$ (item 1 and 4).

3.10.3 Excluded cases

We start by discussing the case $c_{22} = c_{02} = 0$, under the assumption that $d_{22} = d_{20} = 0$ does not hold. We can compute all g_{ij}'s as in the general case. Due to $Q_{40} = d_{20}c_{20}^2$ and $Q_{00} = d_{20}c_{00}^2$, we have to distinguish two cases:

1. $d_{20} = 0$: Therefore we can assume $d_{22} \neq 0$. Now the resultant of Q_{11} and Q_{02} with respect to f_{01} can only vanish w.c. for $d_{02}c_{20}N[5] = 0$:
   a. $c_{20} = 0$: Now we can assume $d_{02} \neq 0$, as otherwise all g_{ij}'s would vanish. Then $Q_{20} = 0$ implies $d_{00} = 0$. Finally, the remaining equations $Q_{11} = 0$ and $Q_{02} = 0$ can be solved for f_{10} and f_{01} (item 1, 2 and 4).
   b. $d_{02} = 0$, $c_{20} \neq 0$: Now we can express f_{01} from $Q_{02} = 0$. Then only $Q_{20} = 0$ remains, which is a homogeneous quadratic equation in f_{21} and f_{10} (item 1).
   c. $N[5] = 0$, $d_{02}c_{20} \neq 0$: As N and the numerator of Q_{20} are homogeneous quadratic expressions in f_{21} and f_{10}, we compute the resultant of N and

Q_{20} with respect to f_{10}. This resultant can only vanish w.c. for (item 2):

$$d_{00}d_{02}c_{11}^2 + 4c_{00}c_{20}d_{00}d_{22} - c_{00}c_{20}d_{11}^2 = 0. \tag{27}$$

2. $c_{00} = c_{20} = 0$, $d_{20} \neq 0$: Then $Q_{20} = 0$ implies $d_{00} = 0$. Now we can assume $d_{02} \neq 0$, as otherwise all g_{ij}'s would vanish. Then Q_{11} and Q_{02} can only vanish w.c. for $f_{01} = f_{10} = 0$ (item 1 and 4).

The discussion of the second excluded case $d_{22} = d_{20} = 0$, under the assumption that $c_{22} = c_{02} = 0$ does not hold, can be done as follows: Again all g_{ij}'s can be computed as in the general case. Due to $Q_{20} = c_{22}d_{00}^2$ and $Q_{00} = c_{02}d_{00}^2$, we only have to discuss the case $d_{00} = 0$. Then Q_{11} can only vanish w.c. for:

1. $c_{20} = 0$: Now we can assume $d_{02} \neq 0$, as otherwise all g_{ij}'s would vanish. Then $Q_{02} = 0$ implies $c_{00} = 0$. Finally, we can solve the remaining equations $Q_{13} = 0$ and $Q_{04} = 0$ for f_{10} and f_{01} (item 1 and 4).
2. $f_{10} = 0$, $c_{20} \neq 0$: Then $Q_{13} = 0$ implies $d_{02} = 0$. We can solve $Q_{02} = 0$ for f_{01} (item 1 and 4).

The discussion of the special case $c_{22} = c_{02} = d_{22} = d_{20} = 0$ shows that the computation of the g_{ij}'s, as done in the general case, already yields a contradiction (all g_{ij}'s equal zero).

3.11 Discussing type (AA,IV)

W.l.o.g. we can express g_{14} from $Q_{44} = 0$, g_{12} from $Q_{42} = 0$, g_{10} from $Q_{40} = 0$, g_{03} from $Q_{33} = 0$ and g_{01} from $Q_{31} = 0$. Now Q_{04} and Q_{00} can only vanish for:

$$c_{00}d_{22} - c_{02}d_{02} = 0, \qquad c_{00}d_{20} - c_{02}d_{00} = 0. \tag{28}$$

Therefore we have to discuss the following cases:

1. $c_{00} = Ac_{02}$, $d_{02} = Ad_{22}$ and $d_{00} = Ad_{20}$ with $A \in \mathbb{R}$: Due to $Q_{02} = Ac_{02}^2d_{11}^2$, we distinguish two cases:
   a. $A = 0$: Now we can assume $c_{20} \neq 0$, as otherwise all g_{ij}'s would vanish. Now Q_{24} and Q_{20} can only vanish w.c. for $f_{10} = 0$. Then $Q_{22} = 0$ implies $c_{02} = 0$ (item 4).
   b. $c_{02} = 0$, $A \neq 0$: Now Q_{13} and Q_{11} can only vanish w.c. for:
      i. $f_{10} = 0$: Then $Q_{24} = 0$ implies $d_{22} = 0$. Finally, $Q_{20} = 0$ yields the contradiction.
      ii. $c_{20} = -Ac_{22}$, $f_{10} \neq 0$: Then Q_{24} and Q_{20} can only vanish w.c. for $f_{30} = 4Af_{10}c_{22}^2/c_{11}^2$. Now $Q_{22} = 0$ yields the contradiction.
2. $c_{00} = c_{02} = d_{22} = 0$: Therefore we can assume $c_{22}d_{20} \neq 0$. Now Q_{24} can only vanish w.c. for:

a. $f_{10} = 0$: Then $Q_{22} = 0$ implies $d_{02} = 0$ and from $Q_{20} = 0$ we get $d_{00} = 0$ (item 1, 2 and 4).
b. $d_{02} = 0$, $f_{10} \neq 0$: Now $Q_{22} = 0$ implies $c_{20} = 0$ and from $Q_{11} = 0$ we get $d_{00} = 0$. This already yields a contradiction as all g_{ij}'s vanish.

3. $c_{00} = c_{02} = d_{20} = 0$: Therefore we can assume $c_{22}d_{22} \neq 0$. Now Q_{20} can only vanish w.c. for:
a. $f_{10} = 0$: Then $Q_{22} = 0$ implies $d_{00} = 0$ and from $Q_{24} = 0$ we get $d_{02} = 0$ (item 1, 2 and 4).
b. $d_{00} = 0$, $f_{10} \neq 0$: Now $Q_{22} = 0$ implies $c_{20} = 0$ and from $Q_{13} = 0$ we get $d_{02} = 0$. This already yields a contradiction as all g_{ij}'s vanish.

3.11.1 Excluded cases

We start by discussing the case $c_{22} = c_{02} = 0$, under the assumption that $d_{22} = d_{20} = 0$ does not hold. In this case we can compute all g_{ij}'s as in the general case. Due to $Q_{02} = d_{22}c_{00}^2$ and $Q_{00} = d_{20}c_{00}^2$, we get $c_{00} = 0$. Then $Q_{11} = 0$ implies $f_{10} = 0$. Moreover, we get $d_{02} = 0$ and $d_{00} = 0$ from $Q_{22} = 0$ and $Q_{20} = 0$ (item 1 and 4).

The discussion of the second excluded case $d_{22} = d_{20} = 0$, under the assumption that $c_{22} = c_{02} = 0$ does not hold, can be done as follows: Again all g_{ij}'s can be computed as in the general case, but this already yields the contradiction as all g_{ij}'s equal zero. We get the same contradiction for the special case $c_{22} = c_{02} = d_{22} = d_{20} = 0$. This finishes the study of all cases.

If we consider those solutions of the case study, which only correspond with item 4 of Theorem 1, one will see that these cases either belong to the case $c_{00} = c_{02} = 0$ or $d_{00} = d_{02} = 0$. We get the remaining conditions $c_{00} = c_{20} = 0$ or $d_{00} = d_{20} = 0$ given in item 4 of Theorem 1 by exchanging the variables t_1 and t_3 in the eleven possible reducible compositions given in Eqs. (5–15).

Such an exchange does not cause a further solution for item 3 of Theorem 1 as the general solution of this case stems from a (SS) case. This finishes the proof of Theorem 1. □

4 Geometric interpretation of reducible compositions

Item c of Corollary 1 is a special case of item 4 of Theorem 1. As item d of Corollary 1 is a special case of item 3 and 4 of Theorem 1, it can also be written as:

- $c_{00} = Ac_{02}, \quad c_{20} = Ac_{22}, \quad d_{02} = Ad_{22}, \quad d_{00} = d_{20} = 0, \quad d_{02}d_{22} \neq 0,$
- $c_{20} = Ac_{22}, \quad d_{00} = Ad_{20}, \quad d_{02} = Ad_{22}, \quad d_{00} = d_{20} = 0, \quad c_{20}c_{22} \neq 0.$

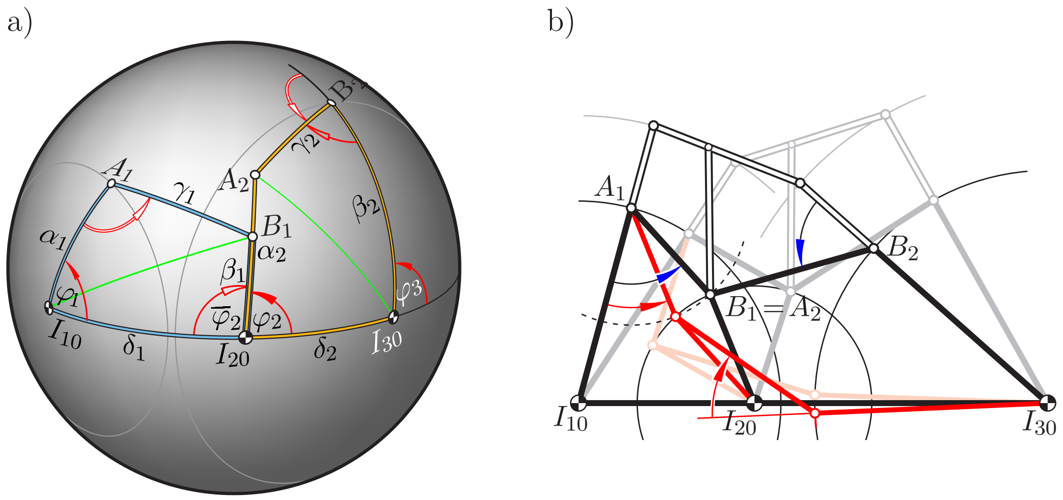

Fig. 2. a) Reducible spherical composition (cf. [12]) obeying Dixon's angle condition $\sphericalangle I_{10}A_1B_1 = \pm\sphericalangle \overline{I}_{30}B_2A_2$. b) Reducible planar composition (cf. [12]) with both components implied by the Dixon's angle condition $\sphericalangle I_{10}A_1B_1 = \pm(\pi - |\sphericalangle I_{30}B_2A_2|)$. (Courtesy H. Stachel)

Therefore we only have to clarify the geometric meaning of the four items of Theorem 1:

ad 1) It was already shown in [4], that spherical four-bars with $c_{00} = c_{22} = 0$ are spherical isograms with $\alpha_1 = \beta_1$ and $\gamma_1 = \delta_1$. By replacing either I_{10} or I_{20} by its antipode, we get the corresponding mechanism with $\alpha_1 + \beta_1 = \pi$ and $\gamma_1 + \delta_1 = \pi$, which is given by $c_{20} = c_{02} = 0$ (cf. [9]).

ad 2) It was shown in [12], that the given algebraic conditions are equivalent to Dixon's angle condition $\sphericalangle I_{10}A_1B_1 = \pm\sphericalangle \overline{I}_{30}B_2A_2$ (cf. Fig. 2a). By replacing either I_{10} or I_{30} by its antipode, we get the corresponding focal mechanism given by the condition $\sphericalangle I_{10}A_1B_1 = \pm\sphericalangle I_{30}B_2A_2$.

ad 3) It can immediately be seen that both couplers are orthogonal. Moreover it can easily be checked by direct computations that the condition $A = B$ corresponds to the geometric condition that the diagonals A_1I_{20} and $I_{20}B_2$ coincide (cf. Fig. 3a). All these geometric characterizations are invariant with respect to the replacement of vertices by their antipodal points.

ad 4) We distinguish the following cases:

- ⋆ $c_{00} = c_{02} = 0$: A straight forward computation based on Eqs. (2) and (3) yields $\alpha_1 = \delta_1$ and $\beta_1 = \gamma_1$ (cf. Fig. 3b). By replacing either A_1 or I_{20} by its antipode, we get the corresponding mechanism with $\alpha_1 + \delta_1 = \pi$ and $\beta_1 + \gamma_1 = \pi$, which is given by $c_{22} = c_{20} = 0$.
- ⋆ $c_{00} = c_{20} = 0$: This implies $\alpha_1 + \gamma_1 = \pi$ and $\beta_1 + \delta_1 = \pi$. By replacing either B_1 or I_{10} by its antipode, we get the corresponding mechanism with $\alpha_1 = \gamma_1$ and $\beta_1 = \delta_1$, which is given by $c_{22} = c_{02} = 0$.

Due to their geometry, we call these four-bars *spherical deltoids*.

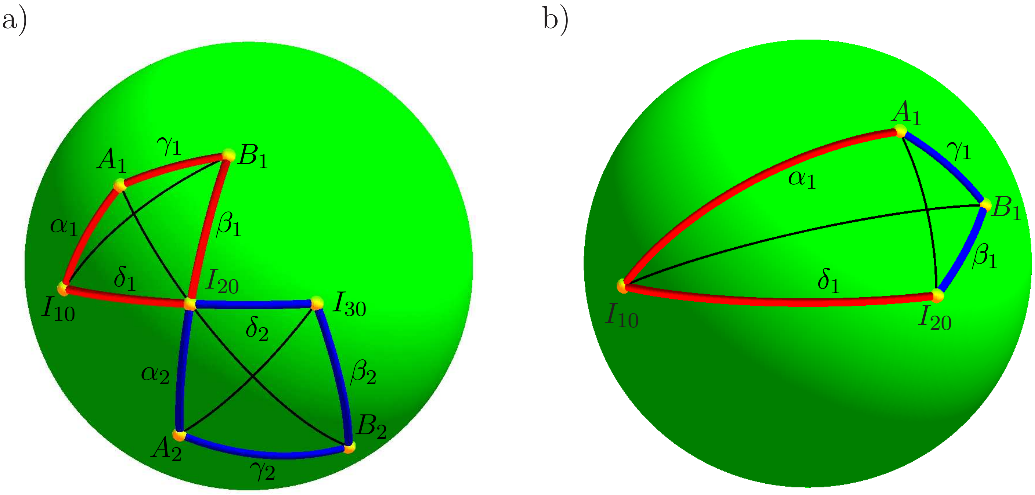

Fig. 3. a) Two orthogonal four-bars, which have one diagonal in common. b) Spherical deltoids are special orthogonal four-bars.

Now the question arises why the four deltoidal cases

$$c_{22} = c_{02} = 0, \quad d_{22} = d_{02} = 0, \quad c_{22} = c_{20} = 0, \quad d_{22} = d_{20} = 0, \tag{29}$$

do not appear during the proof of Theorem 1. The reason for this is the used half-angle substitution $t_i := \tan(\varphi_i/2)$ as none of the angles φ_i can be constant π, which can only be the case for an assembly mode of a spherical deltoid. If we would set $t_i := \cot(\varphi_i/2)$ instead of $t_i := \tan(\varphi_i/2)$ this would imply the following replacements:

$$c_{00} \leftrightarrow c_{22}, \quad c_{20} \leftrightarrow c_{02}, \quad d_{00} \leftrightarrow d_{22}, \quad d_{20} \leftrightarrow d_{02}. \tag{30}$$

One can immediately see that the cases given in item 4 of Theorem 1 correspond with the cases of Eq. (29) with respect to this substitution. Moreover, one can also apply the substitution of Eq. (30) to the conditions given in item c of Corollary 1. One will see that the given conditions correspond to each other. It can also easily be checked, that the conditions of item 1, 2 and 3 of Theorem 1 are invariant with respect to this substitution. As a consequence the corresponding mechanisms of item d of Corollary 1 remain the same under the substitution with exception of the involved deltoidal linkages, which are replaced by their corresponding ones.

Therefore no case was lost during the case study due to the half-angle substitution and therefore Theorem 1 already gives all reducible compositions of spherical four-bar linkages, if one takes those compositions into account, which arise from the given ones by replacing vertices by their antipodal points.

Remark 1 *Clearly, a composition of two spherical four-bar linkages is reducible if one of these linkages is already reducible. Therefore all reducible spherical four-bars (spherical isogram, spherical deltoid, cf. [14]) also have to be included in Theorem 1.* ◇

5 Conclusion

As the transmission function of a planar four-bar mechanism can also be written in the form of Eq. (1), the reducible compositions of planar four-bar mechanisms have the same algebraic characterization as their spherical counterparts (cf. [1,9]). Moreover, if one takes the replacement of vertices by their antipodal points into account, the list of reducible compositions of spherical four-bar mechanisms and the one of their planar analogue can be written within one theorem [6] as follows:

Theorem 2 *If a reducible composition of two planar or two spherical four-bar mechanisms $\mathcal{C}$ and $\mathcal{D}$ is given, then it is one of the following cases:*

I. *Opposite sides of one of the quadrangles $\mathcal{C}$ or $\mathcal{D}$ have the same length.*
II. *$\mathcal{C}$ and $\mathcal{D}$ fulfill Dixon's angle condition $\sphericalangle I_{10}A_1B_1 = \pm(\pi - |\sphericalangle I_{30}B_2A_2|)$ (cf. Fig. 2b).*
III. *The diagonals of each of the four-bars $\mathcal{C}$ and $\mathcal{D}$ intersect orthogonally and the diagonals A_1I_{20} and $I_{20}B_2$ coincide.*
IV. *One of the quadrangles $\mathcal{C}$ or $\mathcal{D}$ is a deltoid.*

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