Rational Offset Surfaces and Related Problems

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Overview

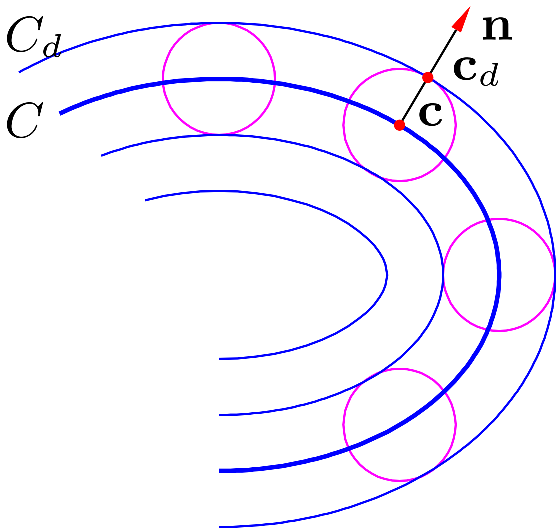
1. Offsets of curves and surfaces
2. Laguerre sphere geometry in $\mathbb{R}^2$ and $\mathbb{R}^3$
3. Models of Laguerre sphere geometry
4. Rational offset surfaces
5. Convolution surfaces – a generalization of offsets
6. Conclusion

Rational Offset Curves

- Rational curve C with parametrization $\mathbf{c}(t) = (c_1, c_2)(t)$
- One-sided oriented offset curve C_d at C distance d admits the parametrization

$$\mathbf{c}_d(t) = \mathbf{c}(t) + d\mathbf{n}(t),$$

  with $\mathbf{n}$ as *unit normal vector* of $\mathbf{c}$.
- Which input curves C possess rational offsets C_d?



Offsets as Envelopes

Offsets (two-sided) are *envelopes* of circles

$$(\mathbf{x} - \mathbf{c}(t))^\top \cdot (\mathbf{x} - \mathbf{c}(t)) = d^2$$

centered at $\mathbf{c}(t)$.

Elimination of the parameter t from

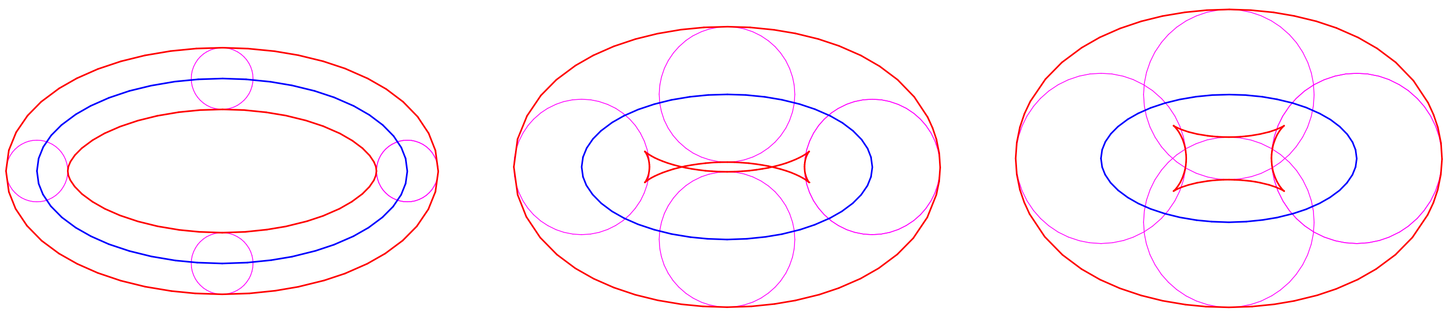
$$\begin{aligned} F(\mathbf{x}, t) : (\mathbf{x} - \mathbf{c}(t))^\top \cdot (\mathbf{x} - \mathbf{c}(t)) &= d^2, \\ F_t(\mathbf{x}, t) : (\mathbf{x} - \mathbf{c}(t))^\top \cdot \dot{\mathbf{c}}(t) &= 0, \end{aligned}$$

leads to an implicit representation $G(\mathbf{x}) = 0$ of the offset C_d.

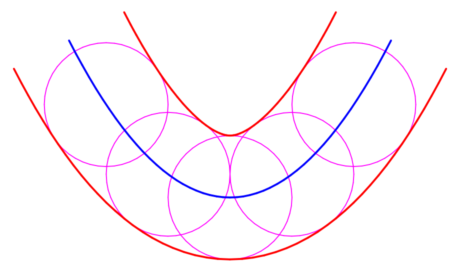
- Offset C_d is rational exactly if its genus equals $g = 0$.
- Objective: Construction of rational parametrizations of rational offset curves and surfaces.

Some Examples

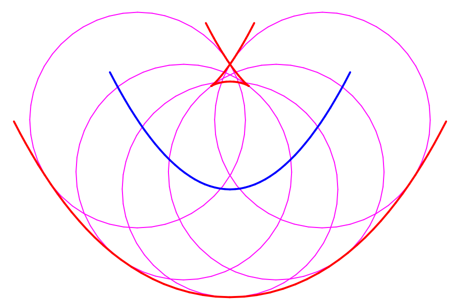
- The offsets C_d of an ellipse C are non-rational algebraic curves of order 8.



- The offsets C_d of a parabola C are rational of order 6. But the standard parametrization $\mathbf{c}(t) = (t, t^2)$ does not result in a rational parametrization of C_d, because of



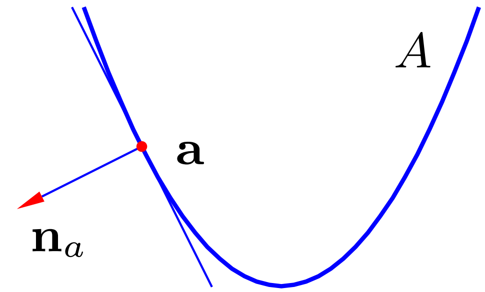
$$\mathbf{n}(t) = \frac{1}{\sqrt{1+4t^2}}(-2t, 1).$$



- Which parametrizations of the parabola lead to rational parametrizations of the offsets curves C_d?

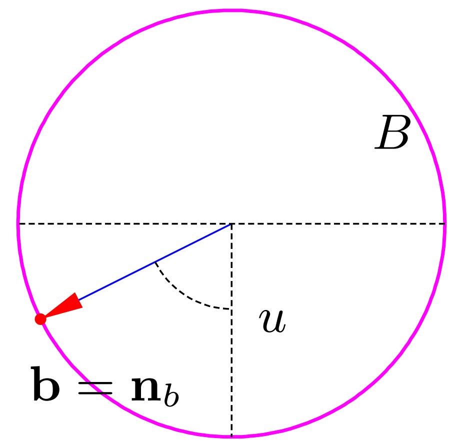
The parabola and its offsets

- A ... parabola with $\mathbf{a}(t) = (t, \frac{1}{2}t^2)$,
- B ... unit circle with $\mathbf{b}(u) = (\sin u, -\cos u)$.
- Normals $\mathbf{n}_a(t) = (-t, 1)$ and $\mathbf{n}_b(u) = \mathbf{b}(u)$.



Correspondence: Require coinciding unit normal vectors $\mathbf{n}_a(t) = \lambda(u)\mathbf{n}_b(u)$. This leads to

$$\begin{aligned} -t &= \lambda n_1 = \lambda \sin u, \\ 1 &= \lambda n_2 = -\lambda \cos u. \end{aligned}$$

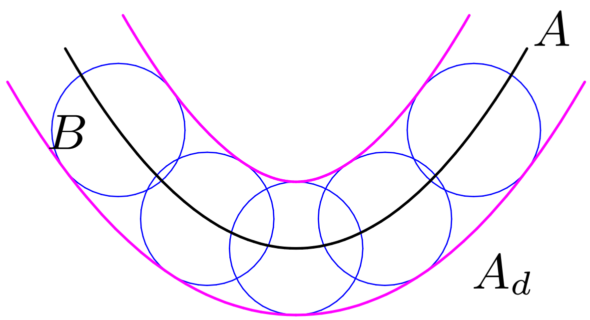


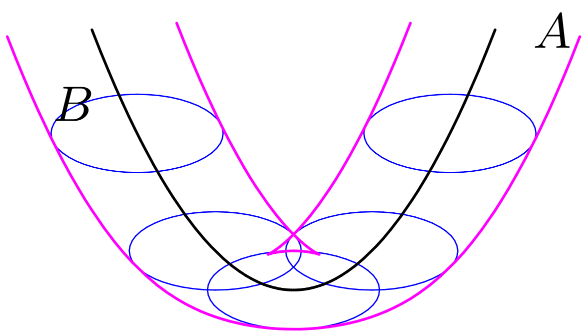
Reparametrization:

$$\lambda(u) = \frac{-1}{\cos u} \Longrightarrow t = \frac{\sin u}{\cos u}$$

$$\mathbf{a}(t(u)) = \left(\frac{\sin u}{\cos u}, \frac{1}{2} \frac{\sin^2 u}{\cos^2 u} \right).$$

The parabola's generalized offsets





- The offsets are given by

$$\mathbf{a}_d(u) = \mathbf{a}(t(u)) + d\mathbf{b}(u).$$

- A rational parametrization ($v = \tan(u/2)$) of B

$$\mathbf{b}(v) = \left(\frac{2v}{1+v^2}, -\frac{1-v^2}{1+v^2}\right)$$

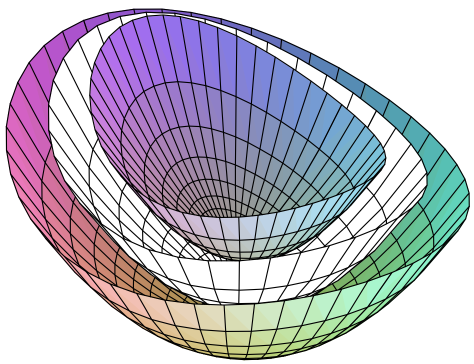
leads to rational parametrizations of A and A_d of degrees 4 and 6, resp.

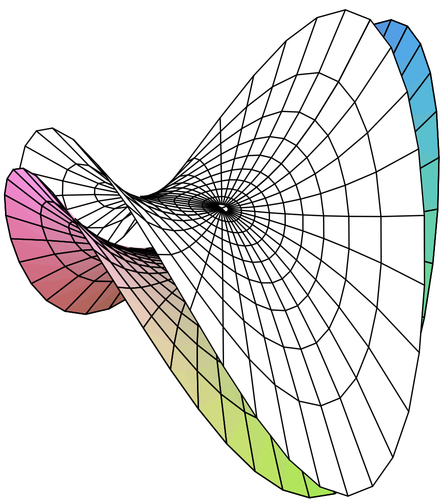
- Reparametrization with respect to coinciding unit normal vectors

$$\mathbf{n}_a = (-2t, 1)^\top = \lambda \mathbf{n}_b,$$

is *rational* for arbitrary rational curves B.

Offsets of Paraboloids





- Paraboloid $A : \mathbf{a}(u,v) = (u, v, \frac{1}{2}u^2 + \frac{c}{2}v^2)$
- Sphere $B : \mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)^\top = (\cos s \cos t, \sin s \cos t, \sin t)^\top$
- Correspondence w.r.t. coinciding unit normals

$$\mathbf{n}_a = (-2u, -2v, 1)^\top = \lambda \mathbf{n}_b,$$

leads to the rational reparametrization

$$u = \frac{-\cos s \cos t}{\sin t}, \qquad v = -\frac{\sin s \cos t}{c \sin t}.$$

- Offsets of Paraboloids are *rational* (W. Lü, '94, '95).

Pythagorean hodograph-curves

Farouki, et al. '90, '91, ...,

Let C be a *polynomial curve* with parametrization $\mathbf{c}$. If the tangent vector is representable as

$$\begin{aligned}\dot{c}_1(t) &= (u(t)^2 - v(t)^2)w(t),\\ \dot{c}_2(t) &= 2u(t)v(t)w(t),\end{aligned}$$

with polynomials $u, v, w \in \mathbb{R}[t]$, one obtains

$$\|\dot{\mathbf{c}}\| = \|\mathbf{n}\| = w(t)(u(t)^2 + v(t)^2).$$

$\Rightarrow C$ has rational *arc length* and its offset curves C_d are rational.

Remark: This approach does not apply to rational curves and not to surfaces.

Literature

Offsets and rational parametrizations:

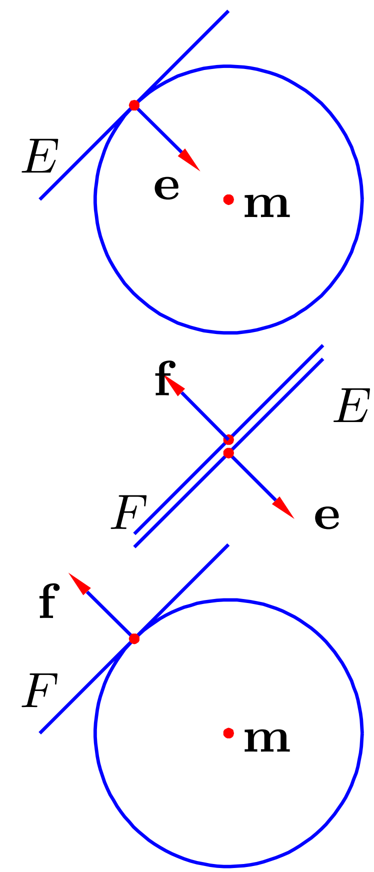
R. Farouki, W. Lü, J.R. Sendra and J. Sendra, H. Pottmann, B. Jüttler, F. Winkler, J. Schicho, L. González-Vega, L. Sampoli, G. Landsmann, R. Krasauskas, Ch. Mäurer, G. Elber, M.S. Kim, ...

2D-Sphere Geometry (Laguerre)

- An oriented (or.) circle C in $\mathbb{R}^2$ is given by

$$C : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0,$$

and the orientation is determined by or. normals. Points are considered as circles of radius 0.



- An oriented line E in $\mathbb{R}^2$ is given by

$$E : e_0 + e_1 x_1 + e_2 x_2 = e_0 + \mathbf{e}^\top \cdot \mathbf{x} = 0.$$

We always assume that $\mathbf{e}^2 = 1$.

- E and C are said to be in oriented contact iff

$$e_0 + e_1 m_1 + e_2 m_2 + r = e_0 + \mathbf{e}^\top \cdot \mathbf{m} + r = 0, \mathbf{e}^2 = 1.$$

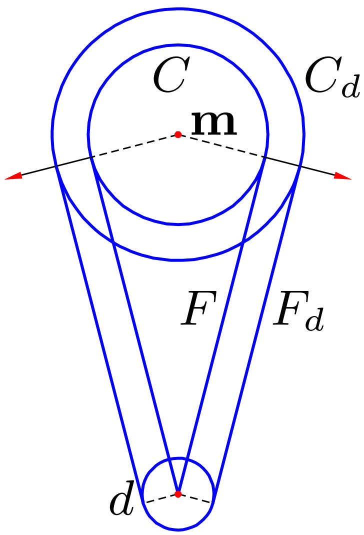
2D-Transformations

A *Laguerre transformation* T consists of two mappings

$$T_C : \mathcal{C} \to \mathcal{C}, \mathcal{T}_{\mathcal{E}} : \mathcal{E} \to \mathcal{E},$$

which are *bijective* on the sets of oriented circles $\mathcal{C}$ and lines $\mathcal{E}$, respectively, and *preserve oriented contact* of circles and lines.

- Motions and similarities are *point-preserving* Laguerre trafos.
- A dilation D maps the circle $C : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0$ onto the circle $C_d : (\mathbf{x} - \mathbf{m})^2 - (r + d)^2 = 0$. D is not point-preserving but maps points to circles of radius d.
- Laguerre transformations are *contact transformations*.



Offsets of 2D-Curves

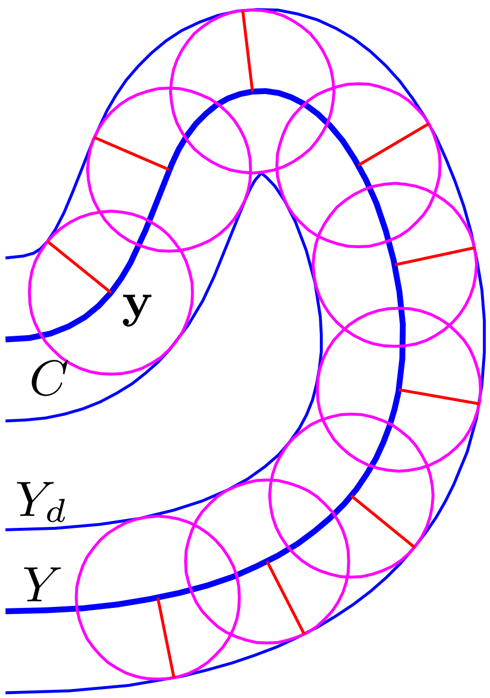
A dilation D maps a curve

$$Y : \mathbf{y}(t) = (y_1, y_2)(t)$$

onto the offset Y_d which is constructed as envelope of the one-par. family of circles

$$C(t) : (\mathbf{x} - \mathbf{y}(t))^2 - d^2 = 0,$$

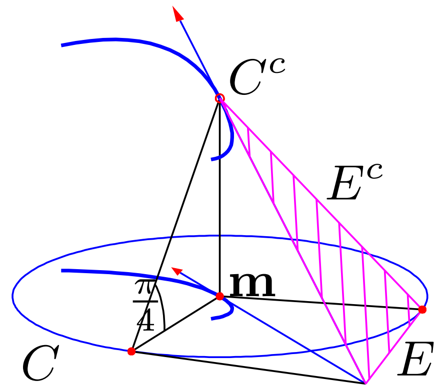
which are centered at Y.



Remark: An oriented curve possesses oriented one-sided offset curves. The offsets of not oriented curves consist locally of two not connected components, the inner part and the outer part.

Cyclographic model–2D case

- An oriented circle $C : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0$ is mapped to the point $C^c = (m_1, m_2, r)$ in $\mathbb{R}^3$. The plane $\mathbb{R}^2$ is embedded in $\mathbb{R}^3$ as plane $x_3 = 0$.



- An oriented line $E : e_0 + \mathbf{e}^\top \cdot \mathbf{x} = 0$ with $\mathbf{e}^2 = 0$ is mapped to the plane

$$E^c : e_0 + e_1 x_1 + e_2 x_2 + x_3 = 0, \ \angle(E^c, \mathbb{R}^2) = \pi/4.$$

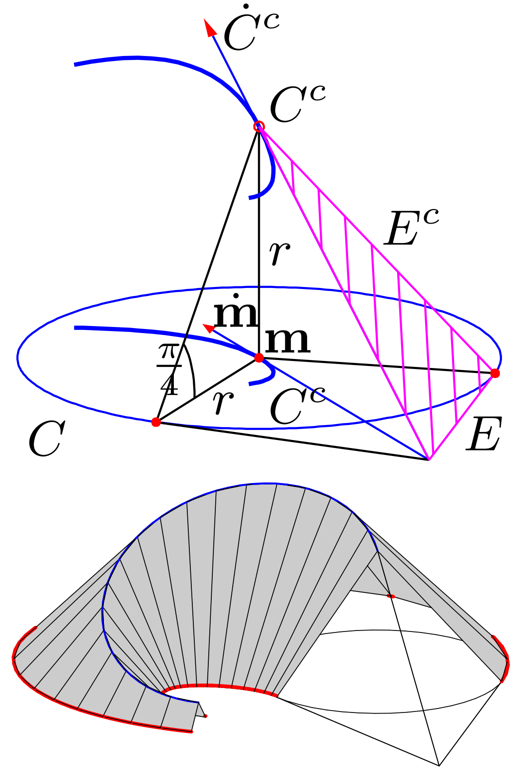
- Oriented contact of C and E is realized by the incidence $C^c \in E^c$.
- $\mathbb{R}^3$ is equipped with the 'scalar product' $\langle \mathbf{x}, \mathbf{y} \rangle_c = \mathbf{x}^\top \cdot I_c \cdot \mathbf{y}$, with $I_c = \mathrm{diag}(1, 1, -1)$. We denote the *Lorenz space* $\mathbb{R}^3$ by $\mathbb{R}^3_1$.
- Laguerre trafos appear in $\mathbb{R}^3_1$ as *affine mappings* $T : \mathbf{x} \mapsto \lambda A \cdot \mathbf{x} + \mathbf{c}$ with $A^\top \cdot A = I_c$, (7-parameter group).

Cyclographic model–2D case

- A one-parameter family of circles

$$C(t) : (\mathbf{x} - \mathbf{m}(t))^2 - r(t)^2 = 0$$

is identified with the curve $C^c(t) = (m_1, m_2, r)(t)$ in $\mathbb{R}^3_1$.



- The circles $C(t)$ possess a real envelope exactly if

$$\begin{aligned} \dot{\mathbf{m}}^\top \cdot \dot{\mathbf{m}} - \dot{r}^2 &\geq 0, \quad \Longleftrightarrow \\ \dot{C}^c \cdot I_c \cdot \dot{C}^c &\geq 0. \end{aligned}$$

where $\dot{x} = dx/dt$.

- The envelope of $C(t)$ is traced out by the *developable surface* of constant slope $\pi/4$ passing through $C(t)^c$.

3D-Sphere Geometry

- An oriented sphere S in $\mathbb{R}^3$ is given by

$$S : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0,$$

and the orientation is determined by oriented normals. Points are considered as spheres of radius 0.

- An oriented plane E in $\mathbb{R}^3$ is given by

$$E : e_0 + e_1 x_1 + e_2 x_2 + e_3 x_3 = e_0 + \mathbf{e} \cdot \mathbf{x} = 0.$$

We always assume that $\mathbf{e}^2 = 1$.

- E and S are said to be in oriented contact iff

$$e_0 + e_1 m_1 + e_2 m_2 + e_3 m_3 + r = e_0 + \mathbf{e} \cdot \mathbf{m} + r = 0, \mathbf{e}^2 = 1.$$

3D-Transformations

- A *Laguerre transformation* T consists of two mappings

$$T_S : \mathcal{S} \to \mathcal{S}, \mathcal{T}_{\mathcal{E}} : \mathcal{E} \to \mathcal{E},$$

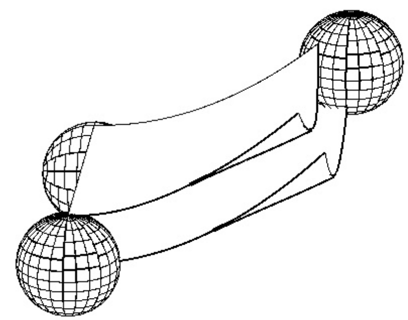
  which are *bijective* on the sets of spheres $\mathcal{S}$ and planes $\mathcal{E}$, respectively, and *preserve or. contact* of spheres and planes.

- Motions and similarities in 3D are *point-preserving* Laguerre trafos.
- A dilation D maps the sphere $S : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0$ onto the sphere $S_d : (\mathbf{x} - \mathbf{m})^2 - (r + d)^2 = 0$.
- Laguerre trafos are *contact transformations*.

Offsets of Curves and Surfaces

A dilation D maps a surface

$$Y : \mathbf{y}(u,v) = (y_1, y_2, y_3)(u,v)$$

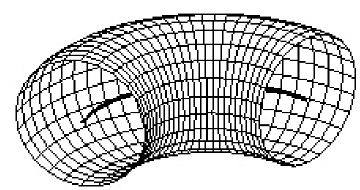


onto the *offset-surface* Y_d which is the envelope of the spheres

$$S(u,v) : (\mathbf{x} - \mathbf{y}(u,v))^2 - d^2 = 0,$$

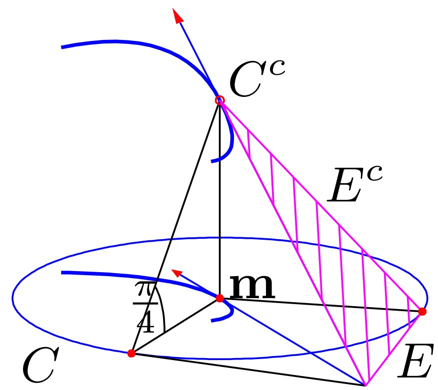
centered at Y.

A dilation D maps a curve $Y : \mathbf{y}(t) = (y_1, y_2, y_3)(t)$ onto the offset Y_d which is the envelope of the spheres $S(t) : (\mathbf{x} - \mathbf{y}(t))^2 - d^2 = 0$, centered at Y. Y_d is called *pipe surface*.



Cyclographic model–3D case

- An oriented sphere $S : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0$ is mapped to the point $S^c = (m_1, m_2, m_3, r)$ in $\mathbb{R}^4$. $\mathbb{R}^3$ is embedded in $\mathbb{R}^4$ as 3-space $x_4 = 0$.



- An oriented plane $E : e_0 + \mathbf{e}^\top \cdot \mathbf{x} = 0$ with $\mathbf{e}^2 = 0$ is mapped to the 3-space

$$E^c : e_0 + e_1 x_1 + e_2 x_2 + e_3 x_3 + x_4 = 0, \ \angle(E^c, \mathbb{R}^2) = \pi/4.$$

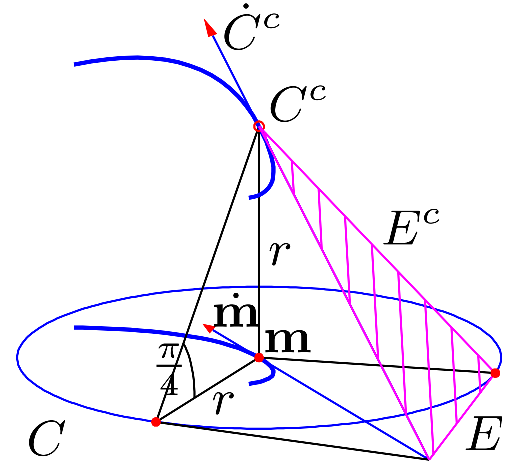
- Oriented contact of S and E is realized by the incidence $S^c \in E^c$.
- $\mathbb{R}^4$ is equipped with the 'scalar product' $\langle \mathbf{x}, \mathbf{y} \rangle_c = \mathbf{x}^\top \cdot I_c \cdot \mathbf{y}$, with $I_c = \text{diag}(1, 1, 1, -1)$. We denote the *Lorenz space* $\mathbb{R}^4$ by $\mathbb{R}^4_1$.
- Laguerre trafos appear in $\mathbb{R}^4_1$ as *affine mappings* $T : \mathbf{x} \mapsto \lambda A \cdot \mathbf{x} + \mathbf{c}$ with $A^\top \cdot A = I_c$, (11-parameter group).

Cyclographic model–3D case

- A one-parameter family of spheres

$$S(t) : (\mathbf{x} - \mathbf{m}(t))^2 - r(t)^2 = 0$$

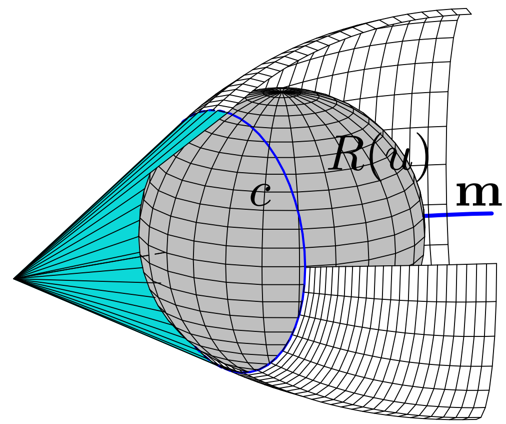
is mapped to the curve $S^c(t) = (m_1, m_2, m_3, r)(t)$ in $\mathbb{R}^4_1$.



- The spheres $S(t)$ possess a *real envelope* exactly if

$$\begin{aligned} \dot{\mathbf{m}}^\top \cdot \dot{\mathbf{m}} - \dot{r}^2 &\geq 0, \quad \Longleftrightarrow \\ \dot{S}^c \cdot I_c \cdot \dot{S}^c &\geq 0. \end{aligned}$$

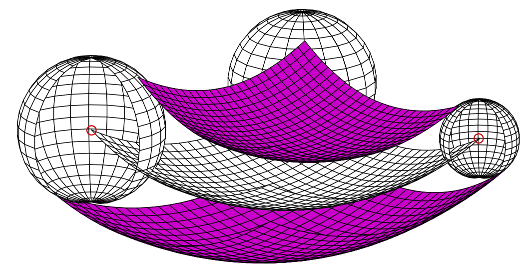
where $\dot{x} = dx/dt$.



- The envelope of a one-par. family of spheres $S(t)$ is called *canal surface*.

Cyclographic model–3D case

- A two-parameter family of spheres



$$S(u,v) : (\mathbf{x} - \mathbf{m}(u,v))^2 - r(u,v)^2 = 0$$

is mapped to the surface

$$S^c(u,v) = (m_1, m_2, m_3, r)(u,v) \in \mathbb{R}^4_1.$$

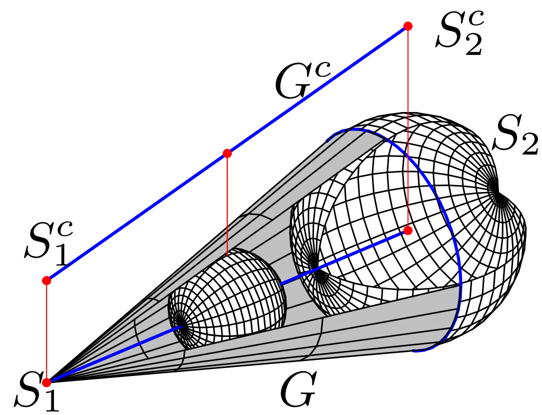
- The spheres $S(u,v)$ possess a real envelope exactly if for all (λ, μ) we have

$$\begin{aligned}(\lambda \mathbf{m}_u + \mu \mathbf{m}_v)^2 - (\lambda r_u + \mu r_v)^2 &\geq 0, \quad \Longleftrightarrow \\ (\lambda S^c_u + \mu S^c_v) \cdot I_c \cdot (\lambda S^c_u + \mu S^c_v) &\geq 0.\end{aligned}$$

holds.

Cyclographic model–Examples

- A cone of revolution G is determined by two oriented spheres S_1, S_2. The center $\mathbf{z}$ of similarity is $G's$ vertex. Thus G^c is a line joining S_1^c, S_2^c.
- A cylinder of revolution G is determined by two congruent spheres. Thus G^c is a horizontal line ($x_4 = \text{const}$).



- A one-par. family of cones of revolution $G(t)$ corresponds to a ruled surface $G(t)^c$ in $\mathbb{R}^4_1$.

Other Models of Laguerre Geometry

Oriented planes in $\mathbb{R}^3$

- Oriented planes are given by

$$\begin{aligned} E(u,v) : e_0(u,v) + \mathbf{e}(u,v)\cdot\mathbf{x} &= 0,\ \ \|\mathbf{e}\|^2 = 1, \\ e_0 + e_1x_1 + e_2x_2 + e_3x_3 &= 0, \end{aligned}$$

  with $\mathbf{e}$ as E's *unit normal vector*.

- Mapping $b : \mathcal{E} \longrightarrow \mathbb{R}^4$

$$E \mapsto E^b = (e_1, e_2, e_3, e_0),\ \text{with}\ e_1^2 + e_2^2 + e_3^2 = 1. \tag{1}$$

- Image points E^b are contained in

$$B : x_1^2 + x_2^2 + x_3^2 = 1,$$

  called the *Blaschke cylinder*. B is a cylinder over the unit sphere S^2.

- $b(E(u,v)) = E(u,v)^b \subset B$ is called *Blaschke image* of A.

- The oriented planes $E \in \mathbb{R}^3$ form a cylinder and not an affine space.

Blaschke Cylinder

- Parallel planes

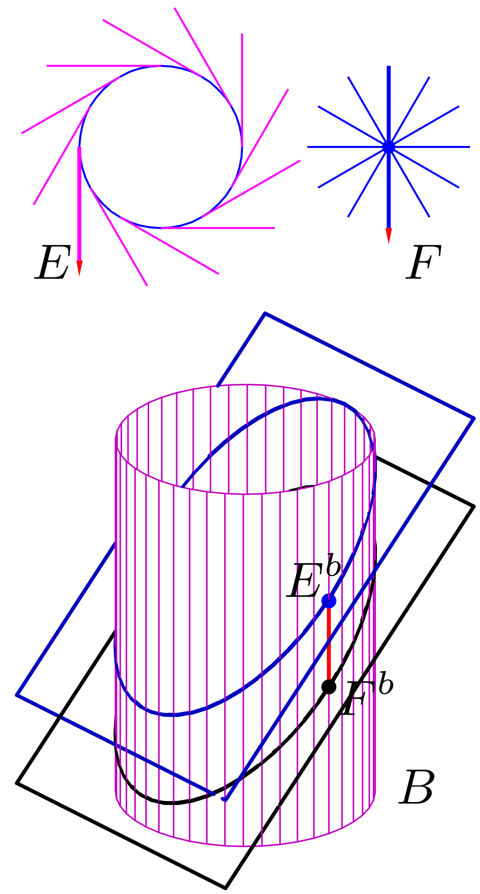
$$E : e_0 + \mathbf{e} \cdot \mathbf{x} = 0, \|\mathbf{e}\| = 1,$$
$$F : f_0 + \mathbf{f} \cdot \mathbf{x} = 0, \mathbf{e} = \mathbf{f}$$

  have image points E^b, F^b in a generating line of B.

- Laguerre trafos T appear as *automorphic projective transformations* of B.

- Tangent planes E of a *sphere* $S \in \mathbb{R}^3$ are mapped onto points E^b in a *hyperplanar intersection* (ellipsoid) of B.

- Tangent planes E of a *cone (or cylinder) of rot.* $S \in \mathbb{R}^3$ are mapped onto points E^b in a *planar intersection* (ellipse) of B.



Blaschke Cylinder

- A non-developable surface S, considered as envelope of its 2-par. family of tangent planes

$$E(u,v): e_0(u,v) + \mathbf{e}(u,v) \cdot \mathbf{x} = 0,$$

corresponds to a surface

$$E^b(u,v): (e_1, e_2, e_3, e_0)(u,v), \text{ with } \mathbf{e}(u,v)^2 = 1$$

in the Blaschke cylinder B.

- A developable surface S considered as envelope of its 1-par. family of tangent planes $E(t): e_0(t) + \mathbf{e}(t) \cdot \mathbf{x} = 0$, corresponds to a curve

$$E^b(t): (e_1, e_2, e_3, e_0)(t), \text{ with } \mathbf{e}(t)^2 = 1,$$

in the Blaschke cylinder B.

Isotropic Model

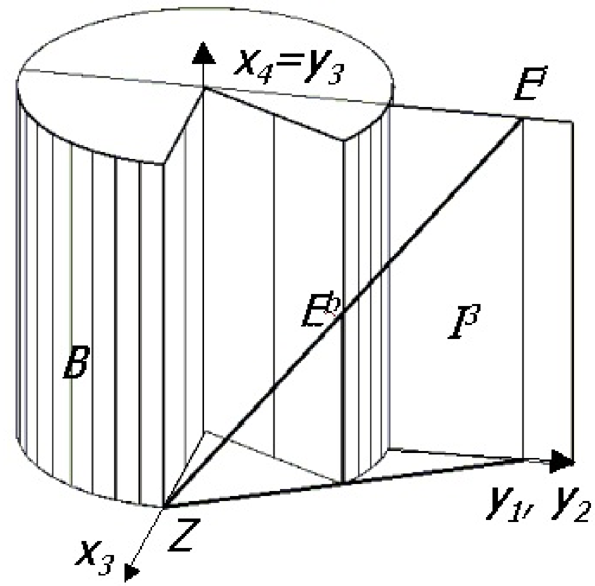
Stereographic projection of B with center $Z = (0, 0, 1, 0)$ and image space $x_3 = 0$ yields the *isotropic model* (affine space) I^3.

- A point $E^b = (e_1, e_2, e_3, e_0)$ maps to the point

  $$E^i = \frac{1}{1 - e_3}(e_1, e_2, e_0) \in I^3.$$

  E^i is called *isotropic image* of E.

- The stereographic projection and its inverse are *rational transformations*.



Models of Laguerre Geometry

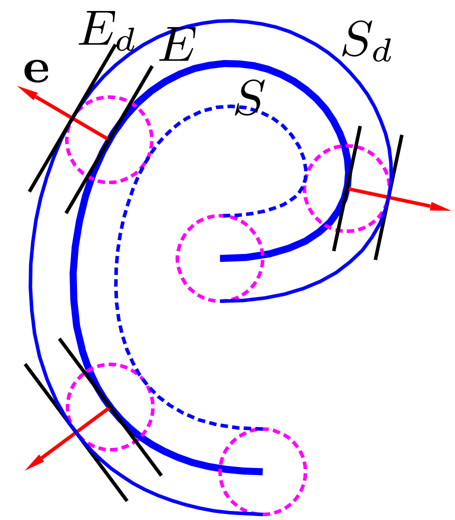
$\mathbb{R}^3$	sphere $S : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0$	oriented plane $E : e_0 + e_1 x_1 + e_2 x_2 + e_3 x_3 = 0,$ $\mathbf{e}^2 = 1$
CM	point $S^c = (m_1, m_2, m_3, r),$	$\pi/4$-hyperplane $E^c : e_0 + e_1 x_1 + e_2 x_2 + e_3 x_3 = 0,$ $\mathbf{e}^2 = 1$
BM	hyperplanar $\cap$ with $S^b : r + m_1 x_1 + \dots$ $\dots + m_3 x_3 + x_4 = 0$	point $E^b = (e_1, e_2, e_3, e_0),\ \mathbf{e}^2 = 1$
I^3	paraboloid of rev. $2y_3 + (y_1^2 + y_2^2)(r + m_3) +$ $2y_1 m_1 + 2y_2 m_2 + r - m_3 = 0$	point $E^i = \frac{1}{1-e_3}(e_1, e_2, e_0)$

Rational Offset Surfaces

A surface S is called *rational offset surface* if it possesses a *rational parametrization* $\mathbf{s}(u,v)$ and *rational unit normal vectors* $\mathbf{e}(u,v)$ in a way that its offset surfaces S_d admit rational parametrizations

$$\mathbf{s}(u,v) + d\mathbf{e}(u,v).$$

- Let $E(u,v) : e_0(u,v) + \mathbf{e}(u,v) \cdot \mathbf{x} = 0$ be tangent planes of S.
- The unit normals $\mathbf{e}(u,v)$ are a rational parametrization of the spherical (Gaussian) image of S.
- The offset surfaces S_d of S are envelopes of the translated planes



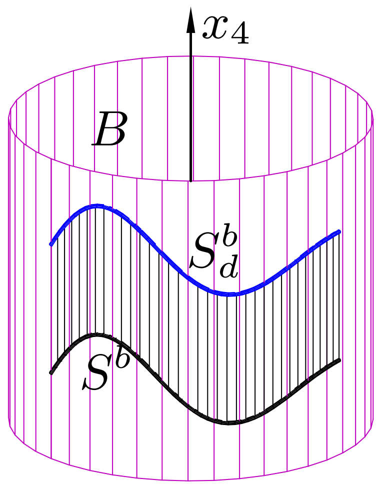
$$E_d(u,v) : e_0(u,v) + d + \mathbf{e}(u,v) \cdot \mathbf{x} = 0.$$

Rational Offset Surfaces–Blaschke Image

- Let $E(u,v)$ and $E_d(u,v)$ be tangent planes of S and S_d, resp.. Their Blaschke images E^b and E^b_d are

$$\begin{aligned} E^b(u,v) &= (e_1, e_2, e_3, e_0)(u,v),\ \mathbf{e}(u,v)^2 = 1, \\ E^b_d(u,v) &= (e_1, e_2, e_3, e_0 + d)(u,v),\ \mathbf{e}(u,v)^2 = 1. \end{aligned}$$

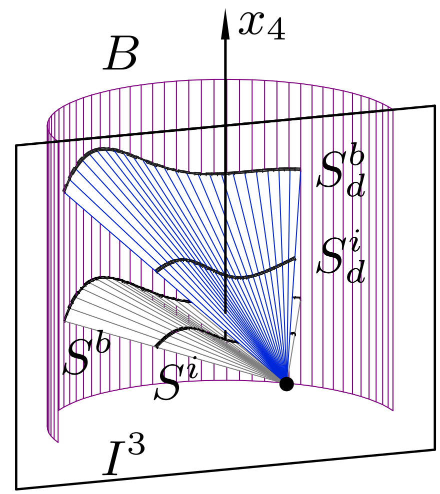
- S^b and S^b_d are rational surfaces $\subset B$.
- S^b_d is a translated version of S^b in x_4-direction.



Rational Offset Surfaces–Isotropic Image

- The isotropic images E^i and E^i_d of S and S_d are rational surfaces in I^3.

$$E^i(u,v) = \frac{1}{1-e_3}(e_1,e_2,e_0)(u,v),$$
$$E^i_d(u,v) = \frac{1}{1-e_3}(e_1,e_2,e_0+d)(u,v).$$



- S^i and S^i_d are rational surfaces in I^3.
- S^i_d is obtained from S^i by

$$y_1 = \tfrac{e_1}{1-e_3} \mapsto y_1$$
$$y_2 = \tfrac{e_2}{1-e_3} \mapsto y_2$$
$$y_3 = \tfrac{e_0}{1-e_3} \mapsto y_3 + \tfrac{d}{1-e_3}.$$

Rational Offset Surfaces in I^3

Theorem: Let $Y(u,v) = (y_1, y_2, y_3)(u,v)$ be an arbitrary rational surface in I^3. The corresponding family of planes in $\mathbb{R}^3$ is

$$E(u,v) : e_0 + e_1 x_1 + e_2 x_2 + e_3 x_3 = 0, \text{ with}$$
$$(e_0, e_1, e_2, e_3) = (2y_3, 2y_1, 2y_2, y_1^2 + y_2^2 - 1)/N,$$
$$\text{and } N = (y_1^2 + y_2^2 + 1).$$

The envelope of the planes $E(u,v)$ is a *rational offset surface*.

Developable Rational Offset Surfaces in I^3

Theorem: Let $Y(t) = (y_1, y_2, y_3)(t)$ be an arbitrary rational curve in I^3. The corresponding family of planes in $\mathbb{R}^3$ is

$$E(t) : e_0 + e_1 x_1 + e_2 x_2 + e_3 x_3 = 0, \text{ with}$$
$$(e_0, e_1, e_2, e_3) = (2y_3, 2y_1, 2y_2, y_1^2 + y_2^2 - 1)/N,$$
$$\text{and } N = (y_1^2 + y_2^2 + 1).$$

The envelope of the planes $E(t)$ is a *developable rational offset surface*.

Rational Offset Surfaces – Constructive Concept

Pottmann '95

- Let $\mathbf{e} = (e_1, e_2, e_3)$ be a rational parametrization of the unit sphere S^2, with polynomials a, b, c in u and v:

$$e_1 = \frac{2ac}{N}, e_2 = \frac{2bc}{N}, e_3 = \frac{a^2 + b^2 - c^2}{N}, \text{ with } N = (a^2 + b^2 + c^2).$$

  Let $h(u, v)$ be an arbitrary rational function.

- The envelope F of the two-par. family of planes

$$E(u, v) : h(u, v) + e_1 x_1 + e_2 x_2 + e_3 x_3 = 0$$

  is a rational offset surface.

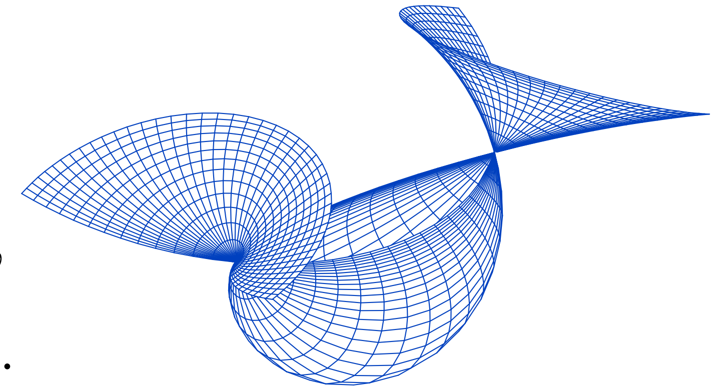
- The offsets F_d of F are envelopes of the planes

$$E_d(u, v) : (h + d) + e_1 x_1 + e_2 x_2 + e_3 x_3 = 0.$$

- If a, b, c are polynomials in t and $h(t)$ is a rational function, F and F_d are *developable.*

Parabolic Dupin Cyclide

Example: Let $a = u, b = v, c = 1$. Then



$$e_1 = \frac{2u}{N}, e_2 = \frac{2v}{N}, e_3 = \frac{u^2 + v^2 - 1}{N},$$

with $N = (u^2 + v^2 + 1)$.

We choose $h(u, v) = q(u, v)/N$ with $q(u, v)$ as quadratic polynomial, $\Longrightarrow$

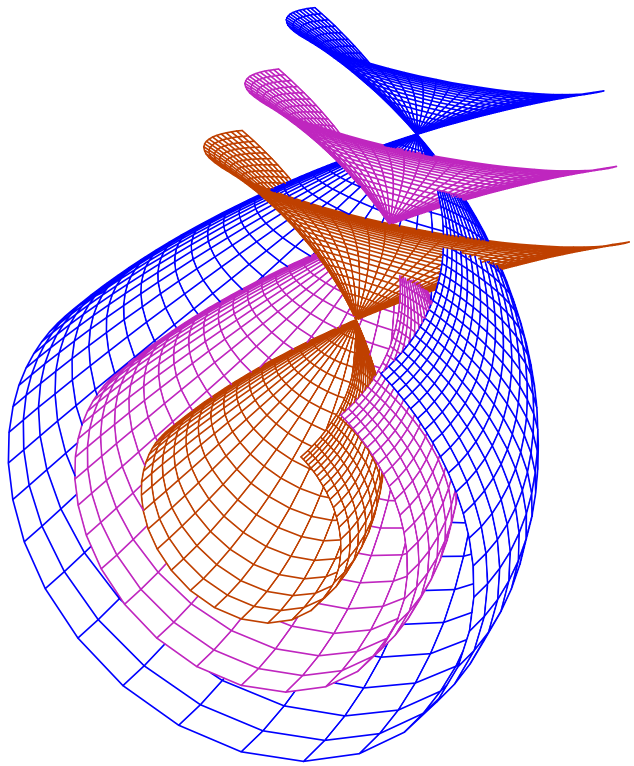
$$E(u, v) : q(u, v) + 2ux + 2vy + (u^2 + v^2 - 1)z = 0.$$

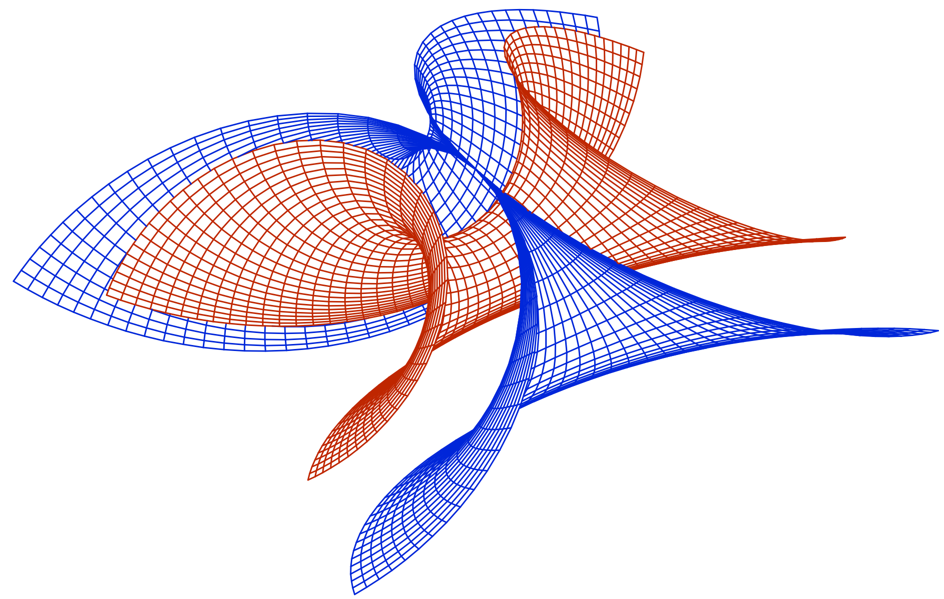
The *isotropic image* is a paraboloid (graph of a quadratic function)

$$E^i(u, v) = (u, v, q(u, v)).$$

The envelope F of planes $E(u, v)$ and all its offsets F_d are parabolic Dupin cyclides (alg.order 3). The real singularities might be different.

Parabolic Dupin Cyclide





Modeling with rational offset surfaces

- Hermite interpolation of G^1-elements (points+tangent planes) with quadratic spline functions (Powell-Sabin elements) over triangles. Rational offset surfaces are composed of par. Dupin cyclides (Pottmann and Peternell).
- Hermite interpolation of G^1-elements (points+tangent planes) and boundary curves by LN-surfaces (Jüttler and Sampoli).
- Interpolation and approximation by rational tensor product B-splines (Schickentanz).

Cones and Cylinders of Rotation

- Let $a(t), b(t), c(t)$ be linear polynomials in t. The spherical image $\mathbf{e} = (e_1, e_2, e_3)$ with

$$e_1 = \frac{2ac}{N}, e_2 = \frac{2bc}{N}, e_3 = \frac{a^2 + b^2 - c^2}{N} \text{ and } N = a^2 + b^2 + c^2,$$

is a circle.

- We choose $h(t) = q(t)/N$ with $q(t)$ as quadratic polynomial. The tangent planes of a cone or cylinder of rot. are

$$E(t) : q(t) + 2acx + 2bcy + (a^2 + b^2 - c^2)z = 0.$$

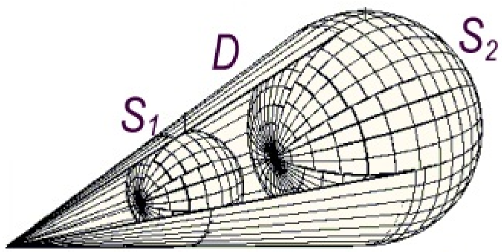
The isotropic image $E^i(t) = (ac, bc, q)(t)$ of $E(t)$ is a (special) conic, a planar section of a paraboloid of rot.

- The envelope F of planes $E(t)$ and all offset surfaces F_d are *cones or cylinders of rotation*. The direction of the generators is $\mathbf{e} \times \mathbf{e}_t$, the vertex of the cone is $E \cap E_t \cap E_{tt}$.

Rational Families of Cones of Rotation

Theorem: Let $D(t)$ be a rational one-par. family of cones of rotation whose spherical image is two-dimensional. The envelope S of $D(t)$ is a rational offset surface.

- A cone of rot. is spanned by two spheres S_1, S_2.



- Two rational families of spheres

$$S_i(t) : (\mathbf{x} - \mathbf{m}_i(t))^2 - r_i(t)^2 = 0, i = 1, 2,$$

where $\mathbf{m}$ and r are rational, determine a rational family of cones of rotation $D(t)$.

The cyclographic image of any *ruled surface* $S \in \mathbb{R}^4$ admits *rational parametrizations*.

Envelopes of Cones of Rotation

- Let $S_i(t) : (\mathbf{x} - \mathbf{m}_i(t))^2 - r_i(t)^2 = 0$, $i = 1, 2$, be two rational families of or. spheres and let $D(t)$ be the corresponding *rational one-par. family of cones of rotation.*

- The isotropic images of the tangent planes of $S_1(t)$ and $S_2(t)$ are two paraboloids of rotation

$$\begin{aligned} F_1 : 2y_3 + (y_1^2 + y_2^2)(r_1 + m_{13}) + 2y_1 m_{11} + 2y_2 m_{12} + r_1 - m_{13} &= 0, \\ F_2 : 2y_3 + (y_1^2 + y_2^2)(r_2 + m_{23}) + 2y_1 m_{21} + 2y_2 m_{22} + r_2 - m_{23} &= 0. \end{aligned}$$

- The intersection $d(t)$ of $F_1(t)$ and $F_2(t)$ is the *isotropic image* of the tangent planes of $D(t)$.

- The curves $d(t) = F_1(t) \cap F_2(t)$ are planar sections of paraboloids of rot. $\Rightarrow d(t)$ are isotropic circles.

Envelopes of Cones of Rotation 2

- The family of curves $d(t)$ is the isotropic image of the envelope F of the cones $D(t)$.
- A parametrization of $d(t)$ is a dual parametrization of F (as set of tangent planes).
- The orthogonal projection $d(t)'$ of $d(t)$ onto the y_1y_2-plane is a family of circles

$$d(t)' : (y_1^2 + y_2^2)(R + M_3) + 2y_1M_1 + 2y_2M_2 + R - M_3 = 0,$$

where $M = m_2 - m_1$, and $R = r_2 - r_1$. The circles $d(t)'$ have rational centers and rational squared radii

$$n(t) = \frac{1}{R + M_3}\,(M_1, M_2)\,,\ s(t) = \frac{M_1^2 + M_2^2 + M_3^2 - R^2}{(R + M_3)^2}.$$

Envelopes of Cones of Rotation 3

- All real components of the envelope F of a rational family of cones of rot. $D(t)$ admit real rational parametrizations.
- The envelope is real if $s(t)$ is non-negative.
- We compute a decomposition

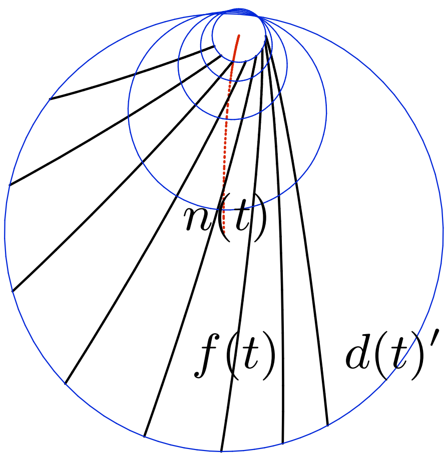
$$s(t) = s_1(t)^2 + s_2(t)^2,$$

  with rational functions $s_1(t)$ and $s_2(t)$.
- Since $s(t) = s_d(t)/(s_n(t)2)$, where $s_d(t)$ is a polynomial, we only have to decompose $s_d(t)$.
- This leads to

$$s_d(t) = \prod_{i=1}^{n} s_0(t - z_i)(t - \bar{z}_i) = (g_1 + \imath g_2)(g_1 - \imath g_2).$$

Envelopes of Cones of Rotation 4

- With $s_1 = g_1/s_n, s_2 = g_2/s_n$, we obtain at first a solution $f(t) = (n_1 + s_1, n_2 + s_2)$ such that $f(t)$ satisfies $d(t)'$ for all t.



- Then a global parametrization $f(t, u)$ which satisfies $d(t)'$ identically for all t and u is computed.
- With

$$-2y_3 = (y_1^2 + y_2^2)(r_1 + m_{13}) + 2y_1 m_{11} + 2y_2 m_{12} + r_1 - m_{13},$$

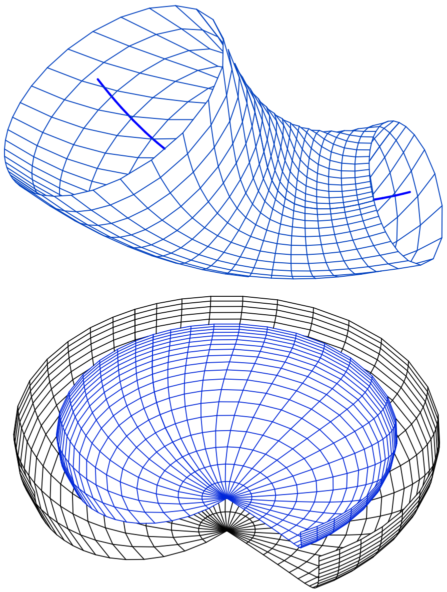
with $y_1 = f_1(t, u), y_2 = f_2(t, u)$. Thus we have found a parametrization of the surface $d(t)$ in I^3, which is a *dual parametrization* of the envelope of $D(t)$.

Corollaries 1

- The envelope S of a rational one-par. family of cones of rotation $D(t)$ is a *rational offset surface*.
- The offsets S_d of rational non-developable ruled surfaces S and all its Laguerre transforms $T(S_d)$ are rational.
- The rationality of offset surfaces is *invariant under Laguerre transformations* (projective transformations of the Blaschke image).

Corollaries 2

- The offsets S_d of rational canal surfaces S admit real rational parametrizations.
- The offsets of non-developable quadrics as *ellipsoids, hyperboloids, paraboloids* and its Laguerre transforms admit *real rational parametrizations*.



Convolution Surfaces

Convolution surfaces - generalized offsets

- Given two surfaces A and B with parametrizations $\mathbf{a}(u, v)$ and $\mathbf{b}(s, t)$ and unit normals $\mathbf{n}_a(u, v)$ and $\mathbf{n}_b(s, t)$, respectively.

- Points $\mathbf{a} \in A$ and $\mathbf{b} \in B$ are said to be corresponding exactly if $\mathbf{n}_a = \mathbf{n}_b$.

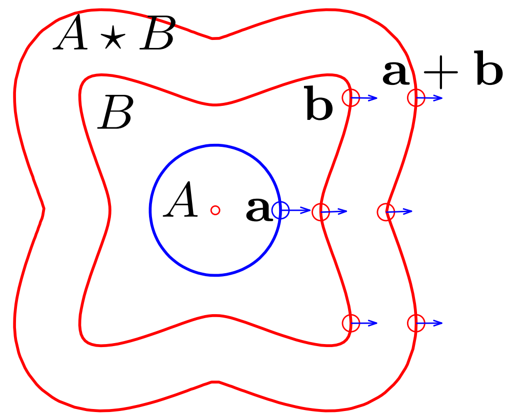
- Convolution surface

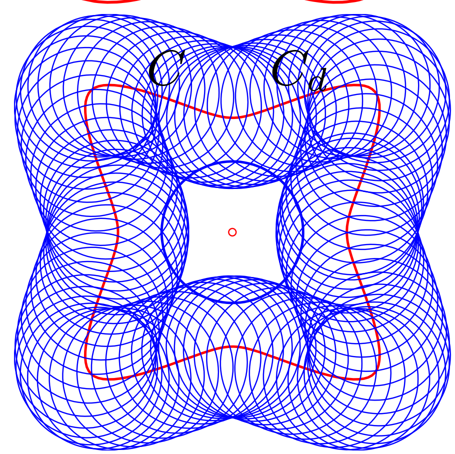
$$C = A \star B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{n}_a = \mathbf{n}_b, \mathbf{a} \in A, \mathbf{b} \in B\}.$$

- Find a reparametrization

$$\phi : (s, t) \mapsto (u, v)$$

such that $\mathbf{n}_a(\phi(s, t)) = \mathbf{n}_b(s, t)$ holds. In general, ϕ is not one-to-one.





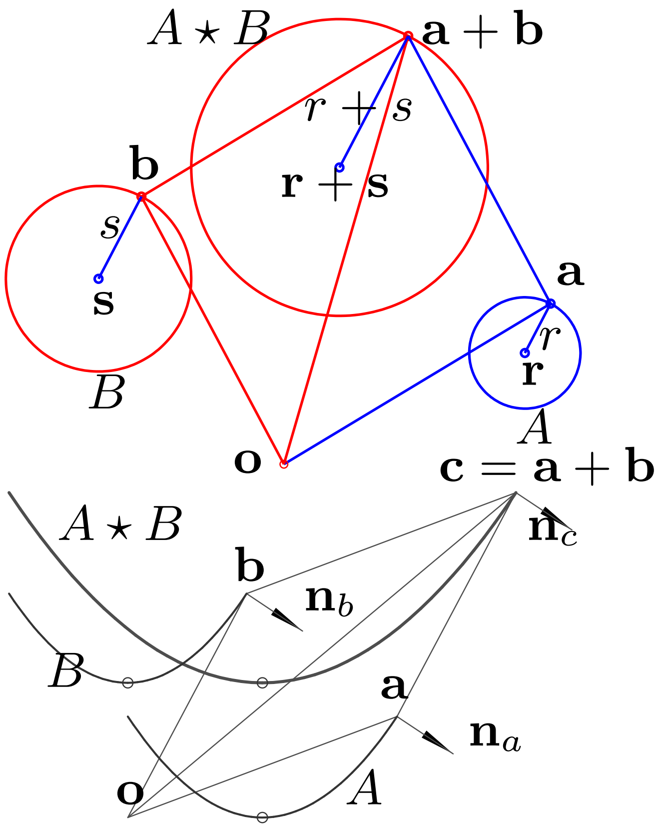
Convolution surfaces of spheres and paraboloids

- The convolution $C = A \star B$ of two spheres

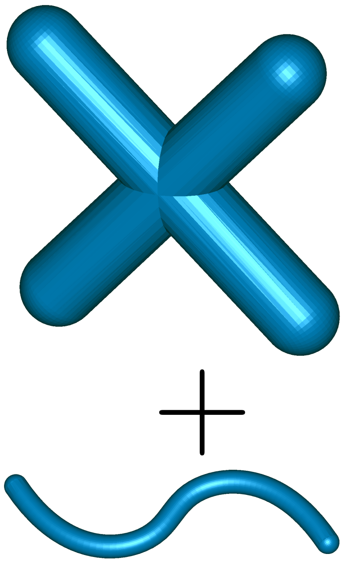
$$A : \mathbf{a}(u,v) = \mathbf{r} + r\mathbf{n}(u,v), \text{ and}$$
$$B : \mathbf{b}(u,v) = \mathbf{s} + s\mathbf{n}(u,v)$$

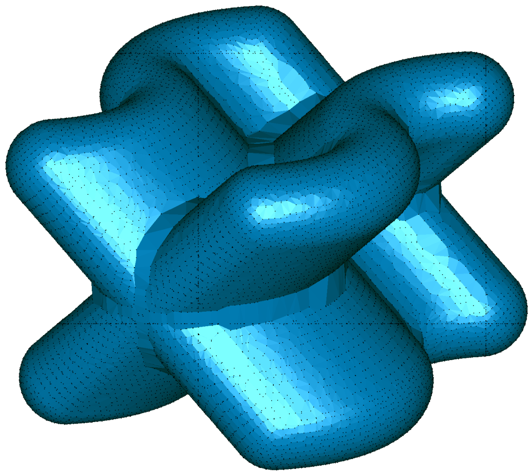
  is a sphere C with $\mathbf{c} = \mathbf{r} + \mathbf{s} + (r+s)\mathbf{n}$.

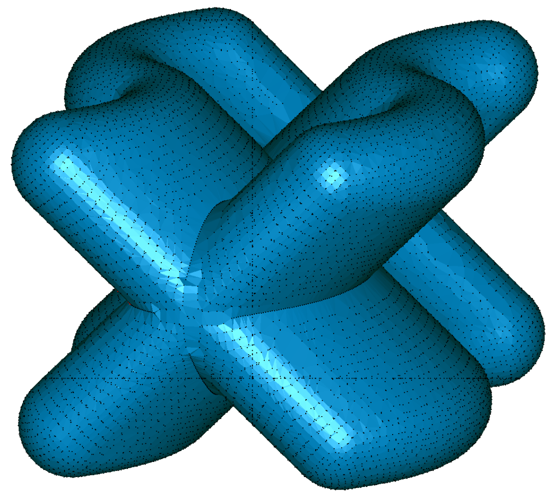
- The convolution $C = A \star B$ of two paraboloids A, B with parallel axes is a paraboloid.



Convolution and Minkowski sum examples

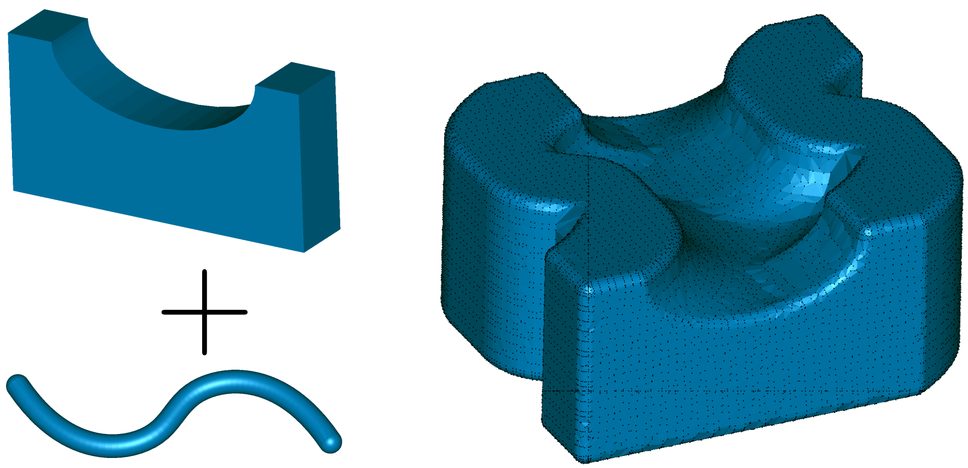


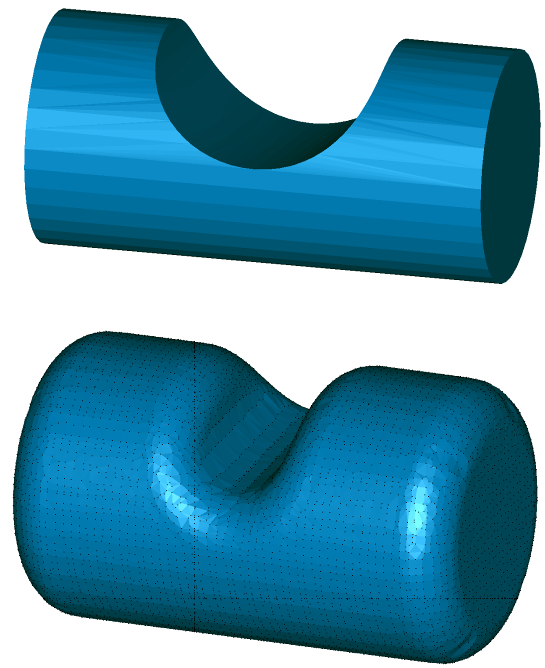




sharpened edges

Minkowski sum, examples 2





Offset of Cad-Model

Quadratic approximation of convolution surfaces

- Given two surfaces A and B

$$\begin{aligned} \mathbf{a}(u,v) &= (u, v, \tfrac{1}{2}(au^2 + cv^2) + O(x^3))^\top, \\ \mathbf{b}(s,t) &= (s, t, \tfrac{1}{2}(\alpha s^2 + 2\beta st + \gamma t^2) + O(x^3))^\top. \end{aligned}$$

- Let $U = (u,v)^\top$, $S = (s,t)^\top$,

$$\mathbf{a}(U) = \begin{bmatrix} U \\ \frac{1}{2}U^\top \cdot D \cdot U + O(x^3) \end{bmatrix}, \text{ and } \mathbf{b}(S) = \begin{bmatrix} S \\ \frac{1}{2}S^\top \cdot E \cdot S + O(x^3) \end{bmatrix}.$$

- Reparametrization (D is invertible)

$$\phi : S \mapsto U = D^{-1} \cdot E \cdot S + O(x^2)$$

- convolution surface C

$$(\mathbf{a}+\mathbf{b})(S) = \begin{bmatrix} F \cdot S + O(x^2) \\ \frac{1}{2}S^\top \cdot E \cdot F \cdot S + O(x^3) \end{bmatrix}, \text{ with } F = I + D^{-1} \cdot E.$$

Regularity of the convolution

The normal curvatures $\kappa^A(\mathbf{v}), \kappa^B(\mathbf{v})$ of A and B in direction of $\mathbf{v}$ are given by

$$\kappa^A(\mathbf{v}) = \mathbf{v}^\top \cdot D \cdot \mathbf{v},\ \kappa^B(\mathbf{v}) = \mathbf{v}^\top \cdot E \cdot \mathbf{v}.$$

$\mathbf{rk}(F) = 2$: The convolution surface C is regular.

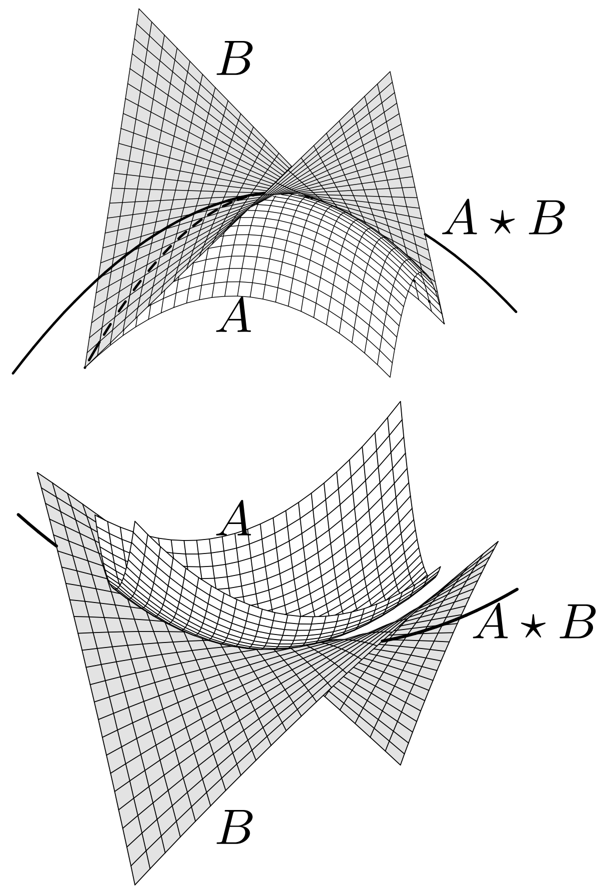
$\mathbf{rk}(F) = 1$: There exists a vector $\mathbf{v} \neq \mathbf{o} = (0,0)^\top$ with

$$F \cdot \mathbf{v} = \mathbf{o} = \mathbf{v} + D^{-1} \cdot E \cdot \mathbf{v}, \Longrightarrow$$

$$-D \cdot \mathbf{v} = E \cdot \mathbf{v} \Longrightarrow -\kappa^A(\mathbf{v}) = \kappa^B(\mathbf{v}).$$

The convolution C is singular.

$\mathbf{rk}(F) = 0$: For any vector $\mathbf{v}$ we have $F \cdot \mathbf{v} = \mathbf{o}$. The convolution is singular and $-\kappa^A(\mathbf{v}) = \kappa^B(\mathbf{v})$.



Convolution of Paraboloids

- A is a paraboloid with $\mathbf{a} = (u, v, u^2 + cv^2)$, $c = \pm 1$, and normals $\mathbf{n}_a = (-2u, -2v, 1)$
- B is a rational surface with parametrization $\mathbf{b}(s, t)$ and normals $\mathbf{n}_b(s, t)$
- The correspondence $\mathbf{n}_a = \lambda \mathbf{n}_b$ (no normalization) gives the reparametrization

$$\phi : (s, t) \mapsto (u, v) = \left(\frac{-n_1}{2n_3}, \frac{-n_2}{2cn_3} \right)$$

- The convolution of a paraboloid and a rational surface is a *rational surface*.

Convolution of Paraboloids

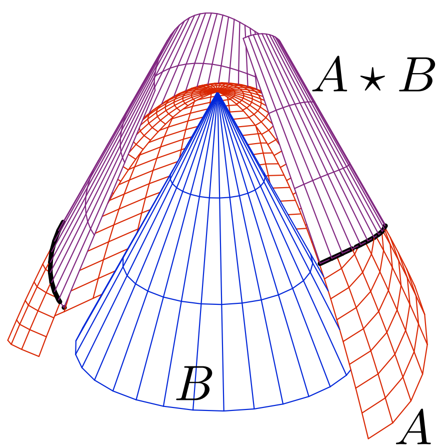
- The convolution $C = A \star B$ is given by

$$\mathbf{c}(s,t) = \left(\frac{-n_1}{2n_3} + b_1, \frac{-n_2}{2cn_3} + b_2, \frac{cn_1^2 + n_2^2}{4cn_3^2} + b_3 \right).$$

- The reparametrization is regular if

$$\det(J\phi) = \frac{\det(\mathbf{n}, \mathbf{n}_s, \mathbf{n}_t)}{4cn_3^2} = \frac{D^2 K}{4cn_3^2}$$

where K is the Gaussian curvature and D is the determinant of the first fundamental form (metric) of B.



- If B is a developable surface ($K = 0$), then $A \star B$ is a developable surface too.

A generalization of paraboloids

An LN-surface (Jüttler,2000) $A : \mathbf{a}(u,v)$ is a parametrized surface with normal vectors $\mathbf{n}(u,v) = (u,v,1)$ and tangent planes

$$T(u,v) : f(u,v) + ux + vy + z = 0,$$

where $f(u,v)$ is a polynomial or a rational function.

- Point representation: $\mathbf{a}(u,v) = (-f_u, -f_v, -f + uf_u + vf_v)^\top$
- The convolution $C = A \star B$ of an LN-surface A and a general rational surface B is rational.
- The reparametrization is the same as for paraboloids.

Convolution of ruled surfaces

Mühlthaler, Pottmann, 2003:

Skew Ruled surfaces A and B with $\mathbf{x}(u,\alpha) = \mathbf{a}(u) + \alpha\mathbf{r}(u)$, and $\mathbf{y}(v,\beta) = \mathbf{b}(v) + \beta\mathbf{s}(v)$. Find $\alpha(u,v), \beta(u,v)$ with $\mathbf{n}_x \parallel \mathbf{n}_y \parallel \gamma\mathbf{r}\times\mathbf{s}$.

- Find $\alpha(u,v), \beta(u,v)$ with $\mathbf{n}_x \parallel \mathbf{n}_y$.
- $\mathbf{n}_x = (\mathbf{a}_u + \alpha\mathbf{r}_u)\times\mathbf{r}$, $\mathbf{n}_y = (\mathbf{b}_v + \beta\mathbf{s}_v)\times\mathbf{s}$.
- Reparametrization

$$\alpha = -\frac{\det(\mathbf{a}_u, \mathbf{r}, \mathbf{s})}{\det(\mathbf{r}_u, \mathbf{r}, \mathbf{s})}, \beta = -\frac{\det(\mathbf{b}_v, \mathbf{r}, \mathbf{s})}{\det(\mathbf{s}_v, \mathbf{r}, \mathbf{s})}.$$

- The convolution of two rational ruled surfaces is a rational surface.

Canal Surfaces

- A canal surface A is the envelope of a one-par. family of spheres

$$R(u) : (\mathbf{x} - \mathbf{m}(u))^2 - r(u)^2 = 0,$$

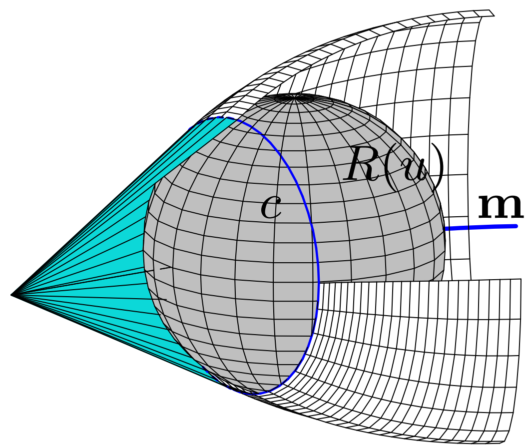
with centers $\mathbf{m}(u)$ and radii $r(u)$.

- A contains the circles

$$c(u) = R(u) \cap \dot{R}(u),$$

where $\dot{R}(u) : (\mathbf{x} - \mathbf{m}) \cdot \dot{\mathbf{m}} + r\dot{r} = 0.$

- Circles c are real if $\dot{\mathbf{m}}^2 - \dot{r}^2 \geq 0.$



Convolution of canal surfaces

- Canal surfaces

$$\begin{aligned} A\ldots \quad & R(u) : (\mathbf{x} - \mathbf{m}(u))^2 - r(u)^2 = 0, \\ B\ldots \quad & S(u) : (\mathbf{x} - \mathbf{n}(u))^2 - s(u)^2 = 0. \end{aligned}$$

- The convolution of each pair of spheres is a sphere

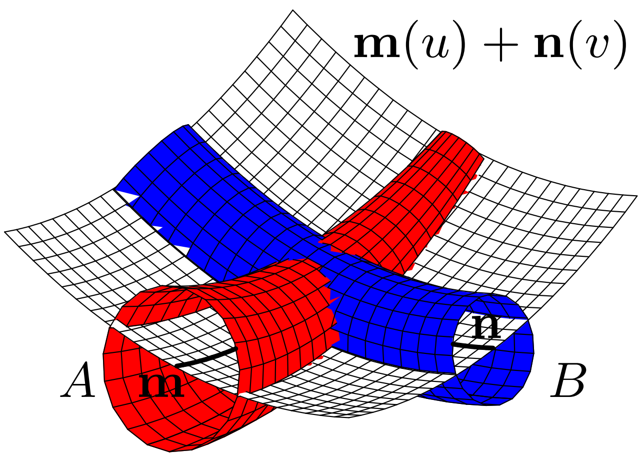
$$T(u, v) : (\mathbf{x} - (\mathbf{m}(u) + \mathbf{n}(v))^2 - (r(u) + s(v))^2 = 0.$$

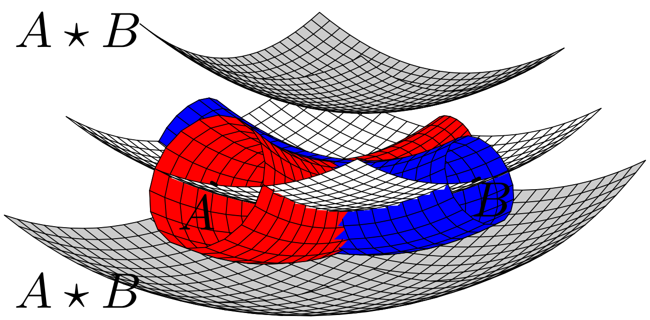
The convolution $C = A \star B$ is the envelope of the two-par. family of spheres $T(u, v)$. This leads to

$$\begin{aligned} T(u, v) : (\mathbf{x} - (\mathbf{m} + \mathbf{n})^2 - (r + s)^2 \quad & = 0, \\ T_u(u, v) : (\mathbf{x} - (\mathbf{m} + \mathbf{n}) \cdot \mathbf{m}_u + (r + s) r_u \quad & = 0, \\ T_v(u, v) : (\mathbf{x} - (\mathbf{m} + \mathbf{n}) \cdot \mathbf{n}_v + (r + s) s_v \quad & = 0. \end{aligned}$$

Convolution of canal surfaces

- The convolution of two canal surfaces $A = R(u)$ and $B = S(v)$ is the cyclographic image of the *translational surface* $T^c(u, v) \in \mathbb{R}^4$ formed by the curves $R(u)^c$ and $S^c(v)$.





The isotropic image of the convolution

- Let $P : \mathbf{p}(u,v)$ and $Q : \mathbf{q}(s,t)$ be two parametrized surfaces.
- Consider P, Q as families of tangent planes

$$E : e_0 + \mathbf{e}^\top \cdot \mathbf{x} = 0, F : f_0 + \mathbf{f}^\top \cdot \mathbf{x} = 0$$

with $\|\mathbf{f}\| = \|\mathbf{e}\| = 1$.

- The surfaces P, Q possess isotropic images

$$P^i : E^i = \frac{1}{1-e_3}(e_1, e_2, e_0), Q^i : F^i = \frac{1}{1-f_3}(f_1, f_2, f_0).$$

- Points $\mathbf{p} \in P$ and $\mathbf{q} \in Q$ are corresponding exactly if the tangent planes

$$E : e_0 + \mathbf{e}^\top \cdot \mathbf{x} = 0, F : f_0 + \mathbf{f}^\top \cdot \mathbf{x} = 0$$

are parallel. (Coinciding unit normals $\mathbf{e} = \mathbf{f}$).

- The convolution $R = P \star Q$ is formed by $\mathbf{p} + \mathbf{q}$.

The isotropic image of the convolution

- The tangent plane G of R at $\mathbf{r} = \mathbf{p} + \mathbf{q}$ is

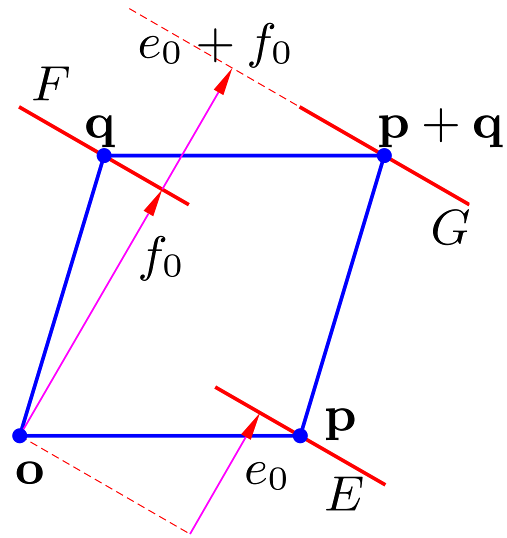
  $$T_r : e_0 + f_0 + \mathbf{g}^\top \cdot \mathbf{x} = 0, \text{ with } \mathbf{g} = \mathbf{e} = \mathbf{f}.$$

  This implies that the *Blaschke image* of C is

  $$R^b = P^b + Q^b = (\mathbf{g}, e_0 + f_0) \in B \subset \mathbb{R}^4.$$

- The *isotropic image* of $R = P \star Q$ is

  $$R^i = \frac{1}{1 - g_3}(g_1, g_2, e_0 + f_0).$$



Conclusion

We have presented some results concerning rational offset surfaces and related problems.

- We used concepts from the geometry of spheres and oriented planes.
- The different models (cyclographic model, Blaschke cylinder, isotropic model) help to understand geometric properties in a better way.

Thank you for your attention!