Equioptic curves of conic sections

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Abstract

Given two plane curves $c_1$ and $c_2$ we call the set of points from which $c_1$ and $c_2$ are seen under equal angle the equioptic curve. We give some basic results concerning equioptic curves in general. Then we pay our attention to the seemingly simple case of equioptic curves of conic sections. Mainly we are interested in upper bounds of the algebraic degree of these curves. Some examples illustrate the results.

1 Introduction

To any given plane curve $c$ the locus $i$ of points where $c$ is seen under a given fixed angle $\phi$ is called the isoptic curve of $c$. Curves appearing as isoptic curves are well studied, see for example [4] and [8] and the references given there. The papers [10, 11] deal with curves having a circle or an ellipse for an isoptic curve.

The name isoptic curve was suggested first by Taylor in [7]. The kinematic generation of isoptic and orthoptic curves is also studied there. The locus of points where a tangent of $c_1$ intersects a tangent of $c_2$ at a certain angle $\phi$ is considered as a generalization of the classical notion of isoptic curves. These investigations also deal with the general forms of pedal curves. Especially the isoptics of concentric cycloids are studied in [9].

Due to the algebraic nature of its definition it is clear that the isoptic of an algebraic curve is also algebraic. The Plücker characteristics of the isoptic of a curve $c$ can be expressed in terms of the characteristics of $c$, cf. [6, 7].
Isoptic curves of conic sections have been studied in [3] and [5]. It turned out that the isoptic curves of ellipses or hyperbolae, i.e., conic sections with center having an equation of the form
\[ \alpha x^2 + \beta y^2 = 1 \] (1)
are quartic curves given by
\[ i : (\alpha + \beta - \alpha\beta(x^2 + y^2))^2 \sin^2 \phi - 4\alpha\beta(\alpha x^2 + \beta y^2 - 1) \cos \phi^2 = 0. \] (2)
where \( \phi \) is the desired optic angle.

It is not mentioned in the literature - though very easy to verify - that the isoptic curves of parabolae with equation
\[ 2py = x^2 \] (3)
are hyperbolae given by
\[ (4x^2 + (p - 2y)^2) \cos^2 \phi - (p + 2y)^2 = 0. \] (4)

Fig. 1 displays some isoptic curves of the three affine types of conic sections.

Special cases of so called orthoptics, which are the isoptics for \( \phi = \frac{\pi}{2} \) are well known in any case: If \( c \) is an ellipse with \( a \) and \( b \) for the lengths of its major and minor axis, then the orthoptic curve is a concentric circle with radius \( \sqrt{a^2 + b^2} \). In case of a hyperbola \( c \) we find a circle concentric with \( c \) with radius \( \sqrt{a^2 - b^2} \) provided that \( a > b \). The orthoptic curve of a parabola is its directrix. This is clear when inserting \( \phi = \frac{\pi}{2} \) into the respective equations of isoptics.\(^1\)

Further it is worth to be noted that any quartic curve that appears as the isoptic curve of a conic section \( c \) is the isoptic curve of a further conic section \( c' \neq c \) at the same time, see [4].

The isoptics of conic sections are spiric curves, see [4, 8], which can be obtained as planar intersections of a torus. Naturally these curves are bicircular like the torus, i.e., the curves have double points at the absolute points (of Euclidean geometry) whereas the surface has the absolute conic (of Euclidean geometry) for a double curve, see [4].

\(^1\)It is clear from Eq. (2) that the orthoptic of a conic section with center is also a quartic curve. To be more precise it is a circle of multiplicity 2. Similarly the orthoptic of a parabola is a repeated line.
So far we have collected some facts on isoptic curves and the interested reader may ask himself: What has all this to do with equioptic curves? Consider a point $X$ on the equioptic curve $e(c_1, c_2)$ of two different plane curves $c_1$ and $c_2$. The generic point $X \in e$ is the common point of two tangents of $c_1$ enclosing a certain angle, say $\phi$. So $X$ is a point of the isoptic curve $i_1(\phi)$ of $c_1$ to the angle $\phi$. Since $X$ is also a common point of two tangents of $c_2$ enclosing the same angle $\phi$, it is a point of the isoptic curve $i_2(\phi)$ of $c_2$, cf. Fig. 2. Hence any point of the equioptic curve $e(c_1, c_2)$ of curves $c_1$ and $c_2$ is the intersection of two isoptic curves $i_1(\phi)$ and $i_2(\phi)$, respectively, for a certain value of $\phi$. So we have

$$e(c_1, c_2) = \{i_1(\phi) \cap i_2(\phi) : \phi \in [0, 2\pi]\}.$$  \hspace{1cm} (5)

It will turn out that the curve $e$ has real branches even if $\phi$ is not real and $|\cos \phi| > 1$. Basically, these branches consist of interior points of curves (especially in the case of conic sections).

The aim of this paper is to give some basic results on equioptic curves. We pay our attention to algebraic curves, especially to conic sections. Further we want to show the algebraic way to find the equations of equioptics of a pair of algebraic curves. For that purpose we describe curves in terms of Cartesian coordinates in Euclidean plane $\mathbb{R}^2$. Whenever necessary we use the projective closure and the complex extension of $\mathbb{R}^2$.

\footnote{At first we do not restrict the huge class of plane curves. We only assume that they have tangents at any point, i.e., from the differential geometric point of view we assume that the curves $c_1$ and $c_2$ are of class $C^1$.}
Figure 2: Two curves $c_i$ and a point $X$ on the equioptic curve $e$, the tangents $T_i, T'_i$ and the corresponding contact points $X_i, X'_i$.

The paper is organized as follows: First we collect some facts on equioptic curves of algebraic curves in general in Sec. 2 and then we focus on conic sections and their equioptics in Sec. 3. Unfortunately pairings of different affine types of conic sections need separate treatment. Sec. 4 is devoted to the study of existence and the counting of equioptic points of three given conic sections. Finally we conclude in Sec. 5 and address some open problems.

2 General remarks on equioptics

Let $c_1$ and $c_2$ be two algebraic curves of respective degrees $d_1$ and $d_2$. The equation of either curve shall be given in implicit form by a polynomial $F_i(x, y)$ of degree $d_i$. We try to find an upper bound for the algebraic degree of the equioptic $e(c_1, c_2)$. For this purpose we write down the system of (algebraic) equations determining the equioptic.

Any point $X = [x, y]^T$ on $e(c_1, c_2)$ is the locus of concurrency of two tangents $T_1, T'_1$ of $c_1$ and two further tangents $T_2, T'_2$ of $c_2$. We assign the coordinates $X_i = [x_i, y_i]^T, X'_i = [\xi_i, \eta_i]^T$ to the contact points of the tangents $T_i$ and $T'_i$.
The contact points have to fulfill
\[ F_1(x_1, y_1) = 0, \quad F_1(\xi_1, \eta_1) = 0, \]
\[ F_2(x_2, y_2) = 0, \quad F_2(\xi_2, \eta_2) = 0. \]
Now we introduce the abbreviations \( g_i := \text{grad} F_i(X_i) \) and \( g'_i := \text{grad} F_i(X'_i) \).

The tangents \( T_i \) and \( T'_i \) have to pass through \( X \) which gives further relations between coordinates of contact points and the point \( X \) on \( e \):
\[ \langle g_1, X - X_1 \rangle = 0, \quad \langle g'_1, X - X'_1 \rangle = 0, \]
\[ \langle g_2, X - X_2 \rangle = 0, \quad \langle g'_2, X - X'_2 \rangle = 0. \]
Finally the condition on the tangents \( T_1 \) and \( T'_1 \) to enclose the same angle as \( T_2 \) and \( T'_2 \) is given by
\[ \langle g_1, g'_1 \rangle^2 \cdot ||g_2||^2 \cdot ||g'_2||^2 = \langle g_2, g'_2 \rangle^2 \cdot ||g_1||^2 \cdot ||g'_1||^2. \]
Eqs. (6) and (7) together with Eq. (8) are nine equations in ten unknowns \( X_1, Y_1, \xi_1, \eta_1, X_2, Y_2, \xi_2, \eta_2, x, y \). In order to determine an equation of \( e(c_1, c_2) \) one has to eliminate all but \( x \) and \( y \) from these equations.

Table 1 shows the degrees of the nine equations with respect to the unknowns.

One can easily see from Table 1 that the system of algebraic equations (6), (7), and (8) is solved by successive elimination of variables. In a first cycle we eliminate \( X_1, \xi_1, X_2, \) and \( \xi_2 \) by computing the resultants
\[ \text{Res}((6.1), (7.1), X_1), \quad \text{Res}((6.2), (7.2), \xi_1), \]
\[ \text{Res}((6.3), (7.3), X_2), \quad \text{Res}((6.4), (7.4), \xi_2). \]
\[ d_1(d_1 − 1) \]
\[ 0 \]
\[ d_1(d_1 − 1) \]
\[ 0 \]
\[ d_1(d_1 − 1) \]
\[ d_2(d_2 − 1) \]
\[ 0 \]
\[ 0 \]
\[ d_1(d_1 − 1) \]
\[ d_2(d_2 − 1) \]
\[ 0 \]

Table 2: The degrees of the resultants given in Eq. (9) and Eq. (10).

\[ \text{Res(Res(Res(Res((8), (6.4), \xi_2), (6.3), X_2), (6.2), \xi_1), (6, 1), X_1).} \] (10)

Table 2 collects the degrees of the resultants in the remaining unknowns \( Y_1, \eta_1, Y_2, \eta_2, \) and \( x, y. \)

The entries of the last column are actually \( \max(d_i(d_i − 1), d_i) = d_i(d_i − 1), \)

since in general \( d_i \neq 0. \)

In order to obtain an upper bound for the degree of the equioptic curve, we compute the final resultant of resultants

\[ \text{Res(Res(Res(Res((9.4), (10), \eta_2), (9.3), Y_2), (9.2), \eta_1), (9.1), Y_1).} \] (11)

which is of degree \( d_1^2 d_2^2(d_1 − 1)^2(d_2 − 1)^2. \) Hence we have:

**Theorem 2.1.**

Let \( c_1 \) and \( c_2 \) be two algebraic curves of degree \( d_1 \) and \( d_2, \) respectively. The degree of the equioptic curve \( e(c_1, c_2) \) is at most

\[ d_1^2 d_2^2(d_1 − 1)^2(d_2 − 1)^2. \]

**Remark:**

1. There are several reasons why the actual degree of the equioptic curve can be lower. The degree of resultants computed in (9), (10), and (11) can be lower as expected, for example see [1, 2]. Sometimes resultants can factor and the corresponding components may not be essential. An example will show up in Sec. (3.1) when we deal with equioptics of circles.
2. The geometric definition of the equioptic curve somehow differs from
the algebraic definition. The algebraic formulation of the geometric
properties of the equioptic curves by means of Eqs. (6), (7), and (8)
is not flawless. Especially, after squaring the angle criterion in order to
get (8) it expresses the fact that \( \cos^2 \angle (T_1, T_1') = \cos^2 \angle (T_2, T_2') \). This
implies that either \( \angle (T_1, T_1') = \angle (T_2, T_2') \) or \( \angle (T_1, T_1') = \pi - \angle (T_2, T_2') \).
Consequently the algebraically defined equioptic curve contains branches
where the tangents fulfil the desired condition, though the curves \( c_1 \) and
\( c_2 \) are not actually seen under the same angle.
This phenomenon will not occur when we compute the equioptic of
conic sections. These curves will be obtained by intersecting isoptics to
equal angles.

3. In general the curves defined by the resultant (11) contain the equioptics
defined by (5) and curves which can be called \textit{quasi-equioptics}, i.e., the
locus of points where one curve is seen under the angle \( \phi \) and the other
curve is seen under \( \pi - \phi \).

Furthermore we observe that parasitic branches of the equioptic curve \( e(c_1, c_2) \)
may occur. These are the sets of real points where no real tangents of either
curve may pass through. At these points \( |\cos \phi| > 1 \) and the angle enclosed
by tangents \( T_i \) and \( T_i' \) is imaginary.

We observe the following properties of equioptic curves:

\textbf{Corollary 2.1.}
\textit{Let} \( c_1 \) \textit{and} \( c_2 \) \textit{be two (algebraic) plane curves. Denote the intersection points
by} \( S_i \) \textit{with and the common tangents by} \( L_j \).

1. The equioptic curve \( e(c_1, c_2) \) passes through the common points \( S_i \) of
\( c_1 \) \textit{and} \( c_2 \).

2. The equioptic curve \( e(c_1, c_2) \) contains the intersection points \( L_{ij} :=
L_i \cap L_j \) \textit{of common tangents of} \( c_1 \) \textit{and} \( c_2 \) \textit{if both curves are locally in
the same halfplanes of both tangents} \( L_i \) \textit{and} \( L_j \), \textit{respectively}.

\textbf{Proof.} \hspace{1em} 1. At a point \( S_i \in \{c_1 \cap c_2\} \) the curve \( c_1 \) as well as \( c_2 \) are seen
under the angle of \( 180^\circ \).

2. Let \( L_{ij} = L_i \cap L_j \) be one intersection point of \( i \)-th and \( j \)-th common
tangent of \( c_1 \) \textit{and} \( c_2 \). At the point \( L_{ij} \) the tangent \( L_j \) plays the role of

\[ T_1 \text{ and } L_j \text{ that of } T_1', \text{ say. Further } L_j \text{ also plays the role of } T_2 \text{ and } L_j \text{ that of } T_2'. \text{ Consequently } \angle(T_1, T_1') = \angle(T_2, T_2'). \]

3 Equioptic curves of conic sections

In this section we focus on equioptics of conic sections. We are not only interested in the degree of such curves. We are also looking for special pairings of conic section. The singularities of equioptics will also be payed attention to.

According to Eq. (5) the equioptics of two conic sections \( c_1 \) and \( c_2 \) can be found by writing down the equations of the respective isoptics \( i_1(\phi) \) and \( i_2(\phi) \). Then we eliminate \( \phi \), i.e., we intersect any pair of isoptics to the same angle \( \phi \). From the algebraic degrees of the isoptics we can conclude the following:

Theorem 3.1.
Let \( c_1 \) and \( c_2 \) be two conic sections given by irreducible quadratic equations. Then the algebraic degree of the equioptic curve \( e(c_1, c_2) \) of \( c_1 \) and \( c_2 \) is bounded by the following values:

1. The algebraic degree of the equioptic of two conic sections with center is at most 6.
2. The algebraic degree of the equioptic of a conic section with center and a parabola is at most 6.
3. The algebraic degree of the equioptic of two parabolae is at most 4.

Proof. A conic section of any affine type can be transformed into any conic section of the same affine type by applying an affine mapping. Since we are interested in certain relations on angles measured between tangents we restrict ourselves to equiform mappings. The coordinate representation of an equiform mapping in the Euclidean plane reads

\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix},
\]

which is a Euclidean motion if and only if \( a^2 + b^2 = 1 \). So the only degrees of freedom when mapping one conic section with center to another one are the ratio \( \alpha : \beta \) (cf. Eq. (1)), and the two coordinates of the center.
1. The equioptic of two conic sections with center is obtained by eliminating $\phi$ from two equations of the form (2). Practically this can be done be letting $\cos^2 \phi = K$. The algebraic degrees are not harmed when applying any affine (indeed projective) transformation in order to find the isoptic of conic sections in more general positions. According to Bézout’s theorem we can expect that any two isoptics $i_1(\phi)$ and $i_2(\phi)$ to the same angle $\phi$ have $\deg i_1 \cdot \deg i_2 = 4 \cdot 4 = 16$ points in common. Note that this is also true for the orthoptics since these are circles with multiplicity two. Surprisingly we observe that the resultant of $i_1$ and $i_2$ with respect to $K$ is a polynomial of degree 6 in the unknowns $x$ and $y$, respectively.

The reduction of the degree is caused by the following facts: Since both absolute points of Euclidean geometry are double points on either isoptic $i_j(\phi)$ (for any $\phi$) exactly 8 of the common points coincide with the absolute points. Further, the ideal line splits off with multiplicity 2 from the equation of the equioptic. (This can be seen by computing the homogeneous equation of the equioptic from the homogeneous equations of the isoptics.) Note that from any real ideal point a conic section with center can be seen under the angle of 0°.

2. Any parabola can be obtained from the parabola \( c_1 : 2py - x^2 = 0 \) by a suitable equiform transform of the above given kind. Apply the transform to the parabola as well as to its isoptic hyperbola given in (4). Then eliminate $\phi$ from both, the equations of the isoptic $i_1$ of the parabola $c_1$ and from the isoptic $i_2$ of the conic section $c_2$ with center given in (2). This obviously results in a polynomial whose degree is at most 6 in $x$ and $y$.

3. For a pair of parabolae we go a similar way and end up with intersecting two families of hyperbolae comprising the set of equioptic curves of the two parabolae $c_1$ and $c_2$. So any two isoptics to the same angle intersect in four points (algebraically counted).

Note that the results on degrees of equioptic curves of conic sections from Th. 3.1 undershoot the upper bound given in Th. 2.1 by far. According to Th. 2.1 the degree of $e(c_1, c_2)$ could reach most 16, provided that $c_1$ and $c_2$ are conic sections, i.e., algebraic curves of degree $d_1 = d_2 = 2$.

Remark: The computation of the equioptic curve $e(c_1, c_2)$ of two conic sections $c_1$ and $c_2$ can be performed by eliminating $\phi$ from the equations of the
respective isoptic curves. For variable $\phi$ the curves $i_1(\phi)$ and $i_2(\phi)$ can be viewed as level sets of two functions defined on the common plane of either $c_i$. In this sense the equioptic curve $e(c_1, c_2)$ is the orthogonal projection of the intersection of two graph surfaces to the plane of either $c_i$.

As a consequence of Cor. 2.1 we have:

**Corollary 3.1.**

Assume $c_1$ and $c_2$ are two conic sections. Let the common points be denoted by $S_i$ and the common tangents may be labelled by $L_i$.

1. The four intersection points of $c_1$ and $c_2$ belong to the equioptic curve.

2. The six intersection points $L_{ij} = L_i \cap L_j$ of the four common tangents of the conic sections $c_1$ and $c_2$ are contained in their equioptic curve $e(c_1, c_2)$ if both curves are locally in the same halfspace of both tangents $L_i$ and $L_j$, respectively.
As mentioned earlier in this paper the orthoptic curve of an ellipse or hyperbola is a circle. The orthoptic of a parabola is its directrix. This allows to count the number of orthoptic points, i.e., points from which either curve can be seen under right angles:

**Theorem 3.2.**

For two arbitrarily given conic sections $c_1$ and $c_2$ the number $o$ of points where both curves can be seen under right angles is at most 2. This bound is sharp except the case, when $c_1$ and $c_2$ are parabolae.
Proof. In the case of a conic section with center the orthoptic is a circle, where as in the case of a parabola it is the directrix. In order to clarify the number of orthoptic points one has to discuss the possible intersections of circle and circle, or circle and line, or line and line. □

The shape of the equioptic curve of two confocal conics is regulated by:

**Theorem 3.3.**

Let \((c_1, c_2)\) be confocal conic sections such that \(c_1\) is chosen from one family and \(c_2\) is chosen from the other family.

1. The equioptic curve of a pair \((c_1, c_2)\) of confocal conic sections (with center) is the union of the two-fold ideal line, two further pairs of conjugate complex lines, and a circle containing the four common points of \(c_1\) and \(c_2\).

2. The equioptic curve of a pair \((c_1, c_2)\) of confocal parabolae is the union of a straight line connecting the common points of \(c_1\) and \(c_2\), the ideal line and the pair of isotropic lines through the common focus of \(c_1\) and \(c_2\).

Proof. 1. Assume \(c_1\) and \(c_2\) are given by an equation of the form (1). Let \(\alpha = 1/a^2\), \(\beta = 1/b^2\) and \(a > b\) for \(c_1\). Without loss of generality we can assume that \(a > b\). Further \(\alpha' = 1/(c^2 - b^2)\) and \(\beta' = -1/(a^2 - c^2)\) guarantee that \(c_1\) and \(c_2\) span a confocal family. Write down the isoptics of either conic sections in homogeneous coordinates and eliminate the angle (parameter). This yields

\[
x_0^2(c^2x_0^2 - x_1^2 - x_2^2) \cdot ((a^2 - b^2)x_0^4 - 2(a^2 - b^2)x_0^2(x_1^2 - x_2^2)) + (x_1^2 + x_2^2)^2 = 0.
\]

The first factor corresponds to the two-fold ideal line.

The second factor is the equation of a circle centered at \([0, 0]^T\) with radius \(c\) carrying real points if \(c^2 > 0\). In this case \(c_1\) and \(c_2\) have the four real points

\[
\frac{1}{\sqrt{a^2 - b^2}} \left[ \pm \sqrt{a(c^2 - b^2)}, \pm \sqrt{b(a^2 - c^2)} \right]^T
\]

in common, which are located on the circle. This holds true even if \(c^2 < 0\). Note that the tangents to this circle are bisectors of the angles enclosed by \(c_1\) and \(c_2\) at their common points.
The third factor splits into the equations of the four isotropic lines
\[ x \pm \sqrt{a^2 - b^2} \pm iy = 0 \quad (15) \]
through \( c_1 \)'s common foci.

2. Assume the parabolae are given by \( c_1 : 2y - \frac{x^2}{a} + a^2 = 0 \) and \( c_2 : 2y + \frac{x^2}{b} - b^2 = 0 \). The equioptic curve has the homogeneous equation
\[ x_0(x_1^2 + x_2^2)((a^2 - b^2)x_0 + 2x_2) = 0. \quad (16) \]
The first factor is the equation of the ideal line and the second factor corresponds to the isotropic lines through the common focus \([0, 0]^T\).
The last factor is the equation of a real line which carries the intersection points \([\pm ab, \frac{1}{2}(a^2 - b^2)]^T\) of \( c_1 \) and \( c_2 \).

![Figure 5: Equioptic curves of pairs of confocal conic sections.](image)

The equioptic curve \( e(c_1, c_2) \) of two conic sections \( c_1 \) and \( c_2 \) carries singularities. It is possible to find some of them immediately:

**Theorem 3.4.**

1. The absolute points of Euclidean geometry are singular points on \( e(c_1, c_2) \).
2. The intersection points of the orthoptics of either conic section are singular points on \( e(c_1, c_2) \).
Proof. 1. Any isoptic curve of a conic section has double points at the absolute points. As the points of the equioptic appear as the intersection of two isoptics to the same angle both curves share their singularities and so the family of intersection points also contains these points.

2. The isoptic curves to ellipses and hyperbolae for the angle $\frac{\pi}{2}$ are circles actually having multiplicity 2 as is clearly seen from (2). The orthoptic of a parabola is its directrix which has also multiplicity 2 which follows from (4) by substituting $\phi = \frac{\pi}{2}$. Obviously their common points have at least multiplicity 2 and therefore they are singular.

Figure 6: Singular points $N_i$ on an equioptic curve $e(c_1, c_2)$ of two conic sections. Left: Two conics in general position. Right: The equioptic of a parabola $c_2$ with one of its osculating circles $c_1$ has a further node at the point of osculation.

Fig. 6 shows the generic case of singular points on an equioptic curve of two conic sections. Further singularities can occur if the conic sections are in higher order contact which is also illustrated in Fig. 6 at hand of an osculating pair.
3.1 Equioptic of two circles

A circle $c$ with center $M = [m, n]^T$ and radius $R$ shall be given in terms of Cartesian coordinates as

$$c: (x - m)^2 + (y - n)^2 - R^2 = 0.$$  \hspace{1cm} (17)

The isoptic $i(\phi)$ is again a circle, centered at $M$, with radius $R \cdot \csc \frac{\phi}{2}$ and thus described by the equation

$$i(\phi): (1 - K)((x - m)^2 + (y - n)^2) - 2^2R = 0,$$  \hspace{1cm} (18)

where $K := \cos \phi$. It is worth to be noted that Eq. 18 is linear in $K$. This will have much influence on the degree of the equioptic curve of two circles.

Now we assume that we are given two circles $c_1$ and $c_2$. Without loss of generality we can assume that $c_1$ is centered at $[0,0]^T$ and has radius $R \in \mathbb{R} \setminus \{0\}$. The circle $c_2$ shall be centered at $[d,0]^T$ with $d \neq 0$ and its radius shall be $r \in \mathbb{R} \setminus \{0\}$.

We are writing down the isoptics $i_1(\phi)$ and $i_2(\phi)$ of $c_1$ and $c_2$, respectively, in terms of homogenous coordinates by letting $x = x_1x_0^{-1}$ and $y = x_2x_0^{-1}$ and eliminate the angle $\phi$. This yields a homogenous equation of the equioptic curve $e(c_1,c_2)$ as

$$e(c_1, c_2): x_0^2((r^2 - R^2)(x_1^2 + x_2^2) + 2dR^2x_0x_1 - d^2R^2x_0^2) = 0.$$  \hspace{1cm} (19)
Figure 8: The quasi-equioptic curve \( \mathcal{E} \) of two circles is passing through the four intersection points of the common tangents of \( c_1 \) and \( c_2 \) but not through the centers of similitude.

This leads to the following result:

**Theorem 3.5.**

Let \( c_1 \) and \( c_2 \) be two circles with radii \( R \) and \( r \) and the distance \( d \) between their centers. The equioptic curve \( e(c_1, c_2) \) of \( c_1 \) and \( c_2 \) is

1. the union of a circle containing the two centers of similitude of \( c_1 \) and \( c_2 \), respectively, and the ideal line with multiplicity 2, if \( d \neq 0 \) and \( r \neq R 

2. the union of the bisector of the centers (with multiplicity 1) and the three-fold ideal line if \( c_1 \) and \( c_2 \) are congruent and

3. the union of the two-fold ideal line and a the pair of isotropic lines through the common center if \( c_1 \) and \( c_2 \) are concentric.

**Proof.**

1. The centers of similitude of \( c_1 \) and \( c_2 \) are given by

\[
S_1 = \left[ \frac{dR}{R+r}, 0 \right]^T \quad \text{and} \quad S_2 = \left[ \frac{dR}{R-r}, 0 \right]^T
\]

and they annihilate the second factor of Eq. (19).

2. Insert \( R = r \) into (19) and note that \( d \neq 0 \).
3. Insert $d = 0$ into (19) and note that $R \neq r$. 

Remark:

1. In Fig. 7 we observe that four intersection points of the common tangents of the two circles $c_1$ and $c_2$ are not located on the equioptic circle $e(c_1, c_2)$. From these four points one circle is seen under an angle of $\phi$ whereas the other curve is seen under the angle $\pi - \phi$. So these points do not belong to the geometrically defined equioptic curve.

There is also a reason why these points are not located on the algebraically determined equioptic curve: Earlier in this paper we have noticed that the equations of the isoptic curves $i_i(\phi)$ of $c_i$ the value $K$ shows up only linear. In the case of the isoptic of an ellipse or hyperbola $K$ appear only in second powers, i.e., $\cos \phi$ appears only in squares. Therefore these equations give the equations for the isoptics to an angle $\phi$ and the angle $\pi - \phi$. This is not the case for the isoptics of circles.

Fig. 8 shows the locus $e$ of points where $c_1$ is seen under the angle $\phi$ and $c_2$ is seen under the angle $\pi - \phi$. We call this curve the quasi-equioptic of $c_1$ and $c_2$, respectively.

2. From points of the ideal line the curves $c_1$ and $c_2$ can be seen at equal angles $\phi = 0$ since any two tangents from ideal points to any curve are parallel.

The five collinear points $C_1$, $C_2$ (centers of $c_i$), $S_1$, $S_2$ (respective centers of similitude), and $E$ (center of the equioptic circle) can be arranged in quadruples in several ways and define cross ratios which are related by

\[ cr(E, C_1, S_1, S_2) \cdot cr(E, C_2, S_1, S_2) = cr(C_1, C_2, S_1, S_2) = -1. \]

### 3.2 Equioptics of conic sections with a circle

This short section is exclusively devoted to the computation of algebraic degrees:

**Corollary 3.2.**

The algebraic degree of the equioptic curve of a conic section and a circle is at most 6.
Proof. At first we derive the equioptic curve \( e(c_1, c_2) \) of a conic section \( c_2 \) with center and a circle \( c_1 \). Without loss of generality we can assume that \( c_2 \) is given by an equation of the form (1) and \( c_1 \) is centered at \([m, n]^T\) and has radius \( R \) and is thus given by Eq. (17). Therefore the isoptics \( i_1(\phi) \) of \( c_1 \) are given by (18) and the isoptics of \( c_2 \) have the equation (2). We eliminate \( \phi \) from (18) and (2) by letting \( K := \cos \phi \) and end up with an algebraic equation of degree 6 in the unknowns \( x \) and \( y \).

The proof is almost the same for the equioptic of a parabola and a circle. \( \Box \)

Remark: We can use the projective closure of \( \mathbb{R}^2 \) and represent the isoptics appearing in the proof of Cor. 3.2 by their respective homogeneous equations. Eliminating \( K \) now results in a homogeneous polynomial of degree 8 which always factors into \( x_0^2 \) and a sextic form. Thus the ideal line (of course with multiplicity 2) is always a part of the equioptic of a circle and conic section with center. This is not the case for the equioptic of a circle and a parabola.

4 Equioptic points

We call a point \( E(c_1, c_2, c_3) \) equioptic point of three curves \( c_1, c_2, \) and \( c_3 \) if there are two tangents, say \( T_i, T'_i \) of either curve \( c_i \) passing through \( E \) such that \( \angle(T_1, T'_1) = \angle(T_2, T'_2) = \angle(T_3, T'_3) \).

It is obvious that equioptic points appear as the intersections of equioptic curves. As a consequence of Th. 3.1 we can give an upper bound for the number of equioptic points of three conic sections:

Theorem 4.1.

Assume that \( c_i \) with \( i \in \{1, 2, 3\} \) are three conic sections. The number \( \nu(c_1, c_2, c_3) \) of equioptic points of the three conic sections \( c_1, c_2, \) and \( c_3 \) is bounded by 36.

Proof. Use Th. 3.1 and apply Bézout’s theorem. \( \Box \)

Remark: The value \( \nu(c_1, c_2, c_3) \) drops if parabolae and circles come into play. The long-winded discussion of the number of equioptic points arbitrary triplets of conic sections (classified with respect to affine or even Euclidean properties) could be postponed to a forthcoming paper.
We can state:

**Lemma 4.1.**
Three generic conic sections do not have orthoptic points. In other words: In general there is no point from which three generic conic sections can be see under the same angle.

*Proof.* We cannot expect that the three circles appearing as orthoptic curves of three conic sections have common points (besides the absolute points of Euclidean geometry). □

For circles there is only the following result:

**Lemma 4.2.**
Three generic circles in arbitrary position with arbitrary radii have no proper equioptic point. They can only be seen from any ideal point under the angle of $0^\circ$.

*Proof.* Three circles in general position with not necessarily equal radii determine three circles as their orthoptic curves. These orthoptic circles are concentric with the given ones and naturally they are of course also in general position. Besides the absolute points of Euclidean geometry these three circles do not share any point. □

## 5 Conclusion

There are a lot of fine details to be studied. A matter of particular interest could be the exact number of equioptic points of the three arbitrary algebraic plane curves. The study of equioptic curves for special configurations of pairs of conic sections is far from being complete. For example one can classify pairs of conic sections from the viewpoint of Euclidean geometry and discuss the corresponding equioptic curves. There are only a few of the singularities detected so far. A huge amount of practical examples showed that in general there a no more singular points than the four discovered. More singularities on the equioptic curve appear if the conic sections are in higher order contact as shown in one example. But this needs a close inspection. What is the number and what are the types of singularities that can occur?
There is still much to do for algebraic curves of higher degree, i.e., for example of cubics, quartics, and so on. Under which circumstances do the degrees of equioptic curves drop? Are singular points on the given curves points of their equioptics? However, as long as the power of computers is not sufficient the algebraic (or computational) approach will shipwreck.

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**References**


