Poristic Loci of Triangle Centers

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Abstract

The one-parameter family of triangles with common incircle and circumcircle is called a porisitic[1] system of triangles. The triangles of a poristic system can be rotated freely about the common incircle. However this motion is not a rigid body motion for the sidelengths of the triangle are changing. Surprisingly many triangle centers associated with the triangles of the poristic family trace circles while the triangle traverses the poristic family. Other points move on conic sections, some points trace more complicated curves. We shall describe the orbits of centers and some other points. Thereby we are able to answer open questions and verify some older results.

Key Words: Poristic triangles, incircle, excircle, non-rigid body motion, poristic locus, triangle center, circle, conic section.

Mathematics Subject Classification (AMS 2000): 51M04

1 Introduction

The family of poristic triangles has marginally attracted geometers interest. There are only a few articles contributing to this particular topic of triangle geometry: [3] is dedicated to perspective poristic triangles, [12] deals with the existence of triangles with prescribed circumcircle, incircle, and an additional

[1]The word poristic is deduced from the greek word *porisma*, which could be tranlated by *deduced theorem*, *cf.* L. Mackensen: *Neues Wörterbuch der deutschen Sprache.* 13. Auflage, Manuscriptum Verlag, München, 2006.

element. Some more general appearances of porisms are investigated in [2, 4, 6, 9, 16] and especially [7] provides an overview on Poncelet's theorem which is the projective version and thus more general notion of porism.

Nevertheless there are some results on poristic loci, *i.e.*, the traces of triangle centers and other points related to the triangle while the triangle is traversing the poristic family. In [14] some invariant lines, circles, and conic sections have been determined. Also triangles with common circumcircle and nine-point circle have been studied by R. Crane in [5]. The poristic loci of triangle centers have not undergone sincere study. For some centers the loci are given in [11], especially the trace of the Gergonne point is treated in [1]. In [13] a result by Weill is reproduced showing that the centroid X_{354} of the intouch triangle is fixed while Δ traces its poristic family.

For most of the centers listed in [10] the respective poristic locus is unknown. In the following we shall derive these loci, at least for some centers that can easily be accessed with our method. For that purpose we impose a Cartesian coordinate system in Sec. 2 which will henceforth be the system of reference. Subsequently we derive paths of points which are not centers in Sec. 3. Afterwards we pay our attention to triangle centers in Sec. 4. First we focus on the centers on the line $\mathcal{L}_{1,3}$,[2] connecting the incenter X_1 with the circumcenter X_3. It carries a lot of centers, some of them stay fixed others do not. We only look at the fixed ones. Triangle centers which are located on the incircle or circumcircle naturally trace these circles. None of these remains fixed, except those on $\mathcal{L}_{1,3}$. Then we shall derive poristic loci of some triangle centers and focus on those that traverse circles and conic sections. The centers and radii or semiaxes of these poristic orbits are given explicitly.

At this point we shall say a few words about techniques used in this work. Computations are done with Maple. Equations of poristic loci mainly use the framework of resultants. The computation of parametrizations of centers is restricted somehow. This will be clear when we see parametrizations of the circumcircle and incircle describing the vertices of Δ and its intouch triangle, respectively. Deriving paths of orthocenters, centroids, circumcenters, midpoints of a pair of triangle centers, as well as paths of triangle centers which appear as intersections of central lines seems to be a very simple task at first glance. But, however, parametrizations become larger and larger and

[2] Triangle centers are labelled according to C. Kimberlings list [10, 11]. Central lines, *i.e.*, lines joining two centers with Kimberling number i and j shall be denonted by $\mathcal{L}_{i,j}$.

the computation of equations exceeds memory capacity and cannot be done in an acceptable amount of time. Therefore centers like the incenter of the intouch triangle (which is X_{177} for the base triangle) cannot be reached with our method.

2 Prerequisites

Let Δ be a triangle with vertices A, B, C. We denote its circumcircle and incircle by u and i, respectively. The circumcenter and the incenter shall be denoted by X_3 and X_1, see [10, 11]. The circumradius R, the radius of the incircle r, and the distance d of X_1 and X_3 are related by

$$d^2 = R^2 - 2rR, \tag{1}$$

see for example [11, p. 40]. The incircle and the circumcircle are circles in a special position. To the best of the author's knowledge there is no english word for that. In german we would say: *"Kreise in Schließungslage"*.

Any two circles u and i define a one-parameter family of triangles all of them having u for the circumcircle and i for the incircle provided that Eq. (1) is fulfilled, *cf.* Fig. 1. Any two triangles out of this family are said to form a poristic pair of triangles.

In the following we want to study the traces of centers and other points related to a triangle traversing its poristic family. For that purpose we use Cartesian coordinates in order to represent points in the Euclidean plane. Without loss of generality we can assume that $X_3 = [0, 0]$ and $X_1 = [d, 0]$. The equations of the circumcircle and the incircle are thus

$$u: \; x^2 + y^2 = R^2, \quad i: \; (x - d)^2 + y^2 = r^2. \tag{2}$$

Aiming at parametrizations of the traces of centers and other points related to the triangle we assume that the line carrying A and B is given by

$$g: \; x \cos t + y \sin t = r + d \cos t \quad \text{with} \quad t \in [0, 2\pi) \tag{3}$$

since $[A, B]$ has to be tangent to i. This allows to parametrize the circular path of points A and B in a proper way. Note that these points are the intersections of g and u and therefore they are given by

$$\begin{aligned} A &= [r \cos t + d \cos^2 t + W \sin t, r \sin t + d \cos t \sin t - W \cos t], \\ B &= [r \cos t + d \cos^2 t - W \sin t, r \sin t + d \cos t \sin t + W \cos t], \end{aligned} \tag{4}$$

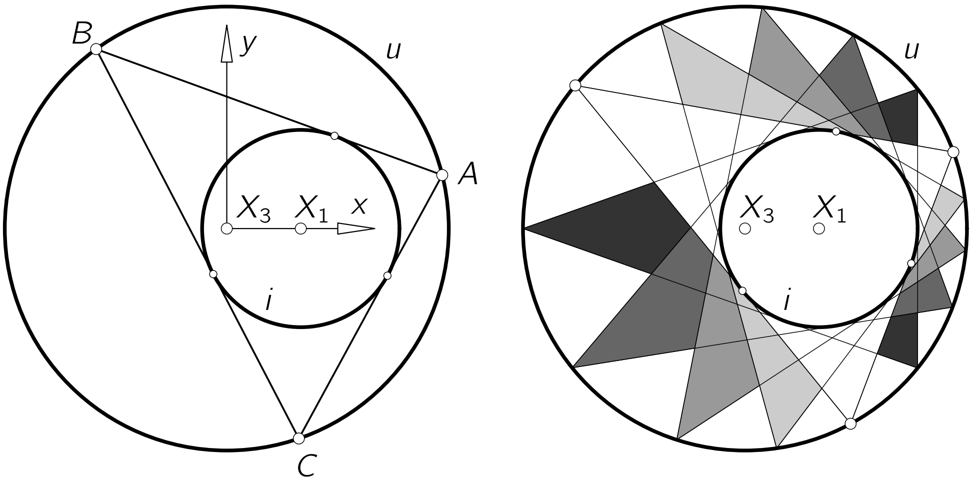

Figure 1: Triangle Δ with incircle i, circumcircle u, and the Cartesian coordinate system imposed on it.

where $W = \sqrt{R^2 - (r + d\cos t)^2}$.[3]

Finally the two tangents from A and B to i which are different from the line $[A, B]$ intersect in Δ's third vertex $C \in u$. This point reads

$$C = \left[\frac{R(2dR - (R^2 + d^2)c_t)}{R^2 + d^2 - 2dRc_t}, \frac{(d^2 - R^2)Rs_t}{R^2 + d^2 - 2dRc_t} \right], \tag{5}$$

whose trace is now described by the same parameter t. Here and in the following c_t and s_t are short hand for $\cos t$ and $\sin t$, respectively.

The triangle Δ and thus any triangle in the poristic family defines some other triangles: The medial triangle Δ_m is built by the midpoints of Δ's sides. We denote the anticomplementary triangle by Δ_a, the excentral triangle by Δ_e, the intouch triangle by Δ_i, the tangent triangle by Δ_t, the orthic triangle by Δ_o, and the extouch triangle by Δ_x.

It is elementary to find the vertices of Δ_a, Δ_e, Δ_m, Δ_o, and Δ_t, if the parametric representation of A, B, and C is known. One vertex from Δ_i is known from the beginning: $B_{AB} = [rc_t + d, rs_t]$, the point of contact of g and i. The remaining vertices of Δ_i can be obtained by reflecting the contact point B_{AB} of the line $[A, B]$ with i in $[A, X_1]$ and $[B, X_1]$, respectively. Though these

[3] Note that r, R, and d are related via Eq. (1). Sometimes we do not eliminate r in order to shorten formulae.

operations are elementary we sketch them in order to make any computation traceable.

Finally we point out that the computation of centers X_i with

$$i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 20\}$$

among many others is elementary with these preparations. The shape of the respective poristic loci will be discussed later in Sec. 4. At this point we should confess that the computation of incenters needs normalization of direction vectors. Luckily we have the incenter of Δ, but, unfortunately we cannot reach incenters of Δ_e and Δ_i.

3 Traces of some points

We give the answer to the question raised in [11, p. 257] by proving the following:

Theorem 3.1. *The trace of the midpoint of any side of a triangle traversing a poristic family is a Limaçon of Pascal.*

Proof. The midpoint M of AB is given as the arithmetic mean of the coordinate vectors of the two points A and B from (4) and reads

$$M = [r \cos t + d \cos^2 t, r \sin t + d \cos t \sin t]. \tag{6}$$

The curve parametrized by (6) is called Limaçon of Pascal, see [15]. Its equations in terms of Cartesian coordinates is obtained by eliminating t and reads

$$m: \ (x^2 + y^2)^2 - 2dx(x^2 + y^2) - ((r^2 - d^2)x^2 + r^2y^2) = 0 \tag{7}$$

for variable choices of r and d such that Eq. (1) is satisfied. □

Fig. 2 shows different shapes of this curve: noded, cusped, or without visible singularity. However, independent on the choice of R and d the point X_3 is a double point on m in any case. The quartic curve m has a cusp at U exactly if $|d| = |r|$. If $|d| < |r|$ U is an isolated double point.

The quartic m touches i twice, *i.e.*, precisely at points $X_{2446} = [d - r, 0]$ and $X_{2447} = [d + r, 0]$. If a midpoint of a side of Δ happens to coincide with one

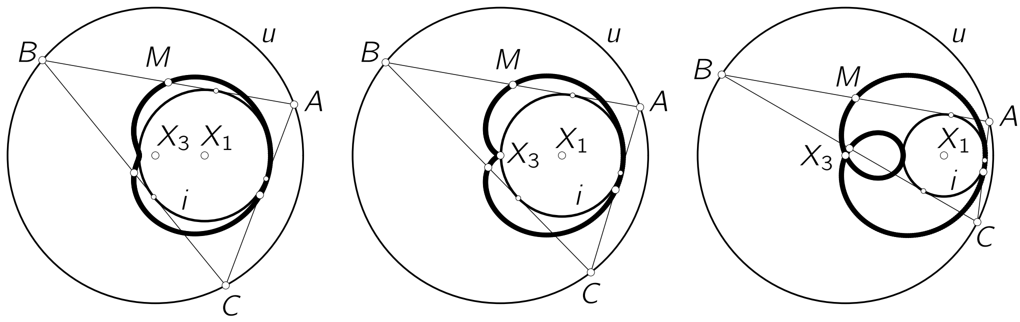

Figure 2: Different shapes of Pascal's limaçon which appears as trace of a side's midpoint.

of these points then Δ is isosceles. Obviously there are two isosceles triangles in the family of poristic triangles.

According to Bézout's theorem the total amount of intersection points of m and i equals 8. The two real contact points X_{2446} and X_{2447} are each of multiplicity two, the remaining four points are the absolute points of Euclidean geometry (a pair of conjugate complex points on the ideal line) each of which has multiplicity two on m and as a common point of m and i. Note that the midpoints of the remaining two sides of Δ hound M on the same curve.

The traces of the excenters (see Fig. 3) of triangles in a poristic family have been studied in [14] with slightly different methods. We observe:

Theorem 3.2. *The three excenters of the triangles in a poristic family trace the same circle e. Its center E is the reflection of X_1 in X_3 and its radius equals $2R$.*

Proof. The normals to $[A, X_1]$ and $[B, X_1]$ at A and B, respectively, intersect at Δ's excenter A_3, opposite to C, for these lines are the internal bisectors at A and B. An elementary computation using (4) and (5) yields

$$A_3 = [2Rc_t - d, 2Rs_t]$$

which obviously parametrizes the circle e with equation

$$e: \ (x+d)^2 + y^2 = 4R^2. \tag{8}$$

Cyclically shifting A, B, and C yields parametrizations of the loci of the other excenters. These parametrizations annihilate Eq. (8). The center of e is

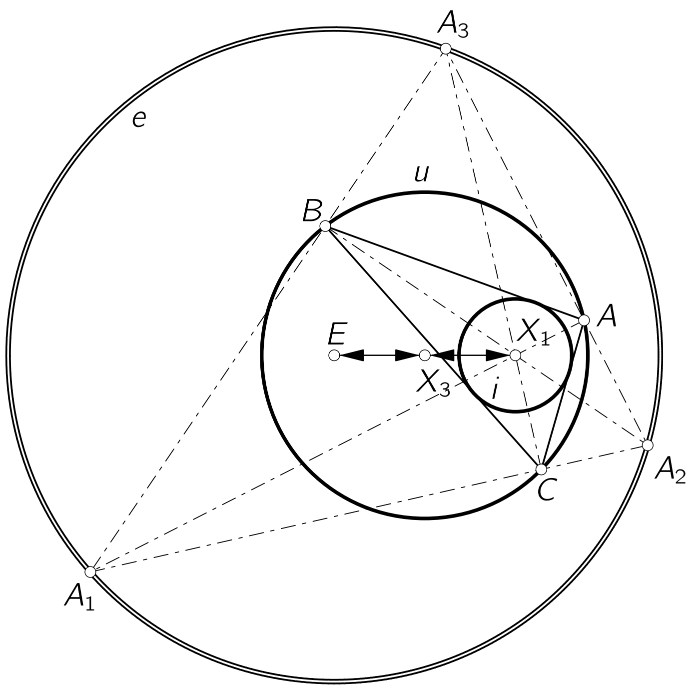

Figure 3: The common path of all three excenters.

$E = [-d, 0]$, *i.e.*, the point X_3 is the midpoint of E and X_1. The radius of e equals $2R$. $\square$

Note that the point E is Δ's center X_{40}, which is frequently called Bevan point (*cf.* [10, 11]) and it remains fixed while Δ goes through the poristic family. The excentral triangles Δ_e together with the triangles Δ form another poristic family of triangles with common circumcircle e and nine-point circle u for Δ is the orthic triangle of Δ_e, see [5].

Similarly we can show:

Theorem 3.3. *The vertices of the tangential triangle Δ_t of Δ move on an ellipse while Δ traverses the poristic family.*

Proof. The vertices T_A, T_B, T_C of Δ_t are constructed as the intersections of the tangents of the circumcircle u at A, B, C, respectively. The trace of the vertex T_C opposite to C is parametrized by

$$T_C = \left[\frac{2R^3 c_t}{R^2 - d^2 + 2dRc_t}, \frac{2R^3 s_t}{R^2 - d^2 + 2dRc_t}\right].$$

This is an ellipse with center $[R^2d/(R^2-2Rr-r^2),0]$ and semiaxes $a = rR^2/(r^2+2Rr-R^2)$, $b = R^2/\sqrt{r^2+2Rr-R^2}$ which can be seen after implicitization. The traces of T_A and T_B have a more complicated parametrization, but, however, they annihilate the same equation. □

Fig. 4 shows the ellipse appearing as the poristic orbit of the vertices of Δ_t.

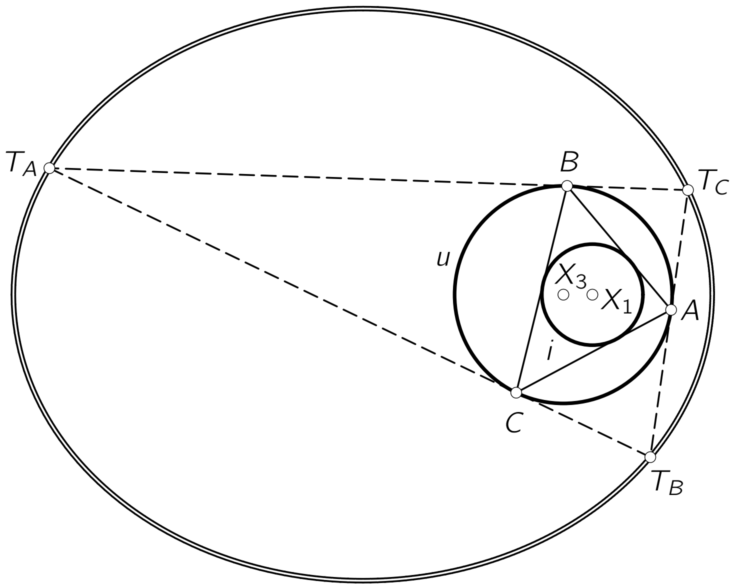

Figure 4: The ellipse traced by the vertices of the tangential triangle.

4 Orbits of some centers

4.1 Centers on $\mathcal{L}_{1,3}$

On the central line $\mathcal{L}_{1,3}$ we find the triangle centers X_i with

$$i \in \{1, 3, 35, 36, 40, 46, 55, 56, 57, 65, 165, 171, 241, 260, 354, 484, 517, 559, 940, 942, 980, 982, 986, 988, 999, 1038, 1040, 1060, 1062, 1082, 1155, 1159, 1214, 1319, 1381, 1382, 1385, 1388, 1402, 1403, 1420, 1429, 1454, 1460, 1466, 1467, 1470, 1482, 1617, 1622, 1697, 1715, 1735, 1754, 1758, 1764, 1771, 1936, 2061, 2077, 2078, 2093, 2095, 2098, 2099, 2223, 2283, 2352, 2446, \ldots, 2449, 2556, 2557, 2564, 2565, 2572, 2573, 2646, 2662, 3057, 3072, 3075, 3245, 3256, 3295, 3303, 3304, 3333, 3336, \ldots, 3340, 3359, 3361, 3428, 3503, 3513, 3514, 3550, 3576, 3579, 3587, 3601, 3612\},$$

cf. [10]. The points X_1 and X_3 are fixed anyway. The circumcenter of Δ's excentral triangle is the point X_{40} and remains fixed as shown in Th. 3.2. The triangle center X_{571} is the ideal point of the line $\mathcal{L}_{1,3}$ and all parallel lines, especially the central lines $\mathcal{L}_{4,8}$ and $\mathcal{L}_{5,10}$.

For some of the centers on $\mathcal{L}_{1,3}$ we can give their precise position and state:

Theorem 4.1. *The triangle centers X_i of Δ with*

$$i \in \{1, 3, 35, 36, 40, 46, 55, 56, 57, 65, 165, 354, 484, 517, 942, 999, 1155, 1159, 1319, 1381, 1382, 1385, 1388, 1420, 1454, 1482, 1697, 2077, 2078, 2093, 2095, 2098, 2099, 2446, 2447, 2646, 3057, 3245, 3256, 3295, 3303, 3304, 3336, \ldots, 3340, 3576, 3579, 3587, 3601, 3612\} \tag{9}$$

remain fixed while Δ traverses its poristic family.

Proof. There is nothing to be done for $X_1 = [d, 0]$, $X_3 = [0, 0]$, and X_{517}, the ideal point of $\mathcal{L}_{1,3}$. The Bevan point X_{40} is the circumcenter of Δ_e and according to Th. 3.2 it is fixed.

The center $X_{65} = [d(R + r)/R, 0]$ ist the orthocenter of Δ_i. $X_{942} = [d(2R + r)/(2R), 0]$ is the midpoint of X_1 and X_{65}. The center $X_{36} = [R^2/d, 0]$ is the inverse of X_{942} in the incircle and the 1^{st} Evans perspector $X_{484} = [R(R + 2r)/d, 0]$ is the reflection of X_1 in X_{36}. Then $X_{35} = [dR/(R + 2r), 0]$ is the inverse of X_{484} in the circumcircle.

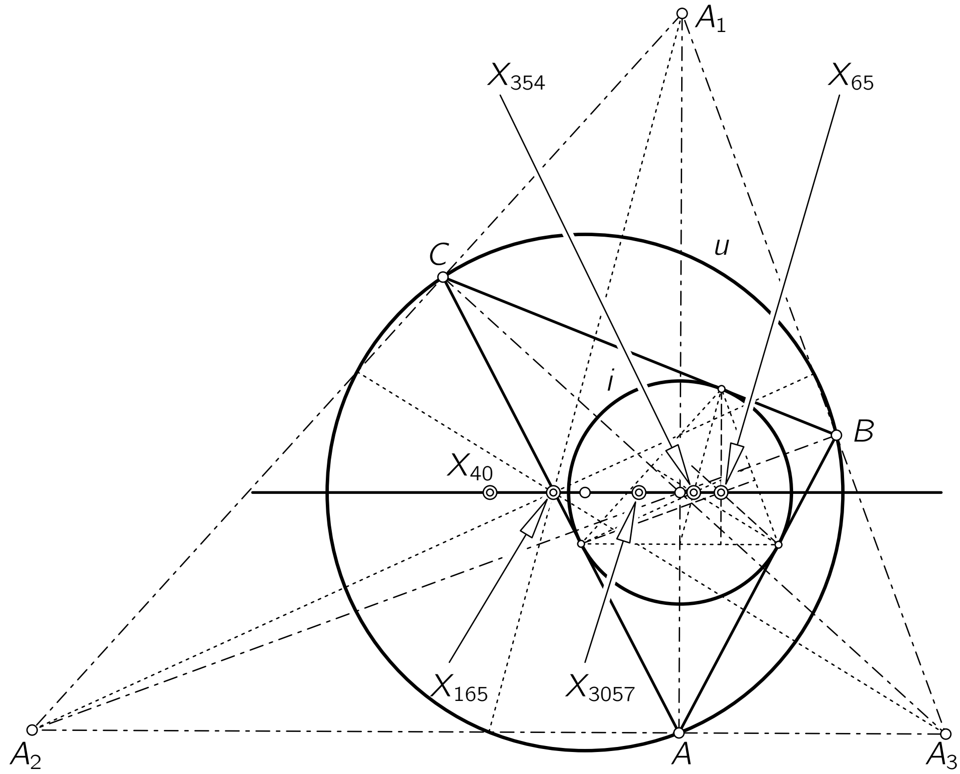

Figure 5: Some centers on the line $\mathcal{L}_{1,3}$ mentioned in Th. 4.1: the circumcenter X_{40} of Δ_e, the orthocenter X_{65} of Δ_i, the centroid X_{165} of Δ_e, the centroid X_{354} (Weill point) of Δ_i, the de Longchamps point X_{3057} of Δ_i.

The centers $X_{55} = [dR/(R+r), 0]$ and $X_{56} = [dR/(R-r), 0]$ are the in- and exsimilicenter of the incircle i and the circumcircle u. They are fixed for u and i are fixed. As the reflection of X_1 in X_{56} we find $X_{46} = [d(R+r)/(R-r), 0]$. X_{57} appears as the intersection of $\mathcal{L}_{1,3}$ and $\mathcal{L}_{2,7}$ and reads $X_{57} = [d(2R+r)/(2R-r), 0]$, which is obviously indpendent of t. The center $X_{165} = [-\frac{1}{3}d, 0]$ is computed as the centroid of Δ_e.

The Weill point X_{354} (cf. [10, 11]) is the centroid of Δ_i and therefore $X_{354} = [d(3R+r)/(3R), 0]$. The center X_{999} is the midpoint of centers X_1 and X_{57} and thus $X_{999} = [2dR/(2R-d), 0]$. The Schröder point $X_{1155} = [R(R+r)/d, 0]$ (cf. [10]) is the inverse of X_{55} in the circumcircle. The Greenhill point X_{1159} (see also [10]) is the intersection of $\mathcal{L}_{1,3}$ and the line parallel to $\mathcal{L}_{1,5}$ through X_7 and consequently $M_{1159} = [4d(R+r)/(4R+r), 0]$.

The Bevan-Schröder point $X_{1319} = [R(R-r)/d, 0]$ (cf. [10]) is the midpoint of X_1 and X_{36}. The center $X_{1385} = [\frac{1}{2}d, 0]$ is the midpoint of X_1 and X_3. The triangle center $X_{1388} = [d(R-2r)/(R-3r), 0]$ is computed as the intersection of $\mathcal{L}_{1,3}$ and $\mathcal{L}_{8,1317}$, with X_{1317} being the reflection of the Feuerbach point X_{11} in the incenter X_1. The points $X_{1381} = [-R, 0]$ and $X_{1382} = [R, 0]$ are the common points of the circumcircle and the line $\mathcal{L}_{1,3}$.

Now we show that $X_{1420} = [d(2R-r)/(2R-3r), 0]$ which is thus also fixed and contained in $\mathcal{L}_{1,3}$: First we observe that $X_{1420} = \mathcal{L}_{1,3} \cap \mathcal{L}_{84,104}$. Now X_{84} is the reflection of X_{1490} in X_3 and $X_{1490} = \mathcal{L}_{1,4} \cap \mathcal{L}_{3,9}$. The triangle center X_{104} is the circumcircle-antipode of X_{100} and thus it is the reflection of X_{100} in X_3 with $X_{100} = \mathcal{L}_{3,8} \cap \mathcal{L}_{56,145}$, where X_{145} is the reflection of X_8 in X_1.

By the way we obtain $X_{1454} = [d(R+r)^2/(R^2+Rr-r^2), 0]$ which lies on $\mathcal{L}_{4,145}$ and $X_{1482} = [2d, 0]$ is the reflection of the circumcenter in the incenter. We find $X_{1697} = \mathcal{L}_{1,3} \cap \mathcal{L}_{8,9} = [d(2R-r)/(2R+r), 0]$. Since X_{2077} is the inverse of X_{40} in the circumcircle we have $X_{2077} = [-R^2/d, 0]$. Analagously we find $X_{2078} = [R^2(2R-r)/(d(2R+r)), 0]$ which is the inverse of X_{57} in the circumcircle.

The triangle center $X_{2093} = [d(2R+3r)/(2R-r), 0]$ is the reflection of X_1 in X_{57}. The reflection of X_3 in X_{57} yields $X_{2095} = [2d(2R+r)/(2R-r), 0]$. The reflection of X_{56} in the incenter X_1 leads to $X_{2098} = [d(R-2r)/(R-r), 0]$. The point X_{2099} can be obtained as reflection of X_{55} in X_1.

The centers $X_{2446} = [d-r, 0]$ and $X_{2447} = [d+r, 0]$ are each others reflections in X_1. Moreover they are the intersections of the incircle i with the line $\mathcal{L}_{1,3}$. X_{2446} is the center closer to X_3, *cf.* [10].

Further $X_{2646} = \frac{1}{2}(X_1 + X_{35}) = [d(R+r)/(R+2r), 0]$. The center $X_{3057} = [d(R-r)/R, 0]$ is the de Longchamps point of Δ_i. This fact is not mentioned in [10]. There X_{3057} only appears as the intersection of lines $\mathcal{L}_{1,3}$ and $\mathcal{L}_{10,11}$.

The center $X_{3245} = [R(R+4r)/d, 0]$ is found as the reflection of X_{36} in X_{484}. Now we show that $X_{3256} = [dR(2R+3r)/(2R^2+Rr+2r^2), 0]$: First note that $X_{3256} = \mathcal{L}_{1,3} \cap \mathcal{L}_{100,226}$. Where X_{226} is the reflection of X_{993} in X_{1125}. The latter point X_{1125} is the midpoint of X_1 and Δ's Spieker point X_{10}. The first one, X_{993}, is the reflection of X_1 in X_{63}, which is the reflection of X_{1478} in X_{10}. The center of the Johnson-Yff circle X_{1478} (*cf.* [10]) is given by $X_{1478} = \mathcal{L}_{1,4} \cap \mathcal{L}_{2,36}$.

Intersecting $\mathcal{L}_{1,3}$ with $\mathcal{L}_{4,390}$ gives $X_{3295} = [2dR/(2R+r), 0]$, where X_{390} comes as a byproduct in a very early stage of the computation: X_{390} is the reflection of the Gergonne point X_7 in X_1. We observe $X_{3303} = \mathcal{L}_{1,3} \cap \mathcal{L}_{12,497} = [3dR/(3R+r), 0]$, with $X_{12} = \mathcal{L}_{1,5} \cap \mathcal{L}_{2,56}$ and $X_{497} = \mathcal{L}_{1,4} \cap \mathcal{L}_{2,11}$. Similarly we find $X_{3304} = \mathcal{L}_{1,3} \cap \mathcal{L}_{11,153} = [3dR/(3R-r), 0]$ with X_{153} being the reflection of X_{20} in X_{100}.

We find the triangle centers X_{3336}, ..., X_{3340}, X_{3361} as intersections of $\mathcal{L}_{1,3}$ with lines $\mathcal{L}_{7,498}$, $\mathcal{L}_{7,499}$, $\mathcal{L}_{7,90}$, $\mathcal{L}_{7,10}$, $\mathcal{L}_{7,145}$, and $\mathcal{L}_{7,1125}$ and obtain $X_{3336} = [d(3R+2r)/(3R-2r), 0]$, $X_{3337} = [d(5R+2r)/(5R-2r), 0]$, $X_{3338} = [d(3R+r)/(3R-r), 0]$, $X_{3339} = [d(4R+3r)/(4R-r), 0]$, $X_{3340} = [d(2R+3r)/(2R+r), 0]$, and $X_{3361} = [d(4R+r)/(4R-3r), 0]$, respectively. We remark that X_{3338} is also the reflection of X_1 in X_{3304}.

We can easily find the centers $X_{3576} = \frac{1}{2}(X_1 + X_{165}) = [\frac{1}{3}d, 0]$ and $X_{3579} = \frac{1}{2}(X_3 + X_{40}) = [-\frac{1}{2}d, 0]$. The center $X_{3587} = [-d(2R+r)/(4R+r)), 0]$ is the intersection of $\mathcal{L}_{1,3}$ and $\mathcal{L}_{84,550}$, where $X_{550} = \frac{1}{2}(X_3 + X_{20})$. The center $X_{3601} = [d(2R+r)/(2R+3r), 0]$ is also located on $\mathcal{L}_{9,21}$, where the Schiffler point X_{21} can be found as intersection of Δ's Euler line with $\mathcal{L}_{7,56}$. Finally $X_{3612} = [d(R+r)/(R+3r), 0]$ is located on $\mathcal{L}_{21,90}$, where $X_{90} = \mathcal{L}_{1,155} \cap \mathcal{L}_{40,80}$. The center X_{155} is the orthocenter of Δ_t and $X_{80} = \mathcal{L}_{1,5} \cap \mathcal{L}_{2,214}$ with $X_{214} = \frac{1}{2}(X_1 + X_{100})$. X_{80} can also be found as the reflection of X_1 in the Feuerbach point X_{11}. $\square$

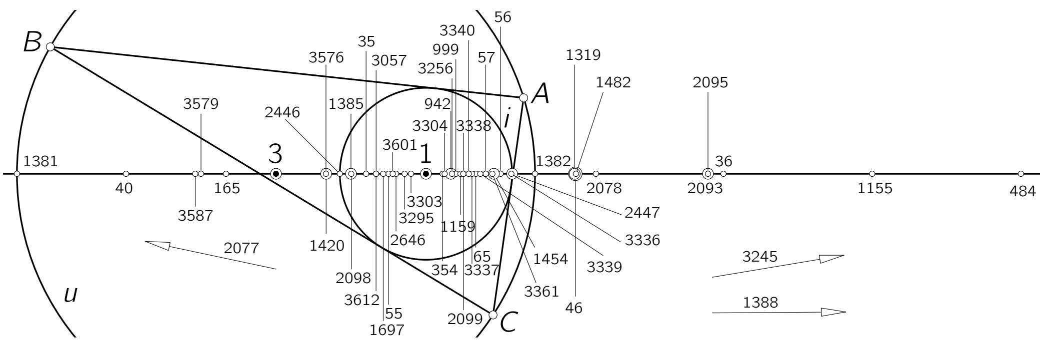

Figure 6: Distribution of fixed centers on $\mathcal{L}_{1,3}$.

Fig. 5 shows some triangle centers on the central line $\mathcal{L}_{1,3}$ which appear as centers of central triangles. Fig. 6 shows the distribution of centers on $\mathcal{L}_{1,3}$ as described in Th. 4.1.

4.2 Centers on the incircle and circumcircle

According to [10] the triangle centers X_i with

$$i \in \{11, 1314, 1315, 1317, 1354, \ldots, 1367, 2446, 2447,\\ 3020, \ldots, 3028, 3317, \ldots, 3328\}$$

are contained in the incircle. Here we can only verify the following result:

Theorem 4.2. *The centers X_{2446} and X_{2447} remain fixed while Δ is running through the poristic family.*

Proof. Actually there is nothing to be done: $X_{2446} = [d - r, 0]$ and $X_{2447} = [d + r, 0]$ are the intersections of the incircle i with the line $\mathcal{L}_{1,3}$, see the proof of Th. 4.1. □

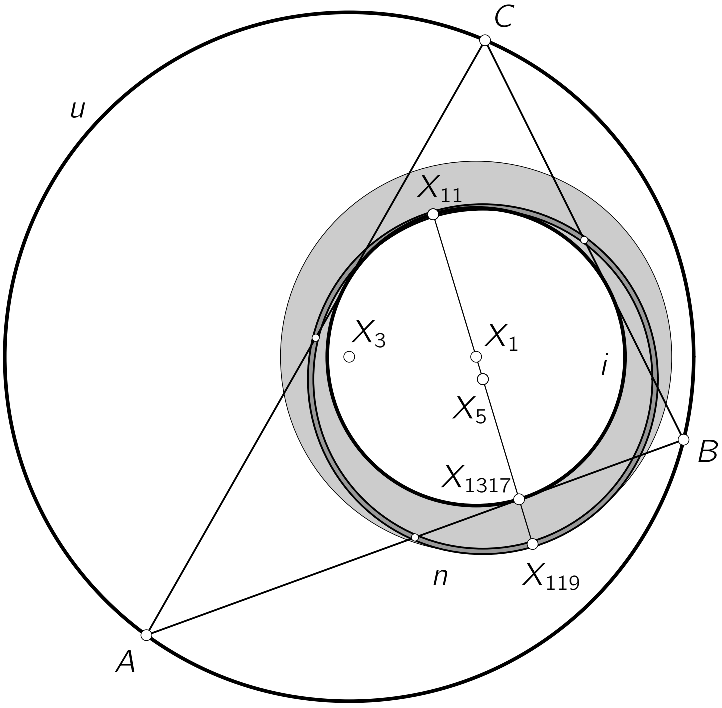

Figure 7: The grey shaded annulus is the locus of all nine-point circles n of triangles in the poristic family.

The point X_{11} known as Feuerbach point is the point of contact of the nine-point circle with the incircle. Thus this point moves on the incircle given in

(2). Since the circumradius R is the same for all triangles in the poristic family the family of corresponding Feuerbach circles consists of congruent circles of radius $R/2$. The nine-point circles of the poristic family are in contact with i and enclose it at any instant. Beside X_{11} the Feuerbach antipode X_{119} is the second point of contact of any nine-point circle n with the outer boundary of their envelope, see Fig. 7. So we can state:

Theorem 4.3. *The nine-point circles of the triangles of a poristic family over coat an annulus bounded by the incircle i and a concentric circle with radius $R - r$.*

From this we can deduce the following result:

Theorem 4.4. *The poristic locus of X_{119} is a circle centered at X_1 with radius $\rho_{119} = R - r$.*

Among the huge amount of known triangle centers X_i only those few with indices

$$i \in \{74, 98, \ldots, 112, 399, 476, 477, 675, 681, 689, 697, 699, 701, 703, 705, 707, 709, 711, 713, 715, 717, 719, 721, 723, 725, 727, 729, 731, 733, 735, 737, 739, 741, 743, 645, 747, 753, 755, 759, 761, 767, 769, 773, 777, 779, 781, 783, 785, 787, 789, 791, 793, 795, 797, 803, 805, 807, 807, 813, 815, 817, 819, 825, 827, 831, 833, 835, 839, 840, 841, 843, 900, 901, 917, 919, 925, 927, 929, \ldots, 935, 953, 972, 1286, \ldots, 1311, 1381, 1382, 1477, 2222, 2249, 2291, 2365, \ldots, 2384, 2687, \ldots, 2770, 2855, \ldots, 2868, 3222, 3563, 3565\}$$

lie on the circumcircle. Here we have:

Theorem 4.5. *Among the triangle centers on the circumcircle u only the points X_{1381} and X_{1382} remain fixed while Δ traverses the poristic family.*

Proof. We refer to the proof of Th. 4.1 where $X_{1381} = [-R, 0]$ and $X_{1382} = [R, 0]$ are mentioned as the intersections of u with $\mathcal{L}_{1,3}$. $\square$

4.3 Centers with circular paths

In the following we describe the orbits of some triangle centers with circular paths. Some of them are points on the circumcircle u, some lie on the incircle i. We show:

Theorem 4.6. *Let Δ be a triangle traversing its poristic family. Then Δ's triangle centers X_i have circular paths for*

$$i \in \{2, 4, 5, 7, 8, 9, 10, 11, 12, 20, 21, 23, 32, 63, 72, 76, 78, 80, 84, 90, 94, 100, 104, 105, 119, 120, 140, 142, \ldots, 145, 149, 153, 186, 191, 200, 210, 214, 226, 323, 329, 347, 355, 376, 381, 382, 388, 390, 392, 399, 442, 495, \ldots, 499, 501, 546, \ldots, 551, 631, 632, 759, 908, 920, 936, 938, 943, 944, 946, 950, 954, 956, 958, 960, 962, 993, 997, 1001, 1004, 1005, 1007, 1125, 1145, 1156, 1158, 1210, 1292, 1317, 1320, 1323, 1324, 1325, 1329, 1376, 1387, 1478, 1479, 1483, 1484, 1490, 1511, 1512, 1519, 1532, 1537, 1538, 1656, 1657, 1698, 1699, 1706, 1737, 1750, 1785, 1837, 1851, 1858, 1898, 1899, 2070, 2071, 2094, 2096, 2478, 2550, 2551, 2886, 2932, 2948, 3036, 3059, 3060, 3085, 3086, 3091, 3110, 3219, 3241, 3243, 3244, 3254, 3305, 3322, 3328, 3358, 3419, 3421, 3434, 3452, 3473, 3474, 3475, 3485, 3486, 3522, 3534, 3543, 3555, 3582, \ldots, 3586, 3589, 3600\}.$$

Each of these centers traces its circular path three times while Δ performs one full turn in the poristic family.

Proof. We demonstrate how to prove the above theorem by means of the trace of X_2: X_2 is the centroid of Δ and therefore a parametrization of the poristic orbit of X_2 is given as the arithmetic mean of the coordinate vectors of A, B, and C from Eqs. (4) and (5), *i.e.*, $X_2(t) = \frac{1}{3}(A+B+C)$. Explicitly we have

$$X_2(t) = \left[\begin{array}{c} \dfrac{d(-4d^2c_t^3R^2 + 4d^2c_t^2R - d(R^2+d^2)c_t + 2R^3)}{3R(R^2+d^2-2dRc_t)} \\ \dfrac{d^2s_t(R^2-d^2+4dRc_t-4R^2c_t^2)}{3R(R^2+d^2-2dRc_t)} \end{array} \right]. \tag{10}$$

This parametrization tells us that X_2 traces its path three times. In order to obtain an equation of it and moreover in order to show that the orbit of X_2 is a circle, we eliminate t by first substituting $c_t = (1-u^2)/(1+u^2)$ and $s_t = 2u/(1+u^2)$. Then we compute the resultant with respect to u of the two polynomials

$$p_x := \mathrm{den}(x_2(u)) - x \cdot \mathrm{num}(x_2(u)), p_y := \mathrm{den}(y_2(u)) - y \cdot \mathrm{num}(y_2(u)),$$

where $x_2(u)$ and $y_2(u)$ are the coordinate functions of $X_2(u)$ and $\mathrm{den}(f/g) = g = \mathrm{num}(g/f)$ give the denominator and numerator of a rational expression. This yields

$$2^{30}d^{12}R^{16}(R^2-d^2)^4(4R^2d^2 - 12xdR^2 + 9y^2R^2 + 9R^2x^2 - d^4)^3$$

and thus

$$c_2 : \; 9R^2(x^2+y^2) - 12dR^2x + d^2(4R^2-d^2) = 0 \tag{11}$$

is an equation of the desired circle. The fact that Eq. (11) appears three times as a factor of the resultant also shows that this circle is traced three times. The latter fact is caused by the so-called improper parametrization of c_2 given in Eq. (11). The circle c_2 is centered at $M_2 = [\frac{2}{3}d, 0]$ and the radius equals $\rho_2 = \frac{1}{3}(R-2r)$. Note that M_2 is a triangle center of Δ (not yet named or labelled, *cf.* [10]) for it is the reflection of $X_3 = [0,0]$ in $X_{3576} = [\frac{1}{3}d, 0]$.

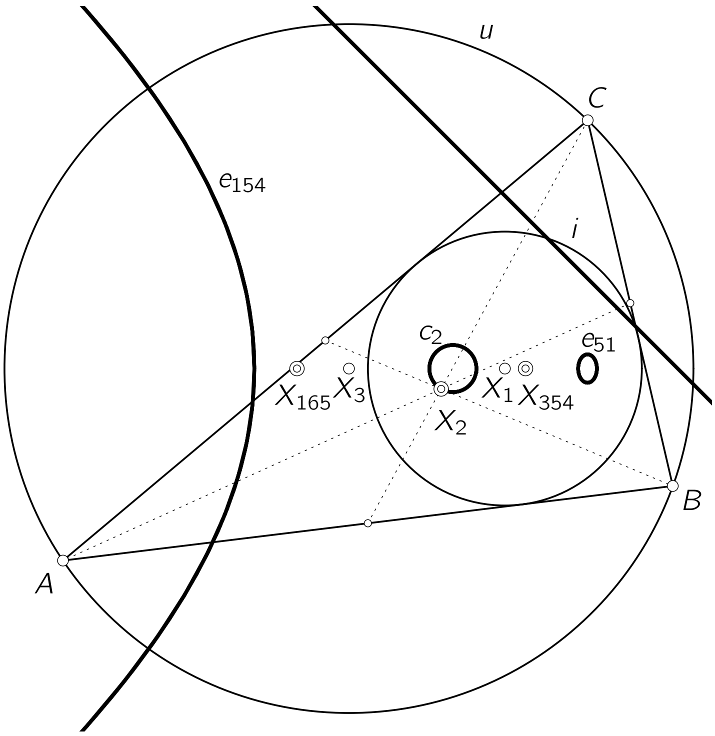

Figure 8: Poristic loci of some centroids: X_2 is the centroid for Δ, Δ_a, and Δ_m at the same time and moves on a circle c_2, *cf.* Th. 4.6. The Weill point X_{354} (centroid of Δ_i) and the centroid X_{165} of Δ_e remain fixed according to Th. 4.1. The centroids X_{51} and X_{154} of Δ_o and Δ_t, respectively, trace conic sections as stated in Th. 4.7.

The method shown so far applies to the orbit of any center listed above. For all other centers we only show how they are related to the vertices of Δ and its deduced triangles Δ_a, Δ_e, Δ_i, Δ_o, Δ_m, Δ_t, and Δ_x in order to find a parametrization of the central orbit.

In the following the poristic path of the center X_i will be denoted by c_i. The center and radius of c_i shall be denoted by M_i and ρ_i.

X_4 is the orthocenter of Δ and thus elementary to find. We have $M_4 = X_{1482}$ and $\rho_4 = R - 2r$. The nine-point center X_5 is the circumcenter of Δ_m and $M_5 = X_1$ and $\rho_5 = \frac{1}{2}\rho_4$. The Gergonne point X_7 moves on c_7 with $M_7 = X_{1159}$ and $\rho_7 = r\rho_4/(4R + r)$. This fits to the results given in [8]. For the trace of the Nagel point we have $M_8 = X_3$ and $\rho_8 = \rho_4$. The Mittenpunkt X_9 leads to $M_9 = [d(2R - r)/(4R + r), 0]$ and $\rho_9 = 2R\rho_4/(4R + r)$. The trace of the Spieker point X_{10} is centered at $M_{10} = X_{1385}$ and has radius $\rho_{10} = \frac{1}{2}\rho_4$. The Feuerbach point is treated earlier, however, it moves on i. Since $X_{12} = \mathcal{L}_{1,5} \cap \mathcal{L}_{2,56}$ we find $M_{12} = X_1$ and $\rho_{12} = r\rho_4/(R + 2r)$. The de Longchamps point X_{20} is the orthocenter of Δ_a and we find $M_{20} = [-2d, 0]$ and $\rho_{20} = \rho_4$.

Since the Schiffler point is given by $X_{21} = \mathcal{L}_{2,3} \cap \mathcal{L}_{7,56}$ we have $M_{21} = [2Rd/(3R + 2r), 0]$ and $\rho_{21} = R\rho_4/(3R + 2r)$. The Far-Out point X_{23} is the inverse of X_2 in the circumcircle and so we find $M_{23} = [6R^3/(d(3R + 2r)), 0]$ and $\rho_{23} = 3R^2/(3R + 2r)$. The 3^{rd} power point X_{32} is the intersection of $\mathcal{L}_{1,4}$ and $\mathcal{L}_{993,1007}$. For the latter two points see below. We find $M_{32} = X_{2099}$ and $\rho_{32} = r\rho_3/(R + r)$.

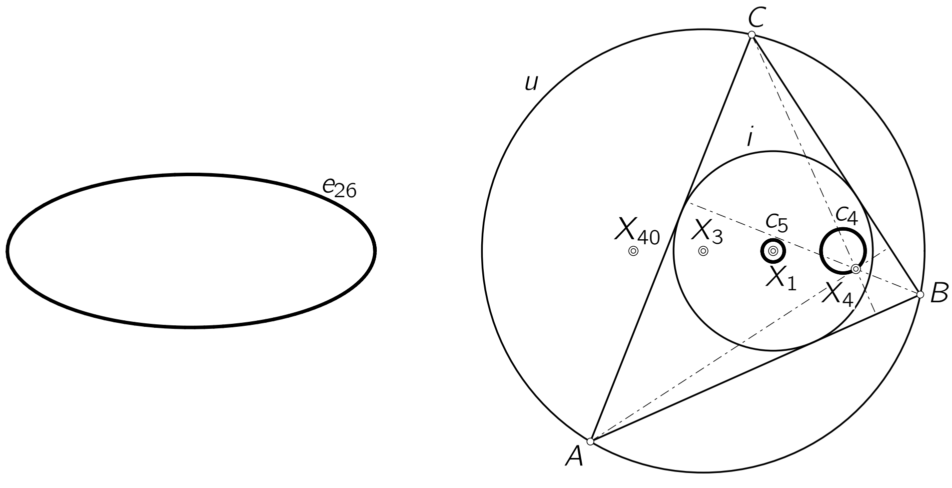

Figure 9: Poristic loci of some circumcenters: Δ's circumcenter X_3 is fixed. X_1 is the circumcenter of Δ_i. The circumcenter of Δ_a is the orthocenter of Δ which is moving on the circle c_4 (*cf.* Th. 4.6). The nine-point center X_5 is the circumcenter of both, Δ_m and Δ_o. The circumcenter X_{26} of Δ_t moves on e_{26} according to Th. 4.7. The Bevan point X_{40} is the circumcenter of Δ_e and is fixed as shown in Th. 3.2.

The center X_{63} is the reflection of X_{1478} in X_{10}. Further $X_{1478} = \mathcal{L}_{1,4} \cap \mathcal{L}_{2,36}$ and therefore $M_{1478} = X_{2099}$ and $\rho_{1478} = \rho_{32}$. Consequently $M_{63} = [-rd/(R+r), 0]$ and $\rho_{63} = R\rho_4/(R+r)$. The point X_{72} is found as the reflection of X_{65} in X_{10} and so $M_{72} = [-rd/R, 0]$ and $\rho_{72} = \rho_4$. The 3^{rd} Brocard point X_{76} is computed as $X_{76} = \mathcal{L}_{3,98} \cap \mathcal{L}_{4,69}$, with X_{98} being the reflection of X_8 in X_{3416} and X_{3416} being the reflection of X_6 in X_{10}. The center X_{69} is the symmedian point of Δ_a and the reflection of X_8 in X_{3416}. Thus we find $M_{76} = X_{1482}$ and $\rho_{76} = \rho_4$.

We note that $X_{78} = \mathcal{L}_{1,2} \cap \mathcal{L}_{210,958}$, where X_{210} is the centroid of Δ_x and $X_{958} = \mathcal{L}_{1,6} \cap \mathcal{L}_{2,12}$. This leads to $M_{78} = [-rd/(R-r)]$, $\rho_{78} = R\rho_4/(R-r)$; $M_{210} = [d(R-r)/)3R), 0]$, $\rho_{210} = \frac{2}{3}\rho_4$; and $M_{958} = [Rd/(2R+r), 0]$, $\rho_{958} = R\rho_4/(2R+r)$. The center X_{80} appears as the reflection of Δ's incenter X_1 in Δ's Feuerbach point X_{11} and therefore $M_{80} = X_1$ and $\rho_{80} = 2r$. We find X_{84} as the reflection of X_{1490} in the circumcenter X_3 with $X_{1490} = \mathcal{L}_{1,4} \cap \mathcal{L}_{3,9}$. So we obtain $M_{84} = [d(2R-r)/r, 0]$, $\rho_{84} = 2R\rho_4/r$ and $M_{1490} = [-d(2R-r)/r, 0]$ $\rho_{1490} = \rho_{84}$.

The trace of $X_{90} = \mathcal{L}_{1,155} \cap \mathcal{L}_{40,80}$ is centered at $M_{90} = [d(R-r)^2/(R^2 - 2Rr - r^2)$ and has radius $\rho_{90} = 2rR\rho_4/(r^2 - 2Rr - R^2)$. For the computation of X_{40}, X_{80}, and X_{155} (the latter being the orthocenter of Δ_t) see the proof of Th. 4.1. Since $X_{94} = \mathcal{L}_{4,143} \cap \mathcal{L}_{23,98}$ we compute $X_{143} = \frac{1}{2}(X_5 + X_{52})$ with X_{52} being the orthocenter of the orthic triangle Δ_o. Thus $M_{143} = [d(R+2r)/R, 0]$ and $\rho_{143} = \rho_4^2/(4R)$. Note that X_{143} is the nine-point center of Δ_o, provided that Δ is acute. We also have $M_{94} = X_{1482}$ and $\rho_{94} = \rho_4$. The Tarry point X_{98} is the reflection of the Steiner point X_{99} in X_3. X_{99} is the common point of u and the Steiner ellipse different from A, B, and C.

For the computation of X_{100} and X_{104} we refer to the proof of Th. 4.1. Then it is easily verified that X_{100}, X_{104} are points on the circumcircle u. Since (X_{105}, X_{1292}) is a pair of antipodal centers on u, their poristic locus equals u. For X_{119} see Th. 4.4. With $X_{120} = \frac{1}{2}(X_4 + X_{1292})$ we find $M_{120} = M_2$ and $\rho_{120} = \frac{1}{3}\rho_4$. Now $X_{140} = \frac{1}{2}(X_3 + X_5)$ and thus $M_{140} = X_{1385}$ and $\rho_{140} = \frac{1}{4}\rho_4$. Note that X_{140} is also the nine-point center of Δ_m.

The Mittenpunkt of Δ_m is denoted by X_{142} and appears as the midpoint of X_7 and X_9 and consequently we have $M_{142} = [3d(2R+r)/(4R+r), 0]$ and $\rho_{142} = (2R+r)\rho_4/(2(4R+r))$. X_{144} comes along as the reflection of X_7 in X_9 and we find $M_{144} = [-6rd/(4R+r), 0]$ and $\rho_{144} = \rho_4(4R-r)/(4R+r)$. The construction of X_{145} is already mentioned in the proof of Th. 4.1. We find $M_{145} = X_{1482}$ and $\rho_{145} = \rho_4$. The center X_{149} appears as the reflection of X_{20} in X_{104} and we observe $M_{149} = X_{1482}$ and $\rho_{149} = R + 2r$. X_{153} is the reflection of X_{20} in X_{100} and we find $M_{153} = X_{1482}$ and $\rho_{153} = 3R - 2r$.

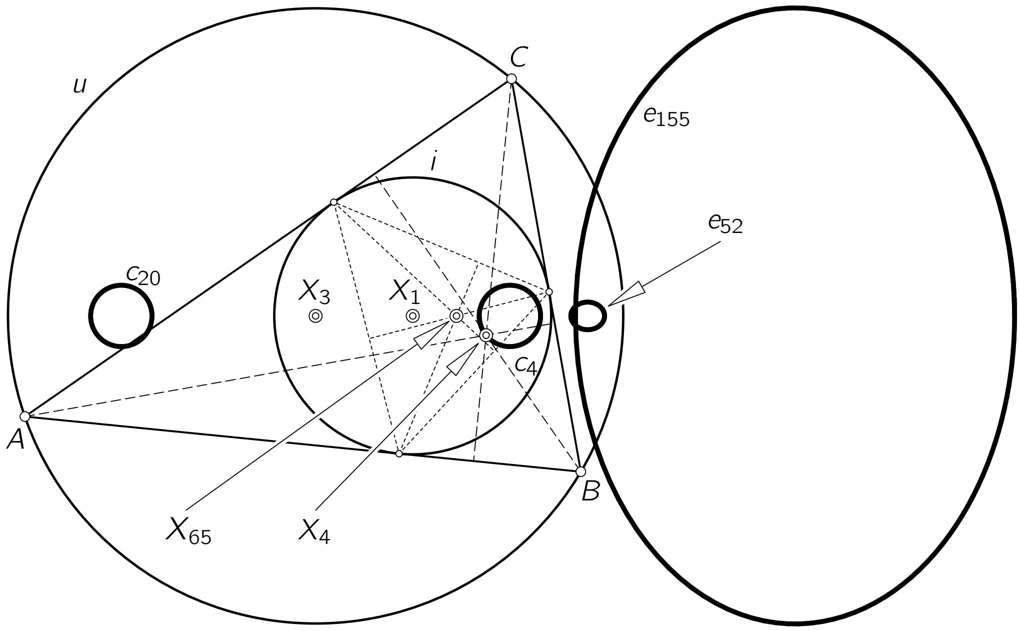

Figure 10: Poristic loci of some orthocenters: Δ's orthocenter X_4 circles along c_4. The de Longchamps point X_{20} runs on the circle c_{20} (*cf.* Th. 4.6). X_1, X_3, and X_{65} are the orthocenter Δ_e, Δ_m, and Δ_i. According to Th. 3.2 X_{65} remains fixed. The orthocenters X_{52} and X_{155} of Δ_o and Δ_t travel along conic sections e_{52} and e_{155}, respectively, *cf.* Th. 4.7.

The center X_{186} is the inverse of X_4 in the circumcircle and so it is no surprise that its poristic path is a circle. It is centered at $M_{186} = [2R^3/(d(3R+2r)), 0]$ and has radius $\rho_{186} = \frac{1}{3}\rho_{23}$. We reflect the incenter X_1 in the Schiffler point X_{21} and arrive at X_{191}. This results in $M_{191} = [d(R - 2r)/(3R + 2r), 0]$ and $\rho_{191} = 2\rho_{21}$. The center X_{200} is the intersection of $\mathcal{L}_{1,2}$ with $\mathcal{L}_{40,64}$, where X_{64} is the reflection of X_{1498} in X_3 and $X_{1498} = \mathcal{L}_{1,84} \cap \mathcal{L}_{4,6}$. X_{200} traces a circle centered at $M_{200} = [-rd/(2R - r), 0]$ and with radius $\rho_{200} = 2R\rho_4/(2R - r)$. Since $X_{214} = \frac{1}{2}(X_1 + X_{100})$ we have $M_{214} = X_{1385}$ and $\rho_{214} = \frac{1}{2}R$.

X_{226} is the reflection of X_{993} in X_{1125}. For the construction of the latter two we refer to the proof of Th. 4.1. So we obtain the data of three poristic traces: $M_{226} = [d(2R + 3r)/(2(R + r)), 0]$, $\rho_{226} = \frac{1}{2}\rho_{32}$; $M_{993} = [dR/(2(R + r)), 0]$, $\rho_{993} = \frac{1}{2}\rho_{63}$; and $M_{1125} = [\frac{3}{4}d, 0]$, $\rho_{1125} = \frac{1}{4}\rho_4$. Reflecting X_{23} in X_{110} gives X_{323} moving on a circle with center $M_{323} = [-6R^3/(d(3R + 2r)), 0]$ and

$\rho_{323} = R(9R+4r)/(3R+2r)$. Reflecting X_{2093} in the Spieker center X_{10} we obtain X_{329} and then $M_{329} = [-4rd/(2R-r), 0]$ and $\rho_{329} = \rho_4$. With $X_{347} = \frac{1}{2}(X_2 + X_5)$ we find $M_{347} = [\frac{5}{6}d, 0]$ and $\rho_{347} = \frac{5}{12}\rho_4$.

For the Fuhrmann center $X_{355} = \frac{1}{2}(X_4 + X_8)$ we find $M_{355} = X_1$ and $\rho_{355} = \rho_4$. Since $X_{376} = \frac{1}{2}(X_2 + X_{20})$ and $X_{381} = \frac{1}{2}(X_2 + X_4)$ we find $M_{376} = [-\frac{2}{3}d, 0]$, $\rho_{376} = \frac{1}{3}\rho_4$ and $M_{381} = [\frac{4}{3}d, 0]$, $\rho_{381} = \frac{2}{3}\rho_4$. The reflection of the circumcenter in the orthocenter yields X_{382} with $M_{382} = [4d, 0]$ and $\rho_{382} = 2\rho_4$.

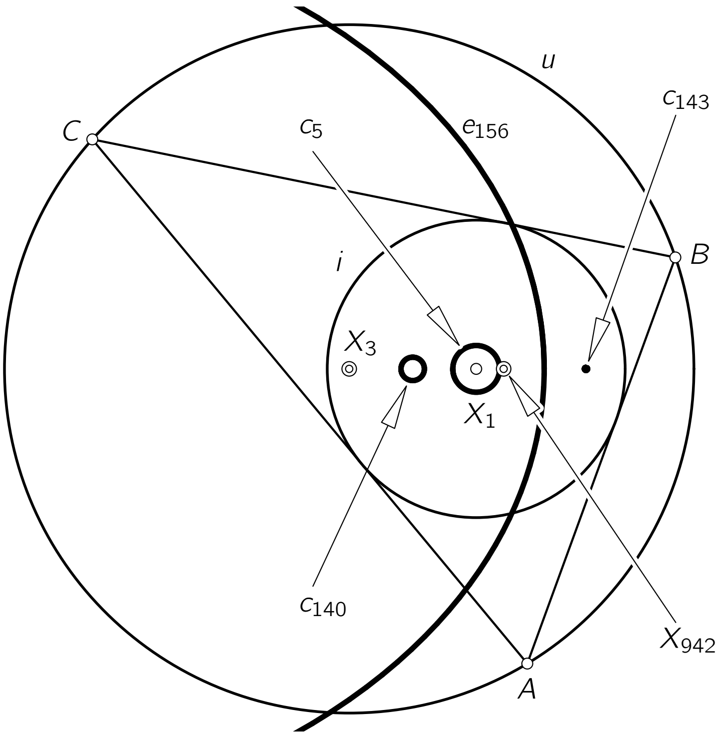

Figure 11: Poristic loci of some nine-point centers: X_5 moves on c_5. The centers X_{140} and X_{143} are the nine-point centers of Δ_m and Δ_o, respectively. According to Th. 4.6 their poristic loci are the circles c_{140} and c_{143}. The nine-point center of Δ_t is the point X_{156}. Its poristic orbit is the conic section e_{156}, *cf.* Th. 4.7. Δ's circumcenter X_3 plays a double-role: It is the nine-point center of Δ_a and Δ_e. The nine-point center of Δ_i is the same point for all triangles in the poristic system, *i.e.*, X_{942} is fixed, see Th. 4.1.

The center $X_{388} = \mathcal{L}_{1,4} \cap \mathcal{L}_{7,8}$ runs on a circle with center $M_{388} = [2d(R+r)/(2R+r), 0]$ and radius $\rho_{388} = r\rho_4/(2R+r)$. We reflect the Gergonne point X_7 in the incenter X_1 in order to obtain X_{390}. So we have $M_{390} =$

$[2d(2R-r)/(4R+r)]$ and $\rho_{390} = \rho_7$. With $X_{392} = \mathcal{L}_{1,6} \cap \mathcal{L}_{9,11}$ we arrive at $M_{392} = M_9$ and $\rho_{392} = \rho_9$. The Parry reflection point X_{399} is the reflection of X_3 in X_{110} therefore we have $M_{399} = X_3$ and $\rho_{399} = 2R$. The complement of the Schiffler point is $X_{442} = \mathcal{L}_{2,3} \cap \mathcal{L}_{11,214}$ and its trace is centered at $M_{442} = [2d(R+r)/(3R+2r), 0]$ and has radius $\rho_{442} = (R+r)\rho_4/(3R+2r)$.

The Johnson midpoint is computed as $X_{495} = \mathcal{L}_{1,5} \cap \mathcal{L}_{4,390}$ and we derive $M_{495} = X_1$ and $\rho_{495} = \frac{1}{2}\rho_{32}$. For $X_{496} = \mathcal{L}_{1,5} \cap \mathcal{L}_{36,550}$ we determine $X_{550} = \frac{1}{2}(X_2 + X_{20})$. This intermediate result yields $M_{550} = X_{40}$, $\rho_{550} = \frac{1}{2}\rho_4$ and $M_{496} = X_1$, $\rho_{496} = r\rho_4/(2\rho_{119})$. With $X_{497} = \mathcal{L}_{1,4} \cap \mathcal{L}_{2,11}$, $X_{498} = \mathcal{L}_{1,2} \cap \mathcal{L}_{3,12}$, and $X_{499} = \mathcal{L}_{1,2} \cap \mathcal{L}_{3,11}$ we find $M_{497} = [2d(R-r)/(2R-r), 0]$, $\rho_{497} = r\rho_4/(2R-r)$; $M_{498} = [d(R+2r)/(R+3r), 0]$, $\rho_{498} = r\rho_4/(R+3r)$; and $M_{499} = X_{1388}$, $\rho_{499} = r\rho_4/(3r-R)$.

We compute $X_{501} = \mathcal{L}_{21,214} \cap \mathcal{L}_{36,58}$ with $X_{58} = \mathcal{L}_{1,21} \cap \mathcal{L}_{3,6}$ which leads to $M_{501} = M_{21}$ and $\rho_{501} = \rho_{21}$. The next five centers are midpoints of centers: $X_{546} = \frac{1}{2}(X_4 + X_5)$, $X_{547} = \frac{1}{2}(X_2 + X_5)$, $X_{548} = \frac{1}{2}(X_5 + X_{20})$, $X_{549} = \frac{1}{2}(X_2 + X_3)$, and $X_{551} = \frac{1}{2}(X_1 + X_2)$. So we find $M_{546} = [\frac{3}{2}d, 0]$, $\rho_{546} = \frac{3}{4}\rho_4$; $M_{547} = M_{347}$, $\rho_{547} = \frac{5}{12}\rho_4$; $M_{548} = X_{3579}$, $\rho_{548} = \frac{1}{4}\rho_4$; and $M_{549} = X_{3576}$, $M_{551} = M_{347}$, $\rho_{549} = \rho_{551} = \frac{1}{6}\rho_4$. X_{631} is the reflection of X_4 in X_{3091}. Therefore we have to determine $X_{3091} = \mathcal{L}_{2,3} \cap \mathcal{L}_{11,153}$. This gives $M_{631} = [\frac{2}{5}d, 0]$, $\rho_{631} = \frac{1}{5}\rho_4$ and $M_{3091} = [\frac{6}{5}d, 0]$, $\rho_{3091} = \frac{3}{5}\rho_4$. Then X_{632} appears as the reflection of X_{3091} in the circumcenter X_3 and we find $M_{632} = [\frac{3}{5}d, 0]$ and $\rho_{632} = \frac{3}{10}\rho_4$. With $X_{759} = \mathcal{L}_{10,21} \cap \mathcal{L}_{58,65}$ we can verify that X_{759} travels on u.

The point Acubens X_{908} is the intersection of $\mathcal{L}_{2,7}$ and $\mathcal{L}_{12,960}$. So we compute $X_{960} = \frac{1}{2}(X_1 + X_{72})$. Since X_{908} is the reflection of X_{1512} in X_{119}, we obtain X_{1512} as reflection of X_{908} in X_{119}. Thus we have $M_{960} = [d(R-r)/(2R), 0]$, $\rho_{960} = \frac{1}{2}\rho_4$; $M_{908} = [-3rR/d, 0]$, $\rho_{908} = \rho_{119}$; and $M_{1512} = [R(2R-r)/d, 0]$, $\rho_{1512} = \rho_{119}$. We find $X_{920} = \mathcal{L}_{1,21} \cap \mathcal{L}_{4,46}$ and therefore we have $M_{920} = [d(R^2 + r^2)/(R^2 - Rr - r^2), 0]$ and $\rho_{920} = rR\rho_4/(R^2 - Rr - r^2)$. If we intersect $\mathcal{L}_{1,2}$ with the lines $\mathcal{L}_{3,9}$ and $\mathcal{L}_{4,7}$ we find X_{936} and X_{938}, respectively. The centers and radii of their paths are $M_{936} = [d(2R-r)/(4R-r), 0]$, $\rho_{936} = 2R\rho_4/(4R-r)$ and $M_{938} = [4dR/(4R-r), 0]$, $\rho_{938} = r\rho_4/(4R-r)$. For $X_{943} = \mathcal{L}_{3,7} \cap \mathcal{L}_{4,12}$ we find $M_{943} = [4dR(R-r)/(4R^2 + 7Rr + 2r^2), 0]$ and $\rho_{943} = rR\rho_4/(4R^2 + 7Rr + 2r^2)$.

The Hofstadter-Trapezoid point X_{944} is the midpoint in between X_{20} and X_{145}. Therefore we have $M_{944} = X_3$ and $\rho_{944} = \rho_4$. The center $X_{946} = \frac{1}{2}(X_1 + X_4)$ traces a circle with center $M_{946} = M_{546}$ and radius $\rho_{946} = \frac{1}{2}\rho_4$. As intercept

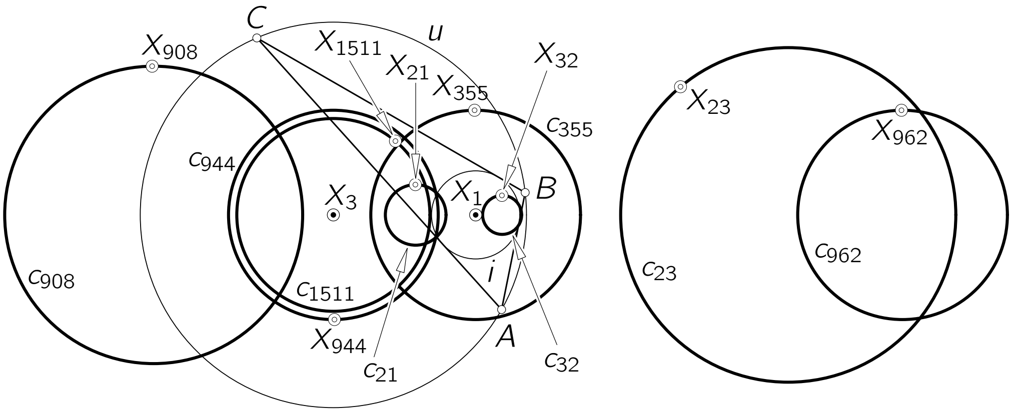

Figure 12: Poristic loci of the Schiffler point X_{21}, the Far-Out point X_{23}, the 3^{rd} power point X_{32}, the Fuhrmann center X_{355}, the point Acubens X_{908}, the Hofstadter-Trapezoid point X_{944}, the Longuet-Higgins point X_{962}, and the Fermat crosssum X_{1511}.

of $\mathcal{L}_{1,4}$ and $\mathcal{L}_{8,9}$ we obtain the point X_{950} and $M_{950} = [d(2R-r)/2R, 0]$ and $\rho_{950} = r\rho_4/(2R)$. The central line $\mathcal{L}_{1,6}$ carries the three centers X_{954} and X_{956}, which also lie in the central lines $\mathcal{L}_{3,7}$ and $\mathcal{L}_{3,8}$, respectively. We find $M_{954} = [4dR/(4R+r), 0]$, $\rho_{954} = Rr\rho_4/(R+r)(4R+r)$ and $M_{956} = X_3$, $\rho_{956} = \rho_{63}$. The Longuet-Higgins point is the reflection of the Nagel point X_8 in the orthocenter X_4. This yields $M_{962} = M_{382}$ and $\rho_{962} = \rho_4$. The midpoint X_{997} of X_1 and X_{200} determines $M_{997} = [d(R-r)/(2R-r), 0]$ and $\rho_{997} = \frac{1}{2}\rho_{200}$.

Since $X_{1001} = \frac{1}{2}(X_1+X_9)$ we have $M_{1001} = [3Rd/(4R+r), 0]$ and $\rho_{1001} = \frac{1}{2}\rho_9$. For $X_{1004} = \mathcal{L}_{2,3} \cap \mathcal{L}_{7,100}$ we compute $M_{1004} = [2Rd(R+r)/(3R^2-Rr-r^2), 0]$ and $\rho_{1004} = R(R+r)\rho_4/(3R^2-Rr-r^2)$. The centers X_{1005} and X_{1007} are located on the Euler line and on the central lines $\mathcal{L}_{9,100}$ and $\mathcal{L}_{4,99}$, respectively. We derive $M_{1005} = [2Rd(2R-r)/(6R^2+5Rr+2r^2), 0]$, $\rho_{1005} = R(2R-r)\rho_4/(6R^2+5Rr+2r^2)$ and $M_{1007} = M_2$, $\rho_{1007} = \frac{1}{3}\rho_4$. The 3^{rd} Ehrmann point $X_{1145} = \frac{1}{2}(X_8+X_{100})$ leads to $M_{1145} = X_3$ and $\rho_{1145} = \rho_{119}$. The center X_{1156} is found as the midpoint of X_9 and X_{100}. Its circular path is centered at $M_{1156} = M_{390}$ and has radius $\rho_{1156} = 9rR/(4R+r)$. The circumcenter of the extouch triangle Δ_x is given by $X_{1158} = \frac{1}{2}(X_{40}+X_{84})$ and its poristic locus is centered at $M_{1158} = [d(R-r)/r, 0]$ and has radius $\rho_{1158} = \frac{1}{2}\rho_{84}$. The center $X_{1210} = \mathcal{L}_{1,2} \cap \mathcal{L}_{3,950}$ yields $M_{1210} = [d(2R-r)/(2R), 0]$ and $\rho_{1210} = \rho_{946}$.

Since X_{1317} is the reflection of X_{11} and X_1 it is easy to find a parametrization of its path which is the incircle. The path of the midpoint $X_{1320} = \frac{1}{2}(X_{145} +$

X_{149}) is centered at $M_{1320} = X_{1482}$ and is congruent to u for $\rho_{1320} = R$. The Fletcher point X_{1323} is the inverse of the Gergonne point X_7 in the incircle and its trace is centered at $M_{1323} = [R(2R - r)/(2d), 0]$ and has radius $\rho_{1323} = \frac{1}{2}r$. The inverse X_{1324} of the Spieker point X_{10} in the incircle moves on a circle centered at $M_{1324} = [R^3/(rd), 0]$ with radius $\rho_{1324} = R^2/r$. The inverse X_{1325} of the Schiffler point X_{21} in the incircle has an orbit centered at $M_{1325} = [2R^2/d, 0]$ with $\rho_{1325} = R$ for its radius. The center $X_{1329} = \frac{1}{2}(X_8 + X_{2098})$, where X_{2098} is known from Th. 4.1 and its proof, respectively, gives $M_{1329} = [d(R - 2r)/(2(R - r), 0]$ and $\rho_{1329} = \frac{1}{2}\rho_4$. The exsimilicenter of the circumcircle and the Spieker circle is given by $X_{1376} = \mathcal{L}_{3,10} \cap \mathcal{L}_{8,56}$. Its poristic locus is centered at $M_{1376} = [Rd/(2R - r), 0]$ and has radius $\rho_{1376} = \frac{1}{2}\rho_{200}$. $X_{1387} = \frac{1}{2}(X_1 + X_{11})$ has a circular path centered at $M_{1387} = X_1$ and $\rho_{1387} = \frac{1}{2}r$. From $X_{1479} = \mathcal{L}_{1,4} \cap \mathcal{L}_{3,11}$ we derive $M_{1479} = X_{2098}$ and $\rho_{1479} = 2\rho_{946}$.

The centers X_{1483} and X_{1484} appear as reflections of X_5 in X_1 and X_5 in X_{11}, respectively. We have $M_{1483} = M_{1484} = X_1$, $\rho_{1483} = \frac{1}{2}\rho_4$, $\rho_{1484} = \frac{1}{2}\rho_{149}$. The Fermat crosssum $X_{1511} = \frac{1}{2}(X_3 + X_{110})$ runs on a circle concentric with u and thus $M_{1511} = X_3$ and $\rho_{1511} = \frac{1}{2}R$. The construction of X_{1519} produces a lot of useful byproducts since X_{1519} is the reflection of X_{1532} in X_{1538}, where X_{1538} is the reflection of X_{1512} in X_{1537}. Since $X_{1537} = \mathcal{L}_{4,145} \cap \mathcal{L}_{11,65}$ and $X_{1532} = \mathcal{L}_{2,3} \cap \mathcal{L}_{12,946}$ we find: $M_{1519} = [R(2R - 3r)/d, 0]$, $M_{1532} = [2R(R - r)/d, 0]$, $M_{1537} = X_{1482}$, $M_{1538} = [R(4R - 5r)/(2d), 0]$ and $\rho_{1519} = \rho_{1532} = \rho_{1537} = \rho_{1538} = \rho_{119}$. The center X_{1656} is the intersection of the Euler line with $\mathcal{L}_{17,18}$. Without explicitly knowing the latter two points we find X_{1656} as the reflection of X_5 in X_{632} and this gives $M_{1656} = [\frac{4}{5}d, 0]$ and $\rho_{1656} = \frac{2}{5}\rho_4$. Reflecting the de Longchamps point X_{20} in the circumcenter X_3 we find X_{1657}. Its trace has center $M_{1657} = [-4d, 0]$ and radius $\rho_{1657} = 2\rho_4$.

The poristic locus of the center $X_{1698} = \mathcal{L}_{1,2} \cap \mathcal{L}_{5,40}$ is the circle with center $M_{1698} = M_{632}$ and has radius $\rho_{1698} = \frac{2}{5}\rho_4$. Since X_{1699} shows up as the reflection of X_{165} in the centroid X_2 we find $M_{1699} = [\frac{5}{3}d, 0]$ and $\rho_{1699} = \frac{1}{3}\rho_4$. The exsimilicenter of the Bevan circle and the Spieker circle is the triangle center X_{1706} which is the reflection of X_{2551} in the Spieker point X_{10}. So we compute $X_{2551} = \mathcal{L}_{4,9} \cap \mathcal{L}_{2,12}$ and find $M_{2551} = [2d(R - r)/(4R - r), 0]$, $\rho_{2551} = \rho_4(2R - r)/(4R - r)$ and $M_{1706} = [d(2R + r)/(4R - r), 0]$, $\rho_{1706} = \rho_{936}$.

The midpoint of X_{36} and X_{80} is labelled X_{1737} and rotates about $M_{1737} = X_{1319}$ at distance $\rho_{1737} = r$.The reflection of X_1 in X_{497} equals the point X_{3586}. This enables us to construct X_{1750} as the reflection of X_{3586} in the orthocenter

X_4. From that we obtain $M_{1750} = [d(6R - r)/(2R - r), 0]$, $\rho_{1750} = 2\rho_{200}$ and $M_{3586} = [d(2R - 3r)/(2R - r), 0]$, $\rho_{3586} = 2\rho_{497}$. The point X_{1785} is the inverse of X_{946} in the incircle. It is circling around $M_{1785} = X_{1319}$ with $\rho_{1785} = r$. For the center $X_{1837} = \mathcal{L}_{1,5} \cap \mathcal{L}_{4,65}$ we find $M_{1837} = X_1$ and $\rho_{1837} = 2\rho_{946}$. The center $X_{1851} = \mathcal{L}_{4,8} \cap \mathcal{L}_{25,105}$, where $X_{25} = \mathcal{L}_{2,3} \cap \mathcal{L}_{6,51}$ and with X_{51} being the centroid of Δ_o and $M_{1851} = X_{1482}$, $\rho_{1851} = \rho_4$. With $X_{1858} = \mathcal{L}_{1,90} \cap \mathcal{L}_{4,65}$ we get $M_{1858} = [d(R^2 + r^2)/R^2, 0)]$ and $\rho_{1858} = 2\rho_{950}$. Reflecting X_{65} in X_{1837} gives X_{1898} and thus $M_{1898} = X_{3057}$ and $\rho_{1898} = 4\rho_{496}$. The point $X_{1899} = \mathcal{L}_{1,98} \cap \mathcal{L}_{4,51}$ is rotating about $M_{1899} = X_{1482}$ at distance $\rho_{1899} = \rho_4$.

The inverse of X_5 and X_{20} in the circumcircle yield X_{2070} and X_{2071} which are rotating about $M_{2070} = [4R^3/(d(3R + 2r)), 0]$ at distance $\rho_{2070} = \frac{2}{3}\rho_{23}$ and $M_{2071} = [-2R^3/(d(3R + 2r)), 0]$ at distance $\rho_{2071} = \frac{1}{3}\rho_{23}$. The reflection of X_2 and X_4 in X_{57} yields X_{2094} and X_{2096}, respectively. From that we conclude that $M_{2094} = [8d(R + r)/(3(2R - r)), 0]$, $\rho_{2094} = \frac{1}{3}\rho_4$ and $M_{2096} = [4rd/(2R - r), 0]$, $\rho_{2096} = \rho_4$. With $X_{2478} = \mathcal{L}_{2,3} \cap \mathcal{L}_{8,210}$ we find $M_{2478} = [2d(R - r)/(3R - r), 0]$ and $\rho_{2478} = \rho_4\rho_{119}/(3R - r)$. The midpoint X_{2550} of the Gergonne and Nagel point determines $M_{2550} = [2d(R + r)/(4R + r), 0]$ and $\rho_{2550} = 2\rho_{142}$.

We find $X_{2886} = \frac{1}{2}(X_1 + X_{3419})$ with X_{3419} as the reflection of X_{55} in X_{10}. This leads to $M_{2886} = [d(R + 2r)/(2(R + r)), 0]$, $\rho_{2886} = \frac{1}{2}\rho_4$ and $M_{3419} = [rd/(R + r), 0]$, $\rho_{3419} = \rho_4$, respectively. The point X_{2932} is the inverse of X_{1145} in the circumcircle. It is rotating about $M_{2932} = X_3$ at distance $\rho_{2932} = R^2/\rho_{119}$. The center X_{2948} comes up as the reflection of X_{3448} in the Spieker center X_{10}. For that we determine X_{3448} as the reflection of X_{20} in X_{74} with the latter being $X_{74} = \mathcal{L}_{20,68} \cap \mathcal{L}_{72,100}$, where X_{68} is the reflection of X_5 in X_{155}. We find $M_{2948} = X_{40}$ and $\rho_{2948} = 2R$.

With $X_{3036} = \frac{1}{2}(X_8 + X_{11})$ we find $M_{3036} = X_{1385}$ and $\rho_{3036} = \frac{1}{2}(3r - R)$. Then $X_{3059} = \mathcal{L}_{7,8} \cap \mathcal{L}_{9,55}$ and we get $M_{3059} = [-dr(R + r)/(R(4R + r)), 0]$ and $\rho_{3059} = 4\rho_{142}$. For $X_{3060} = \mathcal{L}_{2,51} \cap \mathcal{L}_{4,52}$ we find the center and radius of its circular path: $M_{3060} = [4d(R + 2r)/(3R), 0)]$ and $\rho_{3060} = \frac{4}{3}\rho_{143}$. We intersect the line $\mathcal{L}_{1,2}$ with $\mathcal{L}_{4,12}$ and $\mathcal{L}_{4,11}$ and get X_{3085} and X_{3086}, respectively. The centers and radii of the respective poristic loci are: $M_{3085} = [2d(R + r)/(2R + 3r), 0]$, $\rho_{3085} = r\rho_4/(2R + 3r)$ and $M_{3086} = [r(R - 2r)/(2R - 3r), 0]$, $\rho_{3086} = r\rho_4/(2R - 3r)$. The center X_{3110} is the inverse of X_{3286} in the circumcircle and $X_{3286} = \mathcal{L}_{3,6} \cap \mathcal{L}_{7,21}$. So we have $M_{3110} = X_{1385}$ and $\rho_{3110} = \frac{1}{2}d$. We compute $X_{3219} = \mathcal{L}_{2,7} \cap \mathcal{L}_{8,90}$ and find $M_{3219} = [2d(R - r)/(5R + 2r), 0]$ and

$\rho_{3219} = 3R\rho_4/(5R+2r)$. The center $X_{3241} = \frac{1}{2}(X_2+X_{145})$ moves on a circle centered at $M_{3241} = M_{381}$ with radius $\rho_{3241} = \frac{1}{3}\rho_4$.

We reflect X_8 in X_{142} and arrive at X_{3243} with $M_{3243} = [3d(2R+r)/(4R+r), 0]$ and $\rho_{3243} = \rho_9$. The reflection of the Spieker center X_{10} in the incenter X_1 is named X_{3244} and circles around $M_{3244} = M_{546}$ with $\rho_{3244} = \frac{1}{2}\rho_4$. Now X_{3254} is the reflection of the Mittenpunkt X_9 in the Feuerbach point X_{11} and we get $M_{3254} = M_{3243}$ and $\rho_{3254} = 2(R+r)^2/(4R+r)$. The point $X_{3305} = \mathcal{L}_{2,7} \cap \mathcal{L}_{210,1001}$ traces a circle with center $M_{3305} = [d(4R-r)/(7R+r), 0]$ and radius $\rho_{3305} = 3R\rho_4/(7R+r)$. The reflection of X_{3328} in X_1 yields X_{3322}, where X_{3328} is computed as the reflection of X_{1155} in X_{1323}. Note that X_{1155} is the reflection of X_1 in X_{3245}. Now it is easily verified that X_{3322} and X_{3328} run on the incircle. The center $X_{3358} = \frac{1}{2}(X_9+X_{84})$ determines $M_{3358} = [d(4R^2-r^2)/(r(4R+r)), 0]$ and $\rho_{3358} = 2R\rho_4(2R+r)/(r(4R+r))$.

The reflection of X_8 in X_{3419} yields X_{3434} with circular orbit centered at $M_{3434} = [2rd/(R+r), 0]$ and radius $\rho_{3434} = \rho_4$. We construct X_{3452} as the intersection of the central lines $\mathcal{L}_{2,7}$ and $\mathcal{L}_{5,10}$ and find the center of the circular orbit $M_{3452} = [d(2R-3r)/(2(2R-r)), 0]$ and the radius $\rho_{3542} = \frac{1}{2}\rho_4$. This allows to compute X_{3421} as the reflection of X_1 in X_{3452} and we find $M_{3421} = [-2rd/(2R-r), 0]$ and $\rho_{3421} = \rho_4$. From $X_{3474} = \mathcal{L}_{4,46} \cap \mathcal{L}_{7,55}$ we get $M_{3474} = [2d(R+2r)/(2R-r), 0]$ and $\rho_{3474} = \rho_{497}$. On the central line $\mathcal{L}_{1,4}$ we find the next four centers: We intersect with $\mathcal{L}_{8,56}$, $\mathcal{L}_{7,55}$, $\mathcal{L}_{7,21}$, and $\mathcal{L}_{8,21}$ and obtain X_{3473}, X_{3475}, X_{3485}, and X_{3486}, respectively. Their poristic orbits are centered at $M_{3473} = X_{999}$, $M_{3475} = [2d(3R+2r)/(3(2R+r)), 0]$, $M_{3485} = [2d(R+2r)/(2R+3r), 0]$, and $M_{3486} = [2Rd/(2R+r), 0]$ and have radii $\rho_{3472} = \rho_{497}$, $\rho_{3475} = \frac{1}{3}\rho_{388}$, $\rho_{3485} = \rho_{3085}$, and $\rho_{3486} = \rho_{388}$.

The reflection of X_{361} in the circumcenter X_3 leads to X_{3522} with $M_{3522} = [-\frac{2}{5}d, 0]$ and $\rho_{3522} = \frac{1}{5}\rho_4$. The center X_{3534} is the reflection of X_{382} in X_{381} and rotates about $M_{3534} = [-\frac{4}{3}d, 0]$ with $\rho_{3534} = \frac{2}{3}\rho_4$. Reflecting X_{3534} in X_5 we find X_{3543} and $M_{3543} = [\frac{10}{3}d, 0]$ and $\rho_{3543} = \frac{5}{3}\rho_4$. The Dosa point X_{3555} is the reflection of X_{72} in the incenter X_1 and circles about $M_{3555} = [d(2R+r)/R, 0]$ at distance $\rho_{3555} = \rho_4$. On the central line parallel to the Euler line through the Feuerbach point X_{11} we find X_{3582} and X_{3583} by intersecting with $\mathcal{L}_{1,2}$ and $\mathcal{L}_{1,4}$, respectively. This yields circular orbits with centers $M_{3582} = [R(3R-4r)/(3d), 0]$ and $M_{3583} = [R(R-4r)/d, 0]$ and radii $\rho_{3582} = \frac{2}{3}r$ and $\rho_{3583} = 2r$, respectively. Since $X_{3584} = \mathcal{L}_{1,2} \cap \mathcal{L}_{11,547}$ we find $M_{3584} = [d(3R+4r)/(3(R+2r)), 0]$ and $\rho_{3584} = \frac{2}{3}\rho_{12}$. On the central line $\mathcal{L}_{1,4}$ we find X_{3585} and X_{3586} as intersections with $\mathcal{L}_{5,36}$ and $\mathcal{L}_{30,57}$, respectively.

Their poristic paths are centered at $M_{3585} = [d(R+4r)/(R+2r), 0]$ and $M_{3586} = [d(2R-3r)/(2R-r), 0]$. The respective radii are $\rho_{3585} = 2\rho_{12}$ and $\rho_{3586} = 2\rho_{497}$. For $X_{3589} = \mathcal{L}_{4,5} \cap \mathcal{L}_{8,10}$ we find $M_{3589} = M_2$ and $\rho_{3589} = \frac{1}{3}\rho_4$. Finally the center $X_{3600} = \mathcal{L}_{1,7} \cap \mathcal{L}_{8,57}$ circles around $M_{3600} = [2d(2R + r)/(4R-r), 0]$ at distance $\rho_{3600} = \rho_{938}$. □

4.4 Centers moving on conic sections

In this last section we focus on triangle centers that run on conic sections while Δ is moving through its poristic family. We shall give the semiaxes and center of the poristic paths only for some prominent centers and in the cases where these (centers and axes) are relatively simple functions in R, r, and d. We shall skip the lengthy discussion under which circumstances the poristic loci of triangle centers mentioned here are ellipses or hyperbolae. We can show:

Theorem 4.7. *The triangle centers X_i with*

$$i \in \{6, 22, 25, 31, 42, 51, 52, 58, 64, 81, 154, 155, 156, 182, 185, 374, 375, 378, 386, 387, 389, 500, 573, 575, 576, 609, 612, 948, 959, 961, 970, 975, 991, 1012, 1147, 1216, 1350, 1351, 1386, 1486, 1495, 1498, 1658, 1829, 1834, 1836, 1838, 1871, 1900, 1902, 2097, 2334, 2482, 3240, 3242, 3292, 3332, 3581\}$$

trace conic sections while Δ makes a full turn in the poristic family. These conic sections are centered at points on the central line $\mathcal{L}_{1,3}$. One of their axes coincides with $\mathcal{L}_{1,3}$.

Proof. The center X_6 is the Lemoine point of Δ. Its trace has center $M_6 = [3R^2d/(3R^2-2Rr+r^2), 0]$ and major and minor axes are $a_6 = Rr\rho_4/(3R^2 - 2Rr + r^2)$ and $b_6 = R\sqrt{r}\rho_4/\sqrt{\rho_{119}(2R^2 - 3Rr - r^2)}$.

We compute the Exeter point $X_{22} = \mathcal{L}_{2,3} \cap \mathcal{L}_{51,182}$ with $X_{182} = \frac{1}{2}(X_3 + X_6)$ and X_{51} being the centroid of Δ_o. We find $M_{51} = [d(3R+4r)/R, 0]$ and $a_{51} = r\rho_4/(3R)$ and $b_{51} = \rho_4\rho_{908}/(3R)$. The center X_{25} is the intersection of $\mathcal{L}_{2,3}$ and $\mathcal{L}_{6,51}$. The 2^{nd} Power point X_{31} is collinear with the incenter X_1 and Schiffler's point X_{21} and lies on $\mathcal{L}_{940,1001}$ with $X_{940} = \mathcal{L}_{1,3} \cap \mathcal{L}_{2,6}$. We construct X_{42} as $\mathcal{L}_{1,2} \cap \mathcal{L}_{35,58}$, where X_{58} appears as the intersection of the central lines $\mathcal{L}_{1,21}$ and $\mathcal{L}_{3,6}$. The construction of X_{64} is explained in the proof of Th. 4.6.

The center X_{52} is the orthocenter of Δ_o. It is moving on an ellipse centered at $M_{52} = [d(R+4r)/R, 0]$ and with semiaxes $a_{52} = \rho_4\rho_{119}/R$ and $b_{52} = r\rho_4/R$. The point X_{64} is the reflection of X_{1498} in X_3 and a construction of X_{1498} is given in the proof of Th. 4.6.

On the central line joining the incenter X_1 with the Schiffler point X_{21} we find X_{81} which also lies on $\mathcal{L}_{2,6}$. X_{154} is the centroid of Δ_t, X_{155} is the orthocenter of Δ_t, and $X_{156} = \frac{1}{2}(X_{26} + X_{155})$ is the nine-point center of Δ_t. The center X_{185} is the Nagel point of the orthic triangle Δ_o. Its poristic locus is the ellipse with center $M_{185} = [-d(R-4r)/R, 0]$, its semiaxes are $a_{185} = (2R-r)\rho_4/R$ and $b_{185} = (R+r)\rho_4/R$.

The triangle center X_{374} is the centroid of the pedal triangle of X_9. Its poristic locus is the ellipse centered at $M_{374} = [d(R+r)(8R-r)/(3R(4R+r)), 0]$ with semiaxes $a_{374} = 4\rho_4(R+r)/(3(4R+r))$ and $b_{374} = 2R\rho_4/(4R+r)$, respectively. The centroid of the pedal triangle of the Spieker point is denoted by X_{375}. Its poristic trace has center $M_{375} = [d(4R+3r)/(6R), 0]$ and its semiaxes are $a_{365} = (2R+r)\rho_4/(6R)$ and $b_{375} = (3R-r)\rho_4/(6R)$. X_{378} is determined as the reflection of X_{22} in X_3. We have $X_{386} = \mathcal{L}_{1,2} \cap \mathcal{L}_{3,6}$ and $X_{387} = \mathcal{L}_{1,2} \cap \mathcal{L}_{4,6}$. With $X_{389} = \frac{1}{2}(X_3 + X_{52})$ we find an ellipse with $M_{389} = [d(R+4r)/(2R), 0]$, $a_{389} = \rho_4\rho_{119}/(2R)$, and $\rho_{389} = r\rho_4/R$. The orthocenter of the incentral triangle X_{500} leads to $M_{500} = [d(5R+2r)/(2(3R=2r)), 0]$ and $a_{500} = \rho_{21}$, and $b_{500} = \sqrt{r}\rho_{21}/\sqrt{2R}$. With $X_{573} = \mathcal{L}_{3,6} \cap \mathcal{L}_{4,9}$ we find $M_{573} = [-4d(R+r)/(5R+8r), 0]$, $a_{573} = R\rho_4/(5R+8r)$, and $b_{573} = \rho_4\sqrt{rR}/\sqrt{20R^2+37Rr+8r^2}$. The center X_{575} is the midpoint in between X_3 and X_{576}, where X_{576} is the reflection of X_{182} in X_6. The triangle center X_{609} is the intersection of the central lines $\mathcal{L}_{1,32}$ and $\mathcal{L}_{6,36}$.

The center X_{612} is found as intersection of $\mathcal{L}_{1,2}$ and $\mathcal{L}_{6,210}$. We find the next three centers and thereby the parametrizations of their poristic paths as intersection of central lines: $X_{948} = \mathcal{L}_{1,4} \cap \mathcal{L}_{6,7}$, $X_{959} = \mathcal{L}_{1,573} \cap \mathcal{L}_{2,65}$, and $X_{961} = \mathcal{L}_{2,12} \cap \mathcal{L}_{6,959}$. The center of the Apollonius circle is found as $X_{970} = \mathcal{L}_{3,6} \cap \mathcal{L}_{5,10}$. Its poristic trace is centered at $M_{970} = [-d(R+4r)/(2r), 0]$ and the semiaxes are $a_{970} = \rho_4\rho_{119}/(2r)$ and $b_{970} = \frac{1}{2}\rho_4$. Again three centers are found as intersections of central lines: $X_{975} = \mathcal{L}_{1,2} \cap \mathcal{L}_{9,58}$, $X_{991} = \mathcal{L}_{1,7} \cap \mathcal{L}_{3,6}$, and $X_{1012} = \mathcal{L}_{2,6} \cap \mathcal{L}_{1,84}$. The point X_{1147} is the midpoint of X_3 and X_{155}.

The center X_{1216} appears as the reflection of X_{389} in X_{140} and its poristic locus is centered at $M_{1216} = [d(R-4r)/(2R), 0]$ and the semiaxes are $a_{1216} = (2R-r)\rho_4/(2R)$ and $b_{1216} = (R+r)\rho_4/(2R)$. The points X_{1350} and X_{1351} are found as reflections of X_6 in X_3 and X_{1350} in X_{182}, respectively. X_{1386}

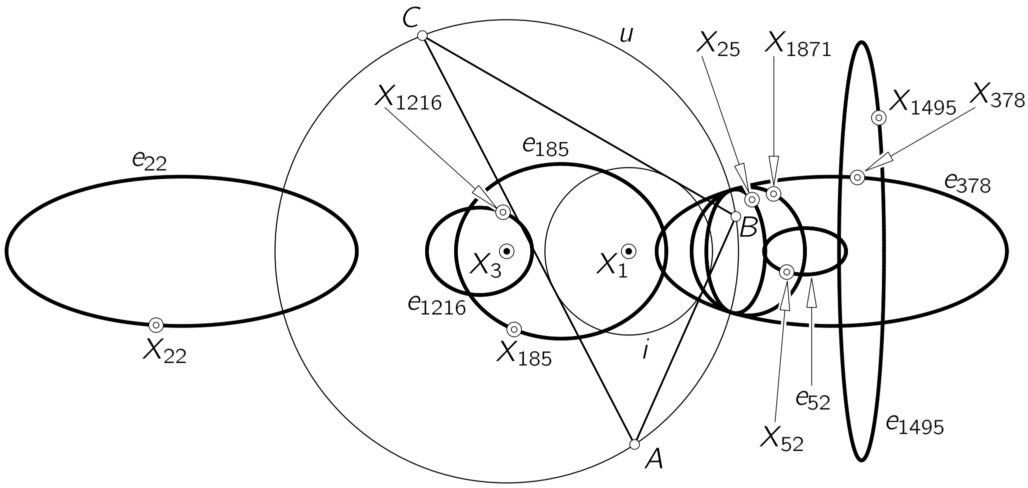

Figure 13: Some ellipses being poristic loci of triangles, *cf.* Th. 4.7.

is the midpoint of X_1 and X_6. The perspector of Δ_t and Δ_i is the center X_{1486}. The triangle center $X_{1495} = \frac{1}{2}(X_{23} + X_{110})$ moves on an ellipse with center $M_{1495} = [3R^3/(d(3R+2r)), 0]$, $a_{1495} = Rr/(3R+2r)$, and $b_{1495} = R(3R+r)/(3R+2r)$. We find $X_{1498} = \mathcal{L}_{1,84} \cap \mathcal{L}_{4,6}$ and $X_{1658} = \frac{1}{2}(X_3 + X_{26})$. Then we find three centers by intersecting central lines: $X_{1834} = \mathcal{L}_{4,6} \cap \mathcal{L}_{12,42}$, $X_{1836} = \mathcal{L}_{4,65} \cap \mathcal{L}_{5,46}$, and $X_{1838} = \mathcal{L}_{1,4} \cap \mathcal{L}_{5,1214}$. The center X_{1214} lies on $\mathcal{L}_{1,3}$ and on $\mathcal{L}_{7,464}$, where $X_{464} = \mathcal{L}_{63,69}$.

On the central line $\mathcal{L}_{4,8}$ we find the centers X_{1829}, X_{1871}, and X_{1900} by intersecting with central lines $\mathcal{L}_{1,25}$, $\mathcal{L}_{5,1848}$, and $\mathcal{L}_{25,35}$, respectively. This yields $M_{1829} = [d(R^2 + 3Rr - r^2)/R^2, 0]$, $a_{1829} = 2r\rho_4/R$, $b_{1829} = \rho_4$; $M_{1871} = [d(3R^2+5Rr-r^2)/(R(2R+r)), 0]$, $a_{1871} = (R+3r)\rho_4/(2R+r)$, $b_{1871} = \rho_4$; and $M_{1900} = [d(R^2+7Rr-r^2)/(R(R+2r)), 0]$, $a_{1900} = 4\rho_{12}$, $b_{1900} = \rho_4$. Reflecting X_{1829} in X_4 we arrive at X_{1902} with $M_{1902} = [d(3R^2-3Rr+r^2)/R^2, 0]$, $a_{1902} = 2\rho_4\rho_{119}/R$, and $\rho_{1902} = \rho_4$.

The triangle center X_{2097} is the reflection of X_6 inX_{57} and $X_{2482} = \frac{1}{2}(X_2 + X_{99})$. We obtain X_{2334} as the common point of the central lines $\mathcal{L}_{1,210}$ and $\mathcal{L}_{6,210}$. The midpoint of X_{69} and X_{145} is identified as center X_{3242}. The point X_{3292} is constructed as the reflection of X_{1495} in X_{110} and its poristic trace is centered at $M_{3292} = [3R^3/(d(3R+2r)), 0]$ and its semiaxes are $a_{3292} = Rr/(3R+2r)$ and $b_{3292} = R(3R+r)/(3R+2r)$. We find $X_{3332} = \mathcal{L}_{1,7} \cap \mathcal{L}_{4,6}$. Finally the center $X_{3581} \subset \mathcal{L}_{3,6}$ lies on the Euler line and we find $M_{3581} = [6R^3/(d(3R+2r), 0]$, $a_{3581} = 2R(3R+r)/(3R+2r)$, and $b_{3581} = 2Rr/(3R+2r)$. $\square$

Fig. 4.4 shows that for certain values of R, r, and d ellipses, parabolae, and hyperbolae appear as poristic trace of the same center.

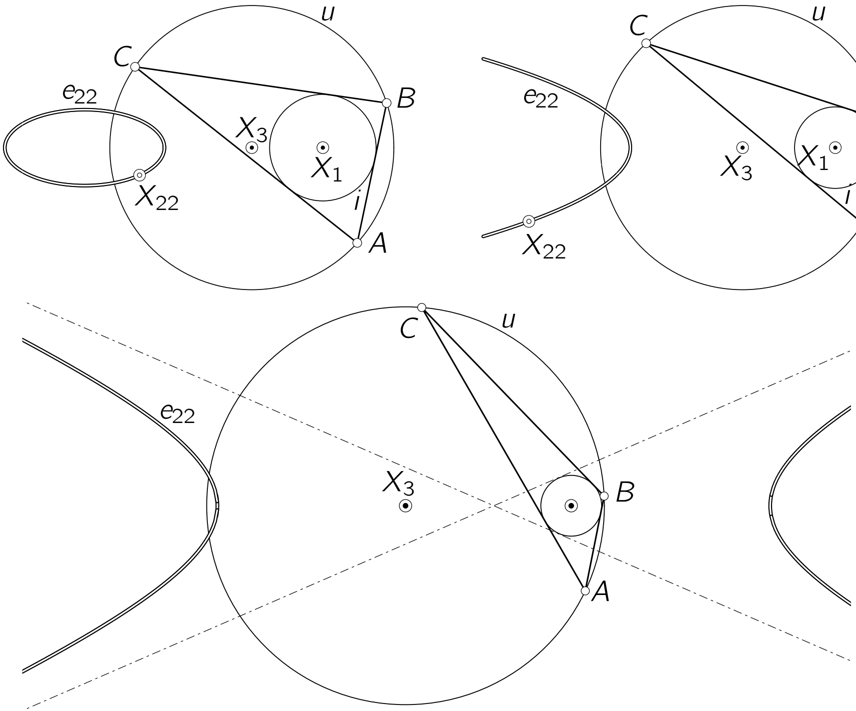

Figure 14: Different shapes of the poristic trace of the Exeter point X_{22}.

5 Final remarks

All the centers mentioned in the proof of Ths. 4.6 and 4.7 are triangle centers for Δ since for any fixed triangle R, r, and d are fixed and so is the relative position of M_i to X_1 and X_3 on $\mathcal{L}_{1,3}$.

The poristic traces of many centers have been parametrized during the computation of the poristic path of all centers mentioned in the theorems. Some of the centers which appear in the construction of centers do not have a conic section for its poristic orbit. The center X_{69} like many others traces an algebraic curve. In most cases the algebraic degree is larger then 4.

In the previous section we skipped the discussion of the affine type of the poristic paths of the centers investigated there. However, it is easy to show that the traces of X_i with $i \in \{22, 64, 154, 156, 609, 1498, 1658, 2482\}$ can be ellipses and hyperbolae as well.

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