Sphere-geometric aspects of bisector surfaces

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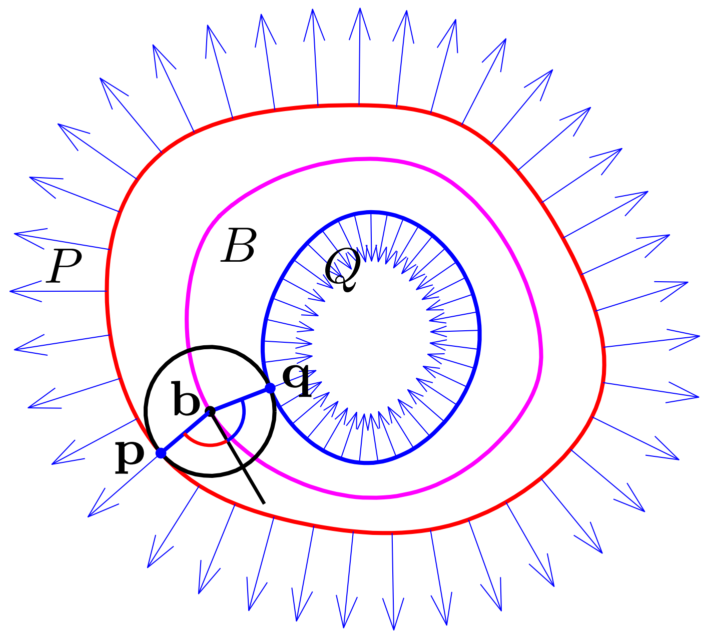
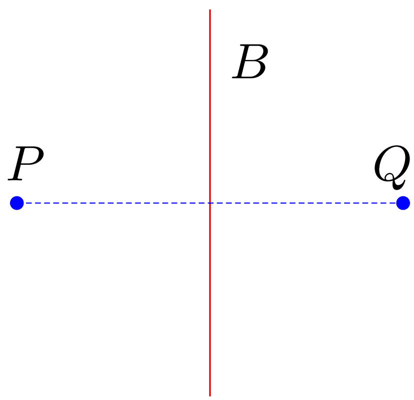
Definition

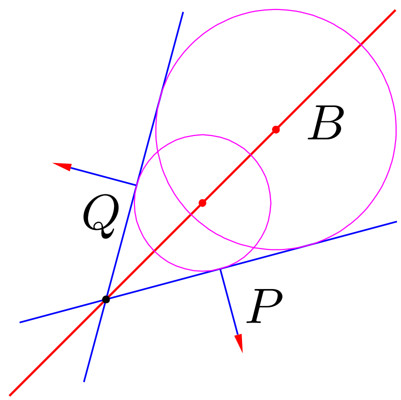
- Smooth oriented objects P and Q in $\mathbb{R}^3$ (points, curves or surfaces)
- The bisector surface B of P and Q is the locus of *centers of spheres* tangent both to P and Q.
- Tangents of B are the bisectors of tangents of P and Q at corresponding points $\mathbf{p}$ and $\mathbf{q}$

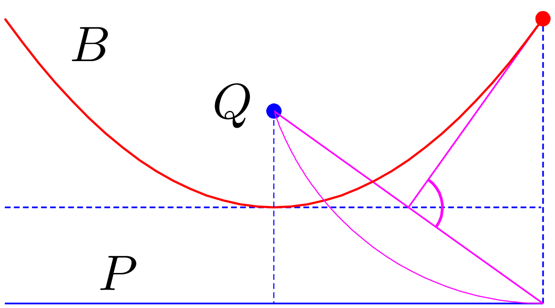


Elementary Examples in $\mathbb{R}^2$

object 1	object 2	bisector
point	point	line of symmetry
or. line	or. line	line of symmetry
point	line	parabola





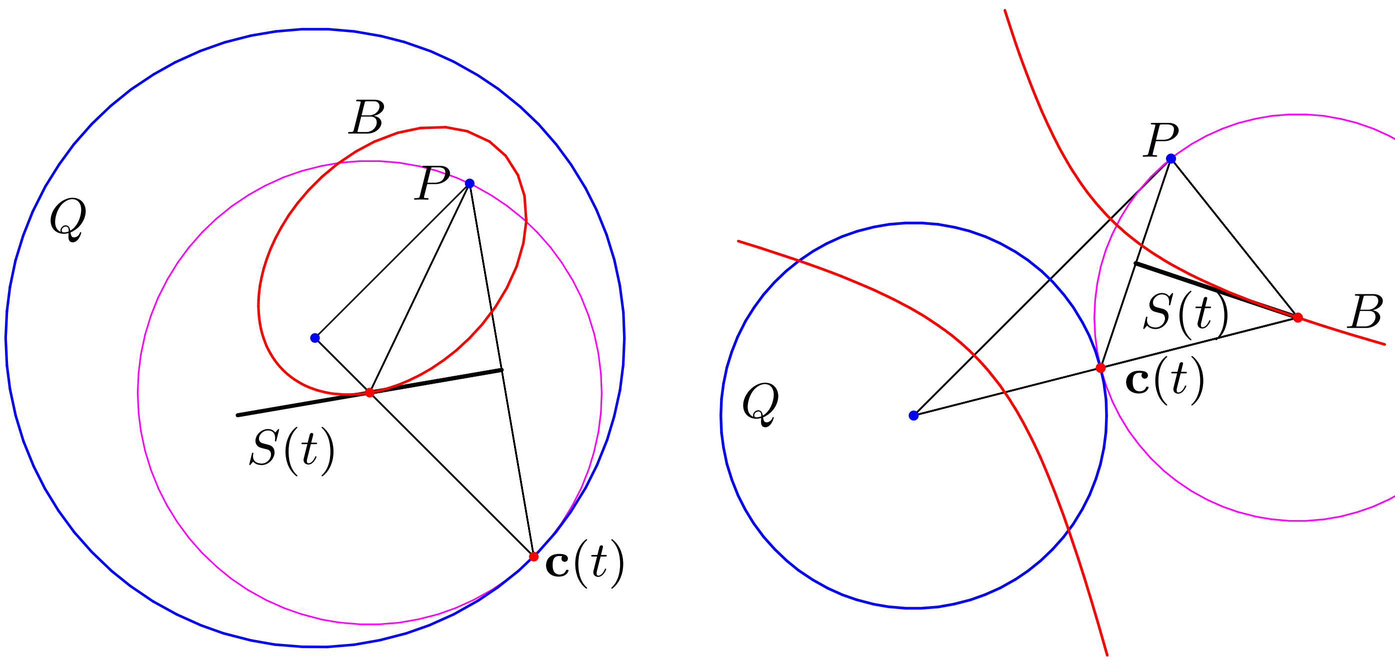


Elementary Examples in $\mathbb{R}^2$

The bisector B of a point P and a circle Q is a conic.

Ellipse: P is inside Q

Hyperbola: P is outside Q



Point-Curve Bisector in $\mathbb{R}^2$

The bisector B of a point $P = (0,0)^\top$ and a curve $Q = \mathbf{c}(t)$ in $\mathbb{R}^2$ is the envelope of all lines of symmetry

$$S(t) : \mathbf{c}(t)^\top \cdot (\mathbf{x} - \frac{1}{2}\mathbf{c}(t)) = 0.$$

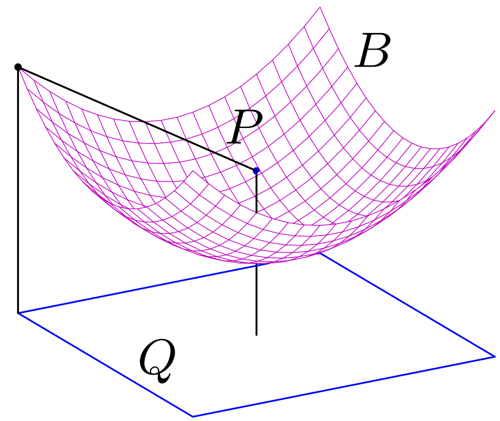
By intersecting Q's normals with the lines of symmetry the bisector B admits the parametrization

$$\mathbf{b}(t) = \mathbf{c}(t) - \frac{1}{2}\frac{\mathbf{c}(t)^\top \cdot \mathbf{c}(t)}{\mathbf{c}(t)^\top \cdot \mathbf{n}(t)}\mathbf{n}(t),$$

where $\mathbf{n} = (-\dot{c}_2, \dot{c}_1)$ is a normal vector of Q.

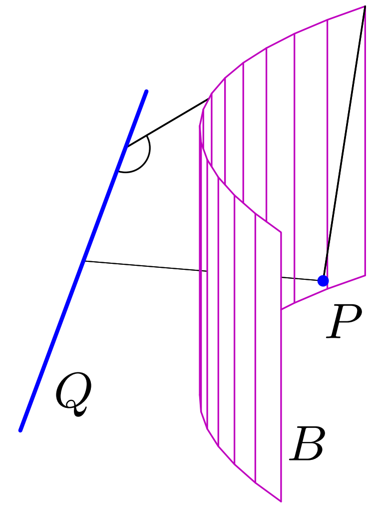
Elementary examples of bisectors in $\mathbb{R}^3$

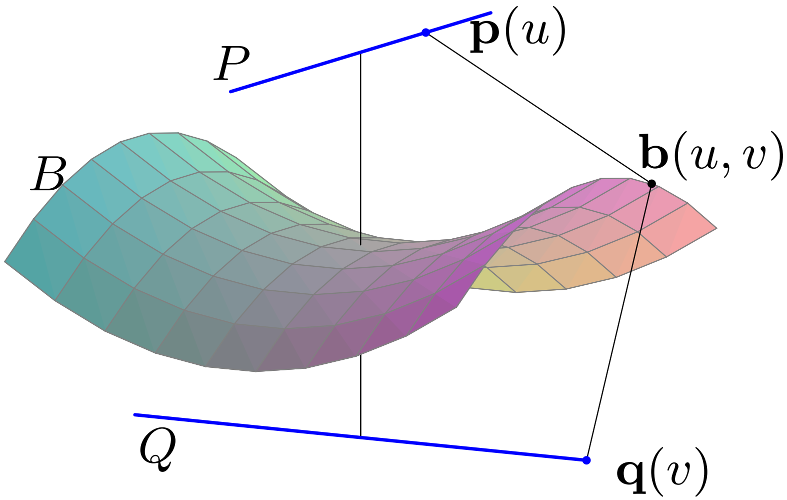
point	point	plane of symmetry
point	plane	paraboloid of revolution
or. plane	or. plane	plane of symmetry

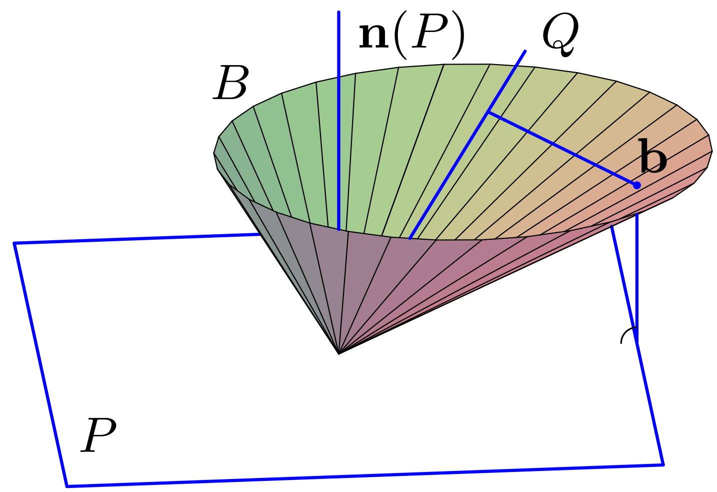


Elementary examples of bisectors in $\mathbb{R}^3$

point	line	parabolic cylinder
line	plane	quadratic cone
line	line	hyperbolic paraboloid





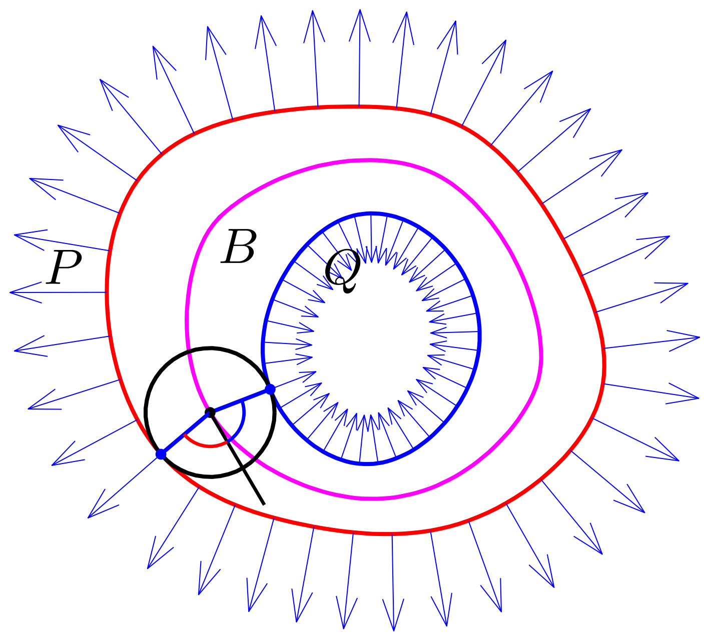


Offset-invariance of bisectors

- Let P_d and Q_d be the offset surfaces of P and Q at oriented distance d.
- The bisector B of P_d and Q_d is the bisector of P and Q.

Geometrical optics

- Consider two smooth surfaces P and Q and their bisector surface B.
- An illumination L_P orthogonal to P is reflected at B to an illumination $\bar{L}_P = L_Q$ perpendicular to Q.
- The bisector B is a mirror surface in that sense.

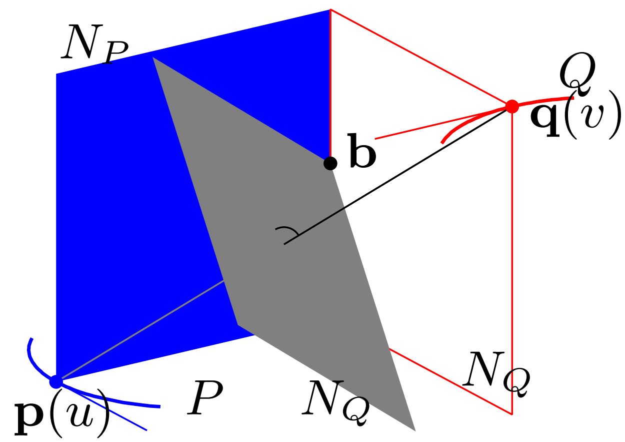


Curve-curve bisector in $\mathbb{R}^3$

- Let P and Q be two curves in $\mathbb{R}^3$ with parametrizations $\mathbf{p}(u)$ and $\mathbf{q}(v)$.
- The bisector surface B is constructed by

$$\mathbf{b}(u,v) = N_P(u) \cap N_Q(v) \cap S(u,v)$$

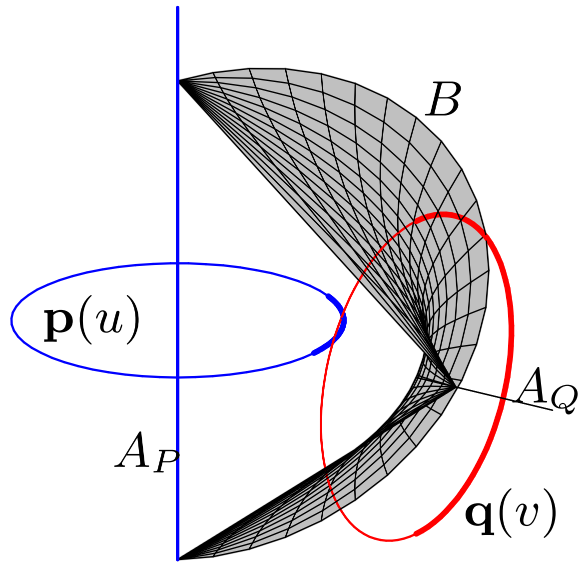
where $N_P \perp P$ and $N_Q \perp Q$ and S is the plane of symmetry of P and Q.



- The bisector construction is *linear*.
- Rational curves P, Q possess a rational bisector B.

Circle-circle bisector in $\mathbb{R}^3$

- Let P and Q be two circles in $\mathbb{R}^3$ with parametrizations $\mathbf{p}(u)$ and $\mathbf{q}(v)$.
- The bisector surface B carries two families of conics in planes through the axes A_P and A_Q of P and Q.
- B is a double Blutel conic surface (Degen, '64, '65, '86, '98).



Some related work

Choi,J.J., Kim,M-S. and Elber, G.: Computing Planar Bisector Curves Based on Developable SSI.

Dutta, D. and Hoffmann, C. On the skeleton of simple CSG objects.

Elber, G. and Kim, M-S.: The Bisector Surface of Rational Space Curves.

Elber, G. and Kim, M-S.: Rational bisectors of CSG Primitives.

Farouki, R.T. and Johnstone, J.K.: The bisector of a point and a plane parametric curve.

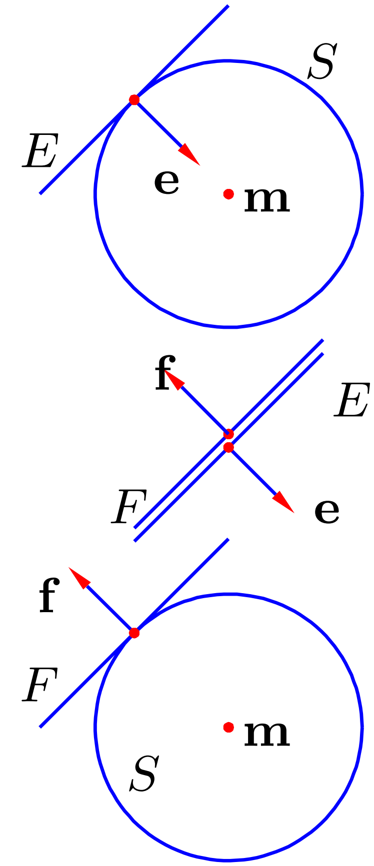
Farouki, R.T. and Johnstone, J.K.: Computing point/curve and curve/curve bisectors.

Farouki, R.T. and Ramamurthy, R.: Specified–Precision Computation of Curve/Curve Bisectors.

Hoffmann, C.: A dimensionality paradigm for surface interrogations,

Literature on *medial axis*, *skeleton* and *voronoi diagrams*.

3D-Sphere Geometry



- An oriented (or.) sphere S in $\mathbb{R}^3$ is given by

$$S : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0,$$

and the orientation is determined by or. normals. Points are considered as spheres of radius 0.

- An oriented plane E in $\mathbb{R}^3$ is given by

$$E : e_0 + e_1 x_1 + e_2 x_2 + e_3 x_3 = e_0 + \mathbf{e} \cdot \mathbf{x} = 0.$$

We always assume that $\mathbf{e}^2 = 1$.

- E and S are said to be in oriented contact iff

$$e_0 + e_1 m_1 + e_2 m_2 + e_3 m_3 + r = e_0 + \mathbf{e} \cdot \mathbf{m} + r = 0, \mathbf{e}^2 = 1.$$

Lie-sphere geometry in $\mathbb{R}^3$

- A *Lie-sphere* is either an *or. sphere* or an *or. plane* or a point in $\mathbb{R}^3$.
- A point-model of *Lie-sphere geometry* is given by the quadric model $\mathcal{L} \in P^5$,
$$\mathcal{L} : 2X_0X_5 + X_1^2 + X_2^2 + X_3^2 - X_4^2 = 0.$$
- Lie-spheres X in $\mathbb{R}^3$ are mapped onto points $\mathbf{X} \in \mathcal{L} \subset P^5$ by the correspondence

$$\begin{array}{ll} \text{sphere } (\mathbf{m}, r) & \mapsto \mathbf{M} = (1, \mathbf{m}, r, -\frac{1}{2}(\mathbf{m}^2 - r^2))\mathbb{R}, \\ \text{point } (\mathbf{p}) & \mapsto \mathbf{P} = (1, \mathbf{p}, 0, -\frac{1}{2}\mathbf{p}^2)\mathbb{R}, \\ \text{plane } (e_0, \mathbf{e}) & \mapsto \mathbf{E} = (0, \mathbf{e}, -1, e_0)\mathbb{R}, \text{ with } \|\mathbf{e}\| = 1. \end{array}$$

- Points $\mathbf{x} \in \mathbb{R}^3$ are mapped onto points $\mathbf{X}$ in $X_4 = 0$.
- Planes E are mapped onto points $\mathbf{E} \in X_0 = 0$.

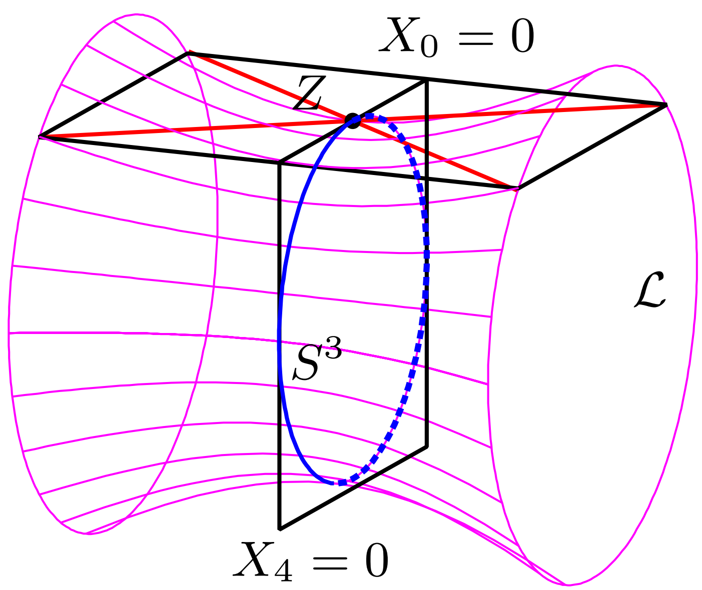
Relations to Laguerre and Möbius geometry

- $\mathbf{Z} = (0,0,0,0,0,1)\mathbb{R}$ is not an image of a sphere, plane or point of $\mathbb{R}^3$ but is considered as 'point' ∞ (one-point compactification).

- The quadratic cone $X_0 = 0, X_1^2 + X_2^2 + X_3^2 - X_4^2 = 0$ consists of images of oriented planes and is referred to as *Blaschke cone (cylinder)*.

- The quadric $\mathcal{L} \cap X_4 = 0$ is projectively equivalent to S^3 and is a point model of the *Möbius geometry* in $\mathbb{R}^3$.

- A line $\in \mathcal{L}$ corresponds to a pencil of touching spheres or parallel planes.



Lie-transformations and oriented contact

- A *bijective mapping* in the set of Lie-spheres which preserves *oriented contact* of spheres is called a *Lie-transformation*.
- The Lie-transformations appear in the quadric model as projective transformations $\mathcal{L} \longrightarrow \mathcal{L}$.
- Lie-transformations are not necessarily point-preserving.
- Point-preserving Lie-transformations are *Möbius transformations*.
- Plane-preserving Lie-transformations are *Laguerre transformations*.

Oriented contact of Lie-spheres

- Polar form of $\mathcal{L}$

$$\langle \mathbf{X}, \mathbf{Y} \rangle = X_0 Y_5 + X_5 Y_0 + X_1 Y_1 + X_2 Y_2 + X_3 Y_3 - X_4 Y_4,$$

- Two Lie-spheres X, Y are in oriented contact exactly if $\langle \mathbf{X}, \mathbf{Y} \rangle = 0$, and $\langle \mathbf{X}, \mathbf{X} \rangle = 0, \langle \mathbf{Y}, \mathbf{Y} \rangle = 0$.

- Any oriented plane is in contact with ∞,

$$\langle \mathbf{E}, \mathbf{Z} \rangle = 0.$$

- Oriented spheres or points are never in contact with ∞,

$$\langle \mathbf{M}, \mathbf{Z} \rangle \neq 0.$$

General bisector construction

- Surfaces P, Q in $\mathbb{R}^3$ with images $\mathbf{P}(u,v)$ and $\mathbf{Q}(s,t)$ in $\mathcal{L} \subset P^5$.
- X is tangent to P and Q exactly if

$$\langle \mathbf{P}, \mathbf{X} \rangle = 0, \quad \langle \mathbf{P}_u, \mathbf{X} \rangle = 0, \quad \langle \mathbf{P}_v, \mathbf{X} \rangle = 0,$$
$$\langle \mathbf{Q}, \mathbf{X} \rangle = 0, \quad \langle \mathbf{Q}_s, \mathbf{X} \rangle = 0, \quad \langle \mathbf{Q}_t, \mathbf{X} \rangle = 0,$$

  and $\langle \mathbf{X}, \mathbf{X} \rangle = 0$ ($\mathbf{X} \in \mathcal{L}$) holds, ($X_0 \neq 0$).

- Eliminating u, v and s, t gives us three equations

$$F(\mathbf{X}) = 0, G(\mathbf{X}) = 0, \langle \mathbf{X}, \mathbf{X} \rangle = 0.$$

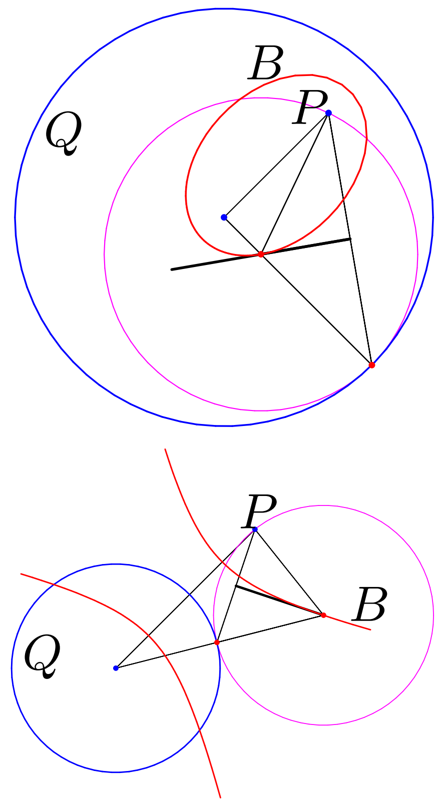
- Bisector pre-image $\mathbf{B}$ in $\mathcal{L}$ is two-dimensional.
- The bisector B is the projection of $\mathbf{B}$ onto $X_4 = X_5 = 0$.

Bisectors of Lie-spheres

- Lie-spheres $P, Q \neq \infty$ in $\mathbb{R}^3$ which are not in oriented contact.
- The bisector pre-image $\mathbf{B}$ is the intersection

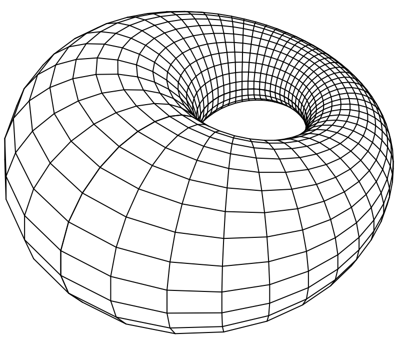
$$\mathcal{L} \cap P^3 : \langle \mathbf{P}, \mathbf{X} \rangle = \langle \mathbf{Q}, \mathbf{X} \rangle = 0.$$

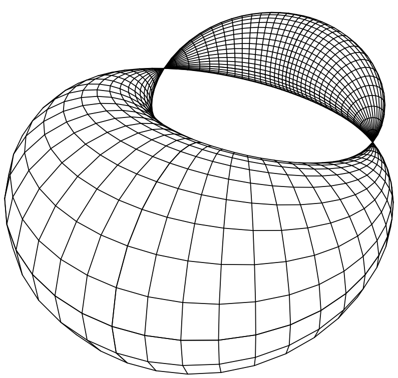
- B is a quadric (of revolution) of index 0 in general.
- B is planar if P, Q are spheres of same radius or oriented planes.

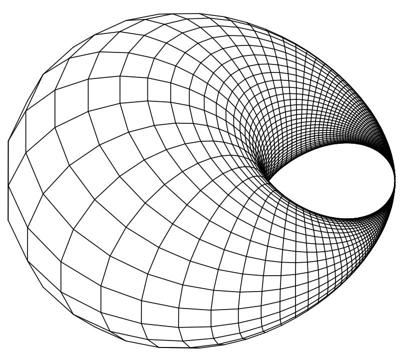


Dupin cyclides

- Lie-spheres P, Q, R and image points $\mathbf{P}, \mathbf{Q}, \mathbf{R}$,
- Plane $E = \mathbf{P} \vee \mathbf{Q} \vee \mathbf{R}$,
- Plane $F = \langle \mathbf{P}, \mathbf{X} \rangle = \langle \mathbf{Q}, \mathbf{X} \rangle = \langle \mathbf{R}, \mathbf{X} \rangle = 0$,
- $E \perp F$: $\mathbf{X} \in E$ and $\mathbf{Y} \in F \Longrightarrow \langle \mathbf{X}, \mathbf{Y} \rangle = 0$
- Conic $\mathbf{C} = E \cap \mathcal{L}$ and conic $\mathbf{D} = F \cap \mathcal{L}$.
- Family of spheres C corr. to $\mathbf{C}$ envelope a *Dupin cyclide* Φ.
- Family of spheres D corr. to $\mathbf{B} = \mathbf{D}$ envelope Φ too.
- Φ admits two generations as canal surface.

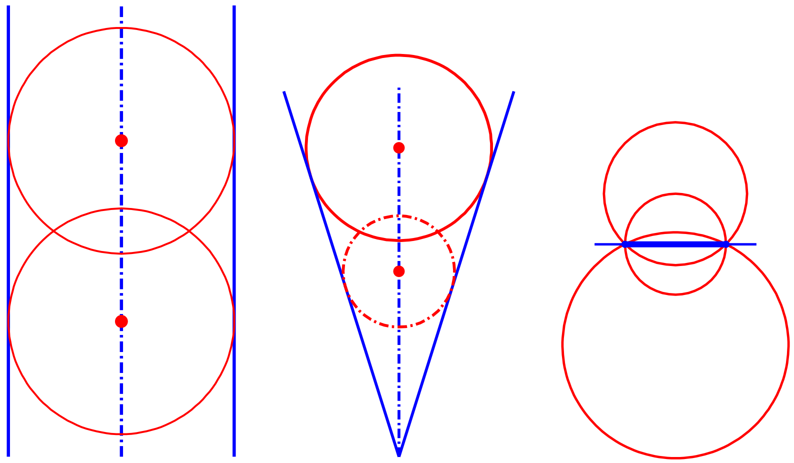






Dupin cyclides– special cases

- Or. planes P, Q and $R \Longrightarrow C$ are tangent planes of a cone (cylinder) of revolution Φ.
- D is the family of spheres touching Φ with centers at Φ's axis.



Lie-canal surfaces

- A one-parameter family of Lie-spheres $C(u)$ is called a Lie-canal surface and corresponds to a curve $\mathbf{C}(u) \in \mathcal{L}$.

- The envelope Φ of $C(u)$ is either a canal surface, a developable surface if $\mathbf{C} \in X_0 = 0$ or a curve if $\mathbf{C} \in X_4 = 0$.

Bisectors of Lie-canal surfaces

- Lie canal surfaces P and Q with image curves $\mathbf{P}(u), \mathbf{Q}(v) \in \mathcal{L}$.
- Contact conditions

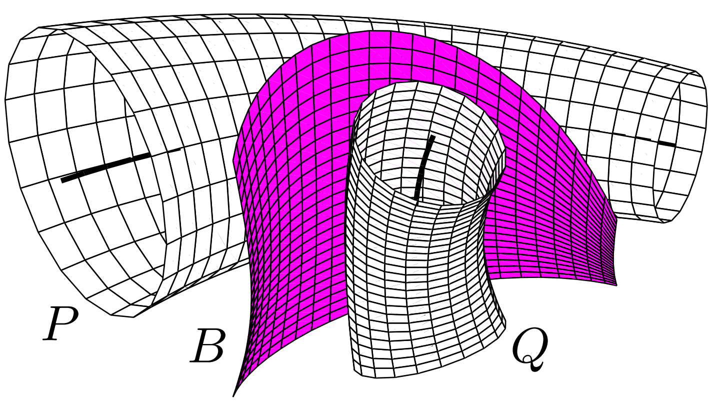
$$\langle \mathbf{P}, \mathbf{X} \rangle = 0, \quad \langle \mathbf{P}_u, \mathbf{X} \rangle = 0,$$
$$\langle \mathbf{Q}, \mathbf{X} \rangle = 0, \quad \langle \mathbf{Q}_v, \mathbf{X} \rangle = 0,$$

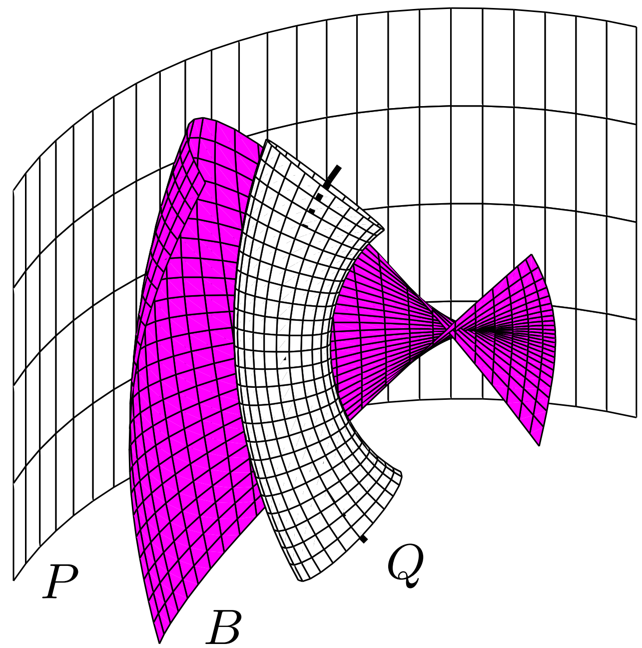
  define a two parameter family of lines $\mathbf{G}(u, v)$ in P^5.

- Bisector pre-image $\mathbf{B}(u, v) = \mathbf{G}(u, v) \cap \mathcal{L}$.

Bisector of two Lie-canal surfaces

- The bisector surface of two Lie canal surfaces $P, Q \in \mathbb{R}^3$ can be constructed in an elementary way (square roots).
- The construction is linear if P and Q are both curves or developable surfaces.





Bisector of Lie-sphere and general surface

- General surface P and Lie-sphere Q with images $\mathbf{P}(u,v)$ and $\mathbf{Q}$.
- Contact conditions

$$\langle \mathbf{P}, \mathbf{X} \rangle = 0, \quad \langle \mathbf{P}_u, \mathbf{X} \rangle = 0, \quad \langle \mathbf{P}_v, \mathbf{X} \rangle = 0,$$
$$\langle \mathbf{Q}, \mathbf{X} \rangle = 0.$$

define a two parameter family of lines $\mathbf{G}(u,v)$ in P^5.

- Bisector pre-image $\mathbf{B}(u,v) = \mathbf{G}(u,v) \cap \mathcal{L}$.

Bisector of Lie-sphere and general surface

- The bisector B of a general surface $P \in \mathbb{R}^3$ and a Lie-sphere Q can be constructed in an elementary way.
- If Q is a point or an or. plane the construction is linear.
- If P is a rational offset surface, B is rational too. (Q is shrunk to a point)

Summary

We have presented a general concept for the geometric interpretation of bisector surfaces of two objects in $\mathbb{R}^3$.

- Points, or. planes and spheres can be treated similarly.
- Elementary construction of the bisector of two Lie canal surfaces.
- Elementary construction of the bisector of a Lie sphere and a general surface.

Thank you for your attention!