Rational Offset Surfaces and Envelopes of Spheres

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Overview

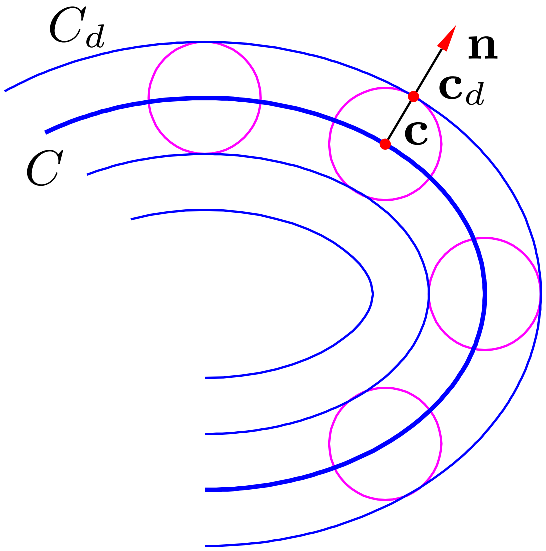
1. Offsets of curves and surfaces
2. Rational Offset Surfaces (PN–Surfaces)
3. Rational Parameterizations
4. Two-par. Families of spheres
5. PN–Surfaces and Two-par. Families of Spheres
6. Conclusion

Offset Curves

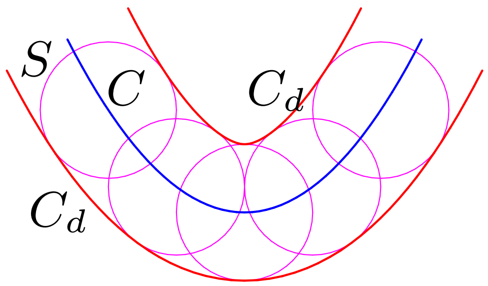
- Rational curve C with parametrization $\mathbf{c}(t) = (c_1, c_2)(t)$
- One-sided oriented offset curve C_d at distance d admits the parametrization

$$\mathbf{c}_d(t) = \mathbf{c}(t) + d\mathbf{n}(t),$$

  with $\mathbf{n}$ as *unit normal vector* of $\mathbf{c}$.
- Which rational input curves C possess rational offset curves C_d?



Offsets as Envelopes



Offsets (two-sided) are *envelopes of circles*

$$S(t) : (\mathbf{x} - \mathbf{c}(t)) \cdot (\mathbf{x} - \mathbf{c}(t)) = d^2$$

centered at C.

Elimination of the parameter t from

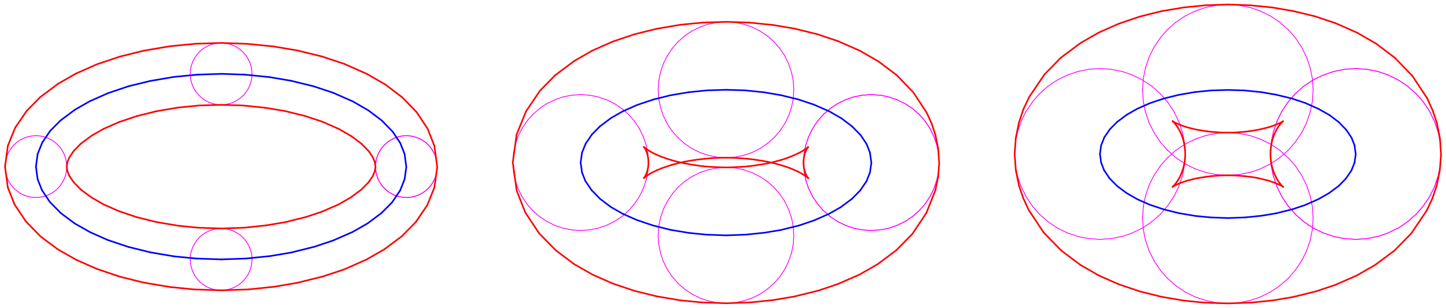
$$\begin{aligned} F(\mathbf{x}, t) : (\mathbf{x} - \mathbf{c}(t)) \cdot (\mathbf{x} - \mathbf{c}(t)) &= d^2, \\ F_t(\mathbf{x}, t) : (\mathbf{x} - \mathbf{c}(t)) \cdot \dot{\mathbf{c}}(t) &= 0, \end{aligned}$$

leads to an implicit representation $G(\mathbf{x}) = 0$ of the offset C_d.

- Offset C_d is rational exactly if its genus $g = 0$.
- Objective: Construction of rational parametrizations of rational offset curves and surfaces.

Some Examples

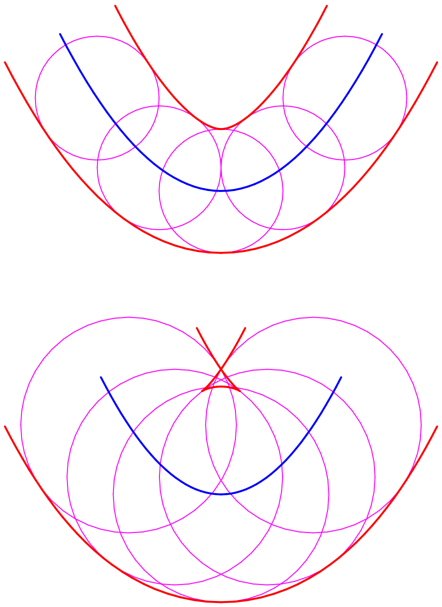
- Offsets C_d of an ellipse C are non-rational algebraic curves of order 8.



- The offsets C_d of a parabola C are rational of order 6. But the standard parametrization $\mathbf{c}(t) = (t, 1/2t^2)$ does not result in a rational parametrization of C_d, because of

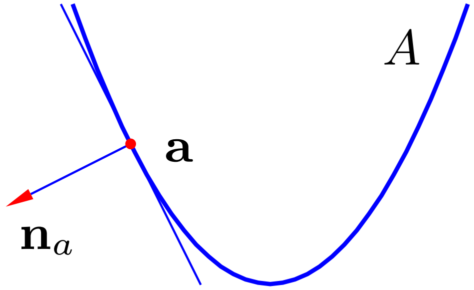
$$\mathbf{n}(t) = \frac{1}{\sqrt{1+t^2}}(-t, 1).$$

- Which parametrizations of the parabola lead to rational parametrizations of the offsets curves C_d?



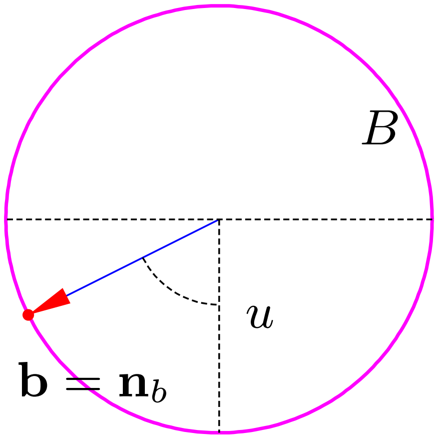
The Parabola and its Offsets

- $A \ldots$ parabola with $\mathbf{a}(t) = (t, \frac{1}{2}t^2)$,
- $B \ldots$ unit circle with $\mathbf{b}(u) = (\sin u, -\cos u)$.
- Normals $\mathbf{n}_a(t) = (-t, 1)$ and $\mathbf{n}_b(u) = \mathbf{b}(u)$.



Correspondence: $\mathbf{n}_a(t) = \lambda(u)\mathbf{n}_b(u) \Rightarrow$

$$-t \quad = \lambda \sin u, \quad 1 = -\lambda \cos u.$$



Reparametrization: $t = \frac{\sin u}{\cos u}$

$$\mathbf{a}(t(u)) = \left(\frac{\sin u}{\cos u}, \frac{1}{2} \frac{\sin^2 u}{\cos^2 u} \right).$$

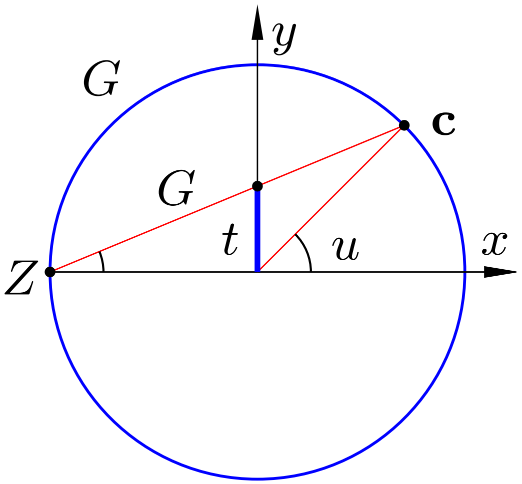
Offsets: $\mathbf{a}_d(u) = \mathbf{a}(t(u)) + d\mathbf{b}(u)$

$$\left(\frac{\sin u}{\cos u}, \frac{1}{2} \frac{\sin^2 u}{\cos^2 u} \right) + d(\sin u, -\cos u).$$

Stereographic Projection of the Unit Circle

- Unit circle $S^1 : x^2 + y^2 = 1$,
- Projection σ with center $Z = (-1, 0)$, maps the y-axis to S^1.
- Projection lines $G(t)$: $\mathbf{x}(t, \lambda) = (-1, 0) + \lambda(1, t)$.
- $S^1 \cap G(t)$ results in $\lambda = \frac{2}{1+t^2}$, $(\lambda = 0)$
- Rational Parametrization of S^1:

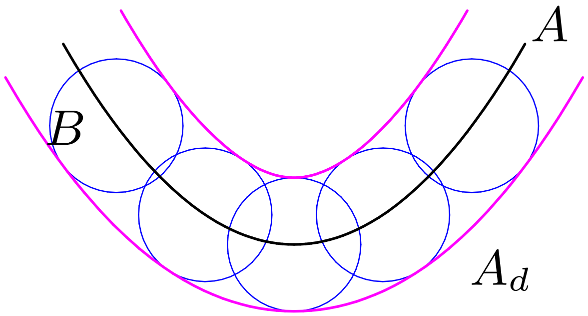
$$\mathbf{c}(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right).$$



- $t = \tan \frac{u}{2}$
- Weierstrass-substitution of the trigonometric functions

$$\cos u = \frac{1-t^2}{1+t^2}, \ \sin u = \frac{2t}{1+t^2}.$$

Rational Offsets of the Parabola



- $\mathbf{a}(t) = (t, \frac{1}{2}t^2)$
- $\mathbf{b}(v) = \left(\frac{2v}{1+v^2}, -\frac{1-v^2}{1+v^2}\right)$
- $t = \frac{2v}{1-v^2}$

- Rational parametrizations of A (degree 4)

$$\mathbf{a}(v) = \left(\frac{2v}{1-v^2}, \frac{2v^2}{(1-v^2)^2}\right),$$

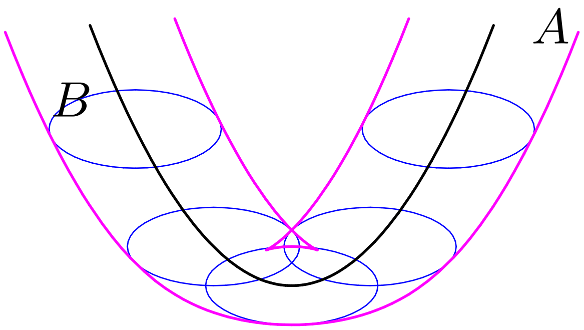
- and the offsets A_d (degree 6)

$$\mathbf{a}_d(v) = \left(\frac{2v}{1-v^2}, \frac{2v^2}{(1-v^2)^2}\right) + d\left(\frac{2v}{1+v^2}, -\frac{1-v^2}{1+v^2}\right).$$

Generalized Offsets of the Parabola

- Parabola $\mathbf{a}(t) = (t, \frac{1}{2}t^2)$ with normal vectors $\mathbf{n}_a(t) = (-t, 1)$.
- Rational curve $\mathbf{b}(u)$ with normal vectors $\mathbf{n}_b(u) = (n_1, n_2)(u)$.
- Correspondence with respect to parallel normal vectors

$$\mathbf{n}_a(t) = (-t, 1) = \lambda(u)(n_1, n_2) = \lambda(u)\mathbf{n}_b(u).$$

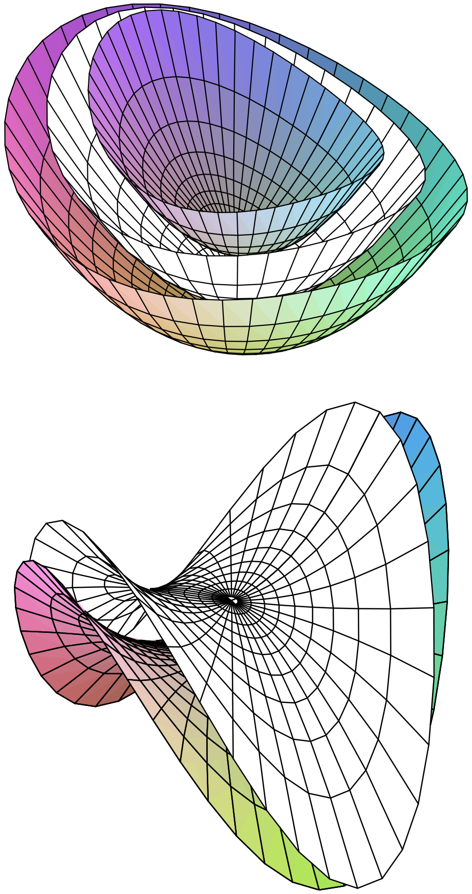


- Rational reparametrization

$$t = \frac{-n_1}{n_2}(u)$$

for arbitrary rational curves B.

Offsets of Paraboloids



- Paraboloid $A : \mathbf{a}(u,v) = (u, v, \frac{1}{2}u^2 + \frac{c}{2}v^2)$
- Sphere B:

  $$\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = (\cos s \cos t, \sin s \cos t, \sin t)$$

- Correspondence w.r.t. parallel normal vectors

  $$\mathbf{n}_a = (-2u, -2v, 1) = \lambda \mathbf{b} = \lambda \mathbf{n}_b,$$

  leads to the rational reparametrization

  $$\begin{aligned} u &= \frac{-\cos s \cos t}{\sin t}, \\ v &= -\frac{\sin s \cos t}{c \sin t}. \end{aligned}$$

- Offsets of Paraboloids are *rational* (W. Lü, ’94, ’95).

Pythagorean hodograph-curves

Farouki, et al. '90, '91, . . . Jüttler, Sir, J. Kosinka ,

Let C be a *polynomial curve* with parametrization $\mathbf{c}$. If the tangent vector is representable as

$$\begin{aligned} \dot{c}_1(t) &= (u(t)^2 - v(t)^2)w(t), \\ \dot{c}_2(t) &= 2u(t)v(t)w(t), \end{aligned}$$

with polynomials $u, v, w \in \mathbb{R}[t]$, one obtains

$$\|\dot{\mathbf{c}}\| = \|\mathbf{n}\| = w(t)(u(t)^2 + v(t)^2).$$

$\Rightarrow$ C has rational *arc length* and its offset curves C_d are rational.

Remark: This approach does not apply to rational curves and not to surfaces.

Literature about rational offsets and related topics

R. Farouki, W. Lü, J.R. Sendra and J. Sendra, H. Pottmann, B. Jüttler, F. Winkler, J. Schicho, L. González-Vega, L. Sampoli, G. Landsmann, R. Krasauskas, Ch. Mäurer, G. Elber, M.S. Kim, J. Kosinka, Z. Sir, B. Bastl, M. Lavicka . . .

Rational Offset Surfaces

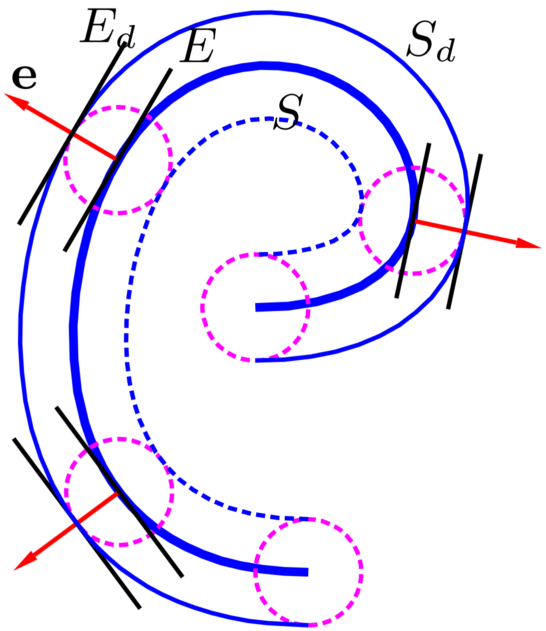
A surface S is called rational offset surface or PN–surface if it possesses a

- rational parametrization $\mathbf{s}(u,v)$ and
- rational unit normal vectors $\mathbf{e}(u,v)$

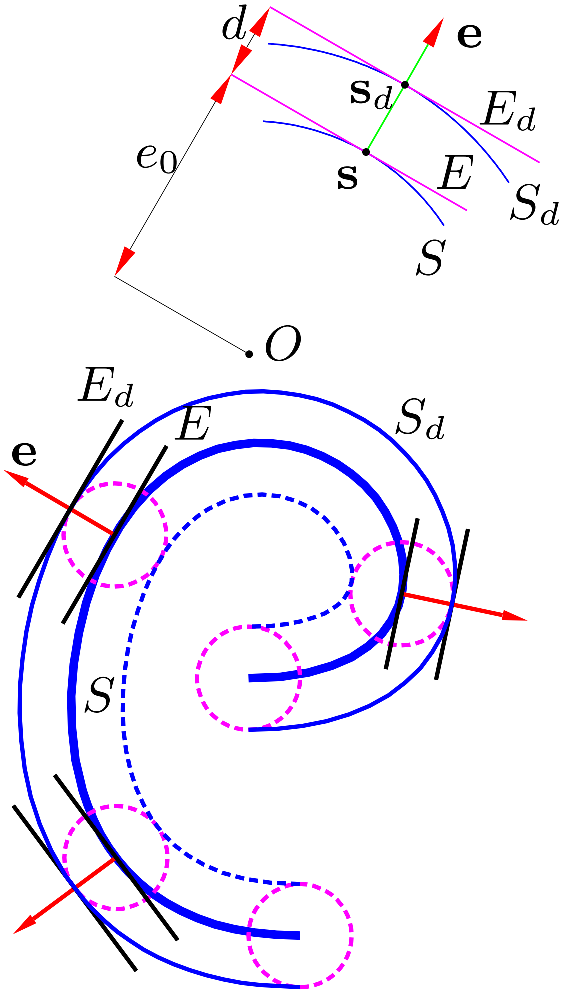
PN . . . Pythagorean Normal vector.

Offset surfaces S_d of S admit rational parametrizations

$$\mathbf{s}_d(u,v) = \mathbf{s}(u,v) + d\mathbf{e}(u,v).$$



The Offset Operation



- Tangent planes of S:

$$E(u,v) : e_0(u,v) + \mathbf{e}(u,v) \cdot \mathbf{x} = 0,$$

with $e_0(u,v) = \mathbf{e}(u,v) \cdot \mathbf{s}(u,v)$.

- Tangent planes of the offset surfaces S_d of S,

$$E_d(u,v) : e_0(u,v) + d + \mathbf{e}(u,v) \cdot \mathbf{x} = 0.$$

- The unit normals $\mathbf{e}(u,v)$ are a rational parametrization of the spherical (Gaussian) image of S.

PN–Surfaces as Envelopes of Tangent Planes

Pottmann '95

- Tangent planes of a PN–surface S are

$$E(u,v) : h(u,v) + \mathbf{e}(u,v) \cdot \mathbf{x} = 0,$$

- where $\mathbf{e} = (e_1, e_2, e_3)$ is a rational unit normal vector field (a rational parametrization of the unit sphere S^2), and
- $h(u,v)$ is a rational function.
- The offsets S_d of S are envelopes of the planes

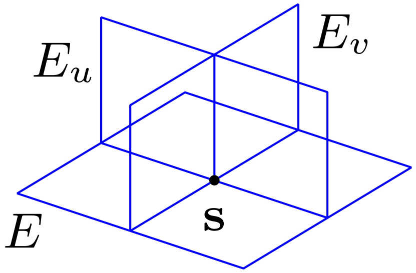
$$E_d(u,v) : (h(u,v) + d) + \mathbf{e}(u,v) \cdot \mathbf{x} = 0.$$

- If a,b, and c are polynomials in t and $h(t)$ is a rational function, S and S_d are *developable PN–surfaces*.

Parameterization of PN–Surfaces

- Let $E(u,v) : \mathbf{e}(u,v) \cdot \mathbf{x} = h(u,v)$ be a two-par. family of planes,
- rational unit normal vector $\mathbf{e}(u,v)$,
- rational function $h(u,v)$.
- A parameterization $\mathbf{s}(u,v)$ of the envelope surface S is obtained by intersecting the planes $E \cap E_u \cap E_v$.
- Thus $\mathbf{s}(u,v)$ is a solution of

$$\begin{aligned} E(u,v) : \mathbf{e}(u,v) \cdot \mathbf{x} &= h(u,v), \\ E(u,v)_u : \mathbf{e}(u,v)_u \cdot \mathbf{x} &= h(u,v)_u, \\ E(u,v)_v : \mathbf{e}(u,v)_v \cdot \mathbf{x} &= h(u,v)_v, \end{aligned}$$



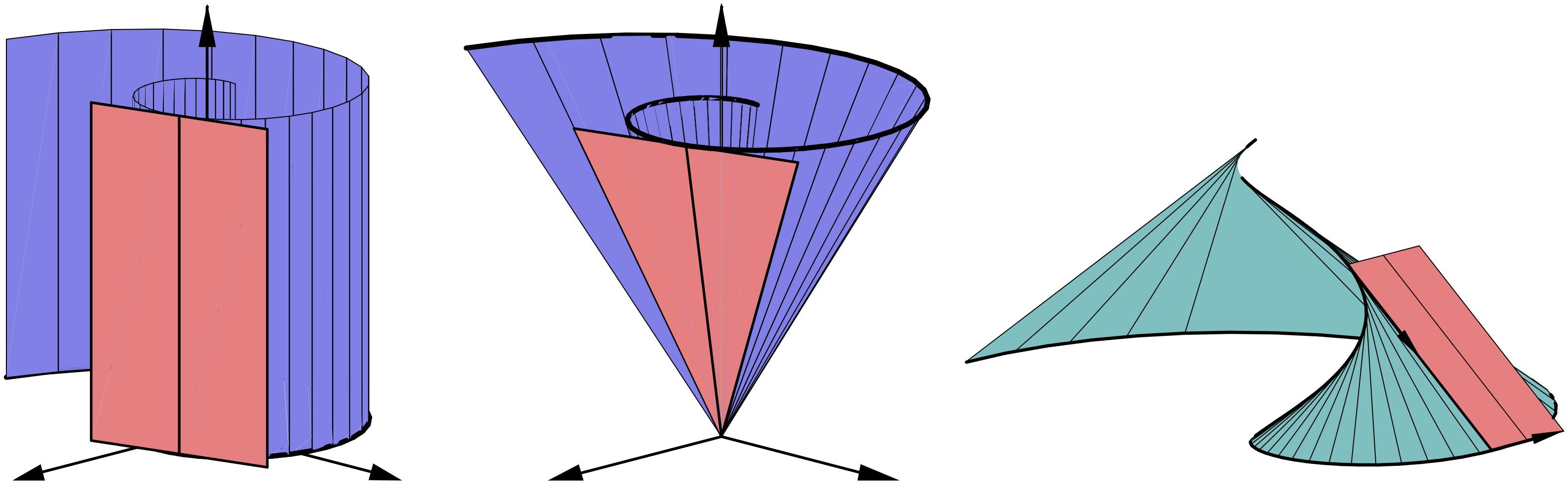
with $z_u = \partial z / \partial u$, $z_v = \partial z / \partial v$.

Developable PN–Surfaces

- If $E(t)$ is a one-par. family of planes, its envelope surface S is a developable surface, parameterized by

  $$\begin{aligned} E(t) : \mathbf{e}(t) \cdot \mathbf{x} &= h(t), \\ E(t)_t : \mathbf{e}(t)_t \cdot \mathbf{x} &= h(t)_t. \end{aligned}$$

  F carries straight lines and is a *cylinder*, a *cone* or the *tangent surface* of a space curve.

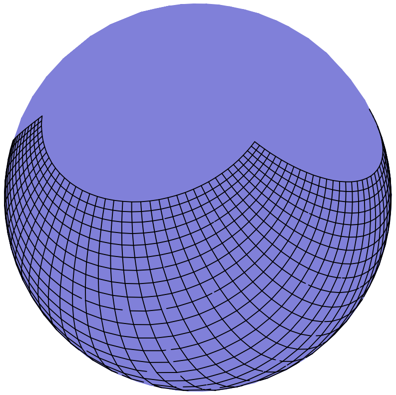


Rational Unit Normal Vector

- Let $\mathbf{e} = (e_1, e_2, e_3)$ be a rational parametrization of the unit sphere S^2, with polynomials a, b, c in u and v:

$$e_1 = \frac{2ac}{N}, e_2 = \frac{2bc}{N}, e_3 = \frac{a^2+b^2-c^2}{N},$$

  with $N = (a^2+b^2+c^2)$.

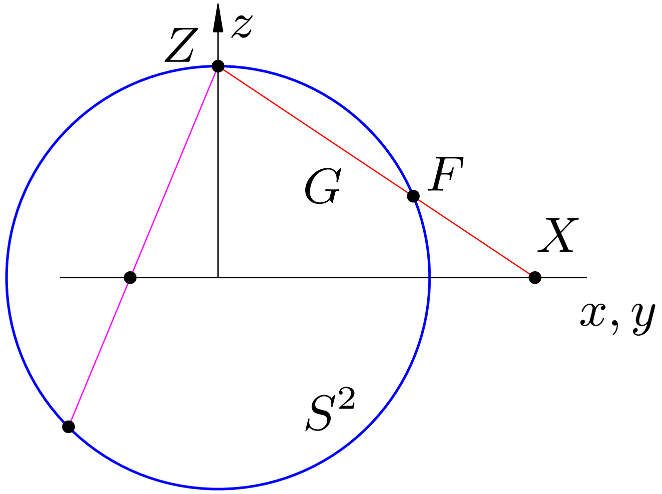


- With $a = u$, $b = v$, and $c = 1$ one obtains

$$e_1 = \frac{2u}{u^2+v^2+1}, e_2 = \frac{2v}{u^2+v^2+1}, e_3 = \frac{u^2+v^2-1}{u^2+v^2+1}.$$

Stereographic Projection of the Unit Sphere

- Unit sphere $S^2 : x^2 + y^2 + z^2 = 1$.
- Projection σ : with center $Z = (0, 0, 1)$
- maps points $X = (u, v, 0)$ of the xy-plane to points $F \in S^2$.



- Projection lines: $G(t)$: $\mathbf{x}(t, \lambda) = (0, 0, 1) + \lambda(u, v, -1)$
- $S^2 \cap G(t)$ leads to $\lambda = \frac{2}{1+u^2+v^2}$, $(\lambda = 0)$
- Rational parametrization of S^2:

$$\mathbf{f}(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{u^2 + v^2 - 1}{1 + u^2 + v^2} \right).$$

Stereographic Projection of the Unit Sphere 2

Projecting the points $X = (a/c, b/c, 0)$ to S^2 results in the rational parameterization

$$\mathbf{e}(u,v) = \left(\frac{2ac}{a^2+b^2+c^2}, \frac{2bc}{a^2+b^2+c^2}, \frac{a^2+b^2-c^2}{a^2+b^2+c^2} \right).$$

This parameterization depends on the center of projection!

Remark: Universal parameterization of S^2

- Choose polynomials a,b,c and d in u and v.
- *Universal parameterization* [Dietz, Hoschek,Jüttler'93, Krasauskas'96,'02]: $\mathbf{m} = (A/D, B/D, C/D)$ of S^2, with

  $$A = 2(ac+bd),\ B = 2(bc-ad),\ C = a^2+b^2-c^2-d^2,\ D = a^2+b^2+c^2+d^2.$$
- The tangent planes of a PN-surface S in $\mathbb{R}^3$ can be represented by $T(u,v) : Ax + By + Cz = h$, with rational $h(u,v)$.
- A parameterization $\mathbf{s}(u,v)$ of S follows by $T \cap T_u \cap T_v$.

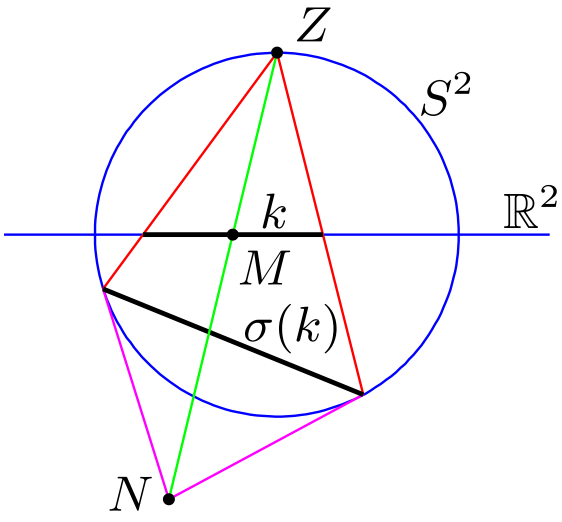
Properties of the Stereographic Projection

- $\mathbf{f}_u \cdot \mathbf{f}_u = \mathbf{f}_v \cdot \mathbf{f}_v$ and $\mathbf{f}_u \cdot \mathbf{f}_v = 0 \Rightarrow$

$$J(\sigma) = g(u,v) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- The stereographic projection σ preserves *angles*.

- Circles $k \in \mathbb{R}^2 : z = 0$ are mapped to circles $\sigma(k) \in S^2$.
- $S^2 \ldots$ model of the 2-dim. *Möbius geometry*.
- The projection σ realizes the transfer between S^2 and the Möbius plane $\mathbb{R}^2$.

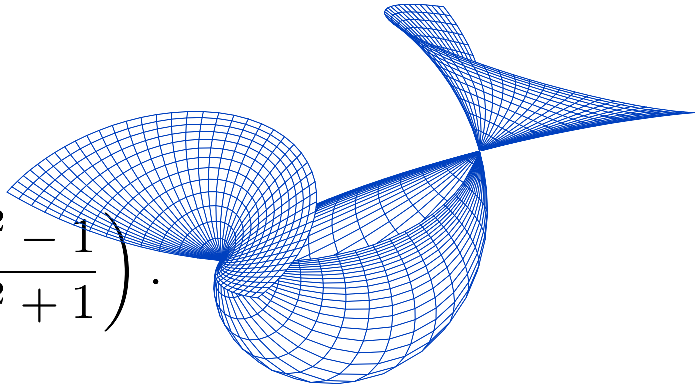


Parabolic Dupin Cyclide

Example: Let $a = u, b = v, c = 1$.

Then

$$\mathbf{e} = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right).$$



We choose $h(u,v) = q(u,v)/N$ with $q(u,v)$ as quadratic polynomial, $\Longrightarrow$

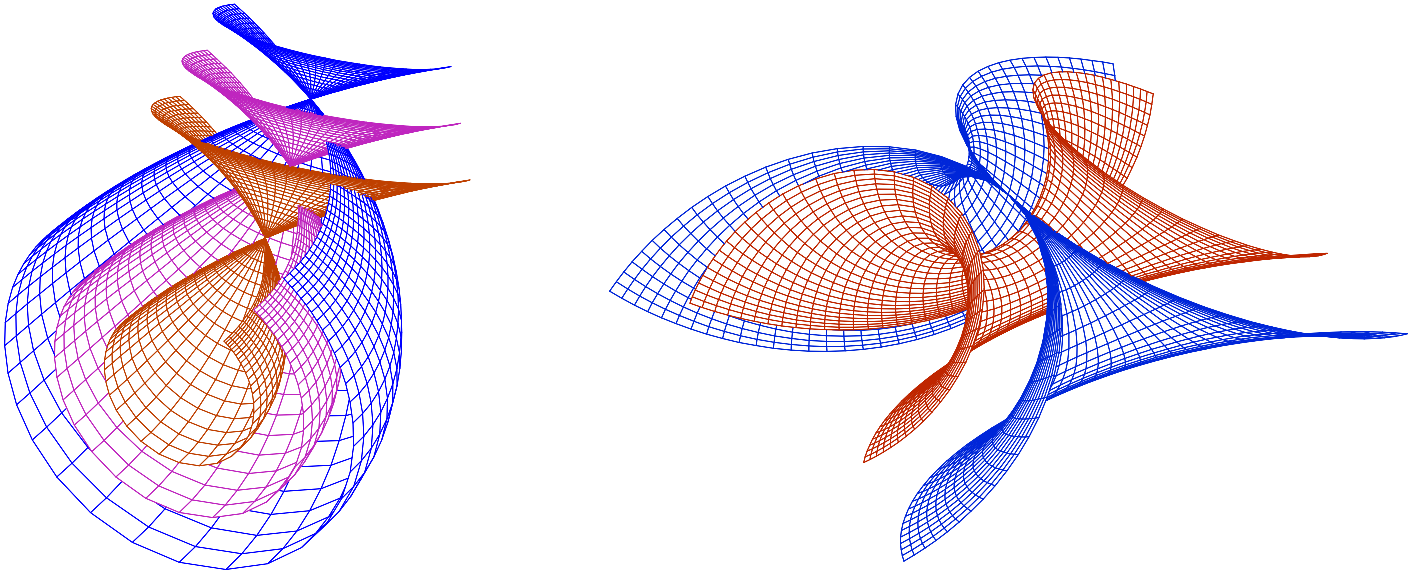
$$E(u,v) : 2ux + 2vy + (u^2+v^2-1)z = q(u,v).$$

The envelope F of planes $E(u,v)$ and all its offsets F_d are parabolic Dupin cyclides (alg.order 3). The real singularities might be different.

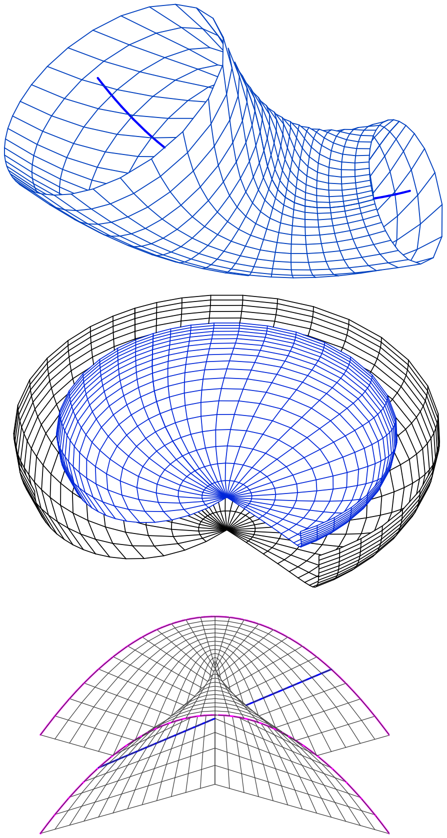
Parabolic Dupin Cyclide

$$E(u,v) : 2ux + 2vy + (u^2+v^2-1)z = u^2 - v^2$$

$$\mathbf{s}(u,v) = \left(\frac{u(2v^2+1)}{u^2+v^2+1}, \frac{-v(2u^2+1)}{u^2+v^2+1}, \frac{u^2-v^2}{u^2+v^2+1} \right)$$

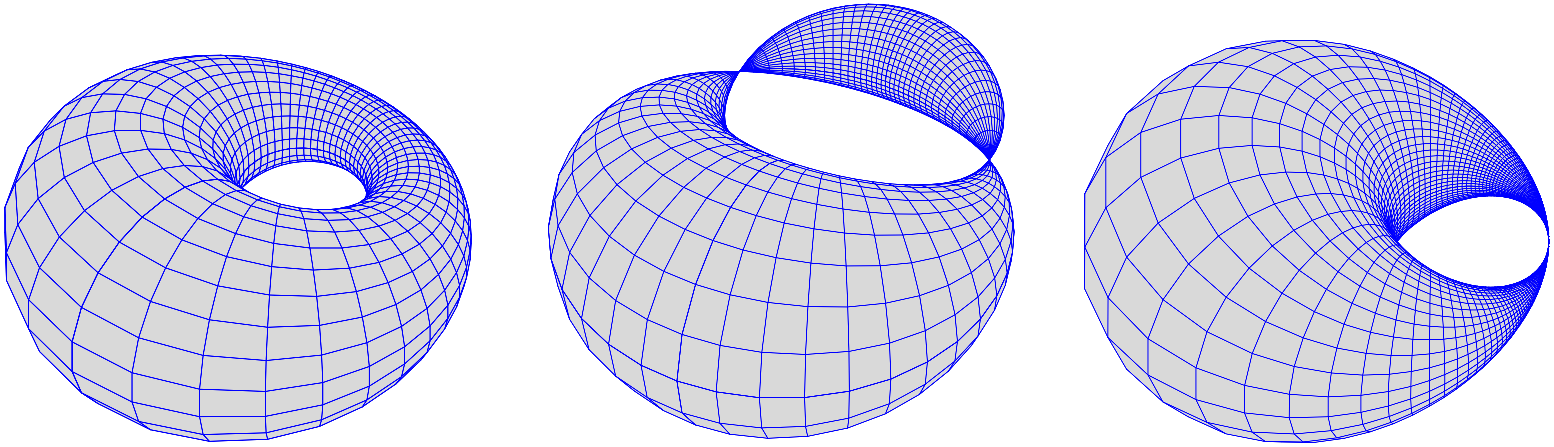


Results about PN–Surfaces



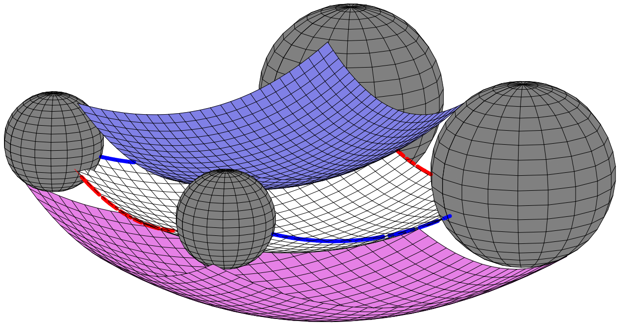
- The offsets S_d of rational canal surfaces S admit real rational parameterizations.
- The offsets of quadrics such as ellipsoids, hyperboloids, paraboloids admit real rational parameterizations.
- The offsets S_d of rational non-developable ruled surfaces are rational.
- The envelope S of a rational one-par. family of cones of revolution $D(t)$ is a *PN–surface*.

Dupin Cyclides of Order 4



Families of Spheres

- Let $S(u,v)$ be a rational two-par. family of spheres in $\mathbb{R}^3$ with centers $\mathbf{m}(u,v)$ and radius function $r(u,v)$.
- When does the envelope surface Φ of $S(u,v)$ admit rational parameterizations?



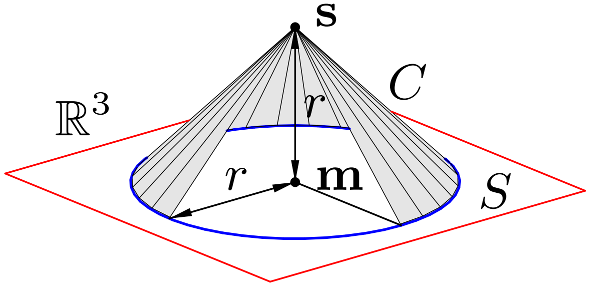
- Characterization of families of spheres with rational envelopes.
- Explicit parameterizations.
- Relations to PN–surfaces.

$\mathbb{R}^4$ as model of the space of spheres

- Cyclographic mapping: Points $\mathbf{s}$ in $\mathbb{R}^4$ → oriented spheres S in $\mathbb{R}^3$

$$\gamma : \mathbb{R}^4 \to \mathcal{S},$$

$$\mathbf{s} = (\mathbf{m}, r) \mapsto \gamma(\mathbf{s}) = S : (\mathbf{x} - \mathbf{m})^2 = r^2.$$



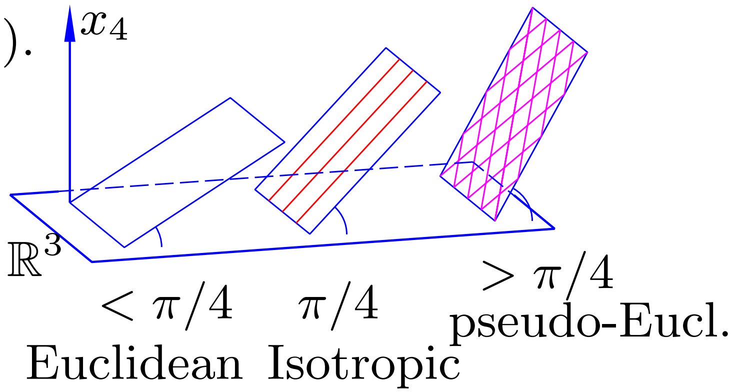
- Minkowski-space $\mathbb{R}^4$:

$$\langle \mathbf{x}, \mathbf{x} \rangle := \mathbf{x}^T D \mathbf{x}, \text{ with } D = \operatorname{diag}(1, 1, 1, -1).$$

- Light cone C with vertex $\mathbf{s}$:

$$C : \langle \mathbf{x} - \mathbf{s}, \mathbf{x} - \mathbf{s} \rangle = 0.$$

- $C \cap \mathbb{R}^3$ is the non-oriented sphere S.

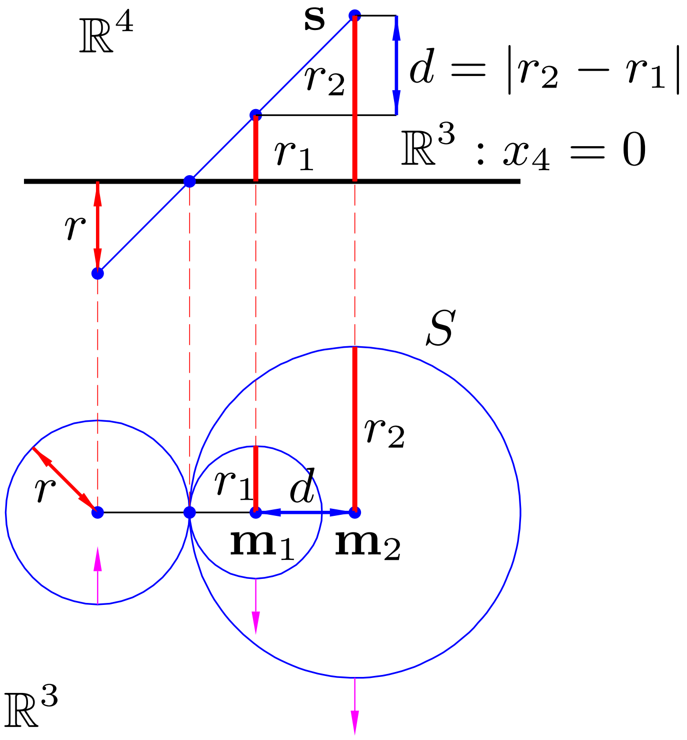


Touching spheres

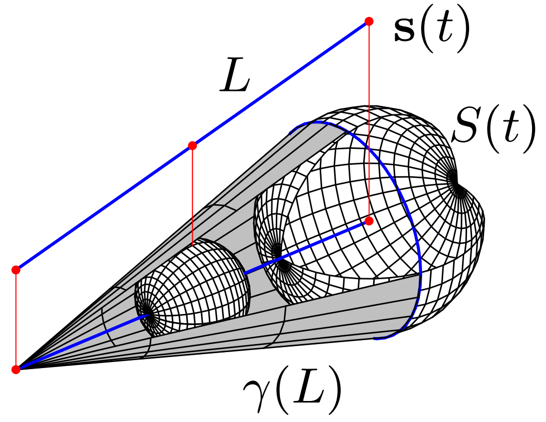
- Two spheres $S_1(\mathbf{m}_1, r_1)$ and $S_2(\mathbf{m}_2, r_2)$ are in oriented contact, if and only if

$$\|\mathbf{m}_2 - \mathbf{m}_1\|^2 = (r_2 - r_1)^2.$$

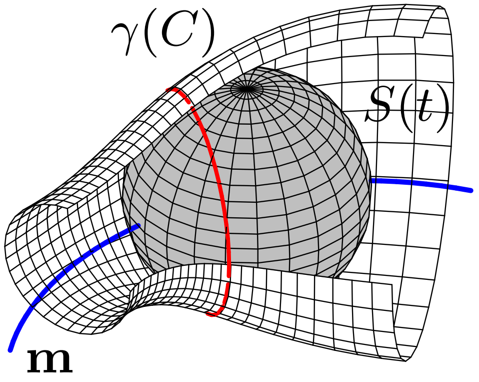
- $\Leftrightarrow \langle \mathbf{s}_2 - \mathbf{s}_1, \mathbf{s}_2 - \mathbf{s}_1 \rangle = 0.$
- The spheres touching a fixed sphere S are represented in $\mathbb{R}^4$ by the points of the light cone $C : \langle \mathbf{x} - \mathbf{s}, \mathbf{x} - \mathbf{s} \rangle = 0.$



One-parameter families of spheres

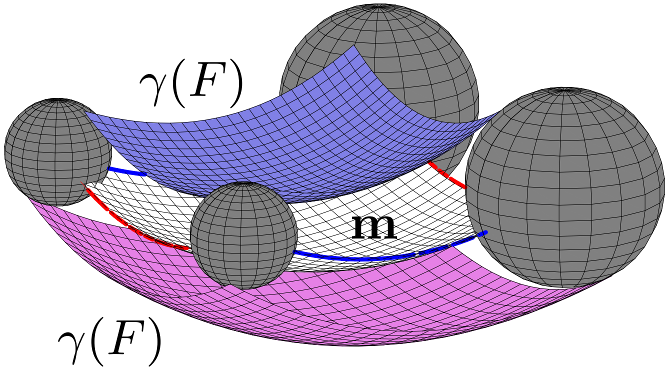


- A curve $C : \mathbf{c}(t)$ in $\mathbb{R}^4$ corresponds to a one-par. family of spheres.
- The envelope $\gamma(C)$ is real, iff $\langle \dot{\mathbf{c}}, \dot{\mathbf{c}} \rangle \geq 0$.
- Line L in $\mathbb{R}^4$ ↦ the envelope $\gamma(L)$ is a cone of revolution.
- Curve C in $\mathbb{R}^4$ ↦ the envelope $\gamma(C)$ is a canal surface.



Two-parameter families of spheres

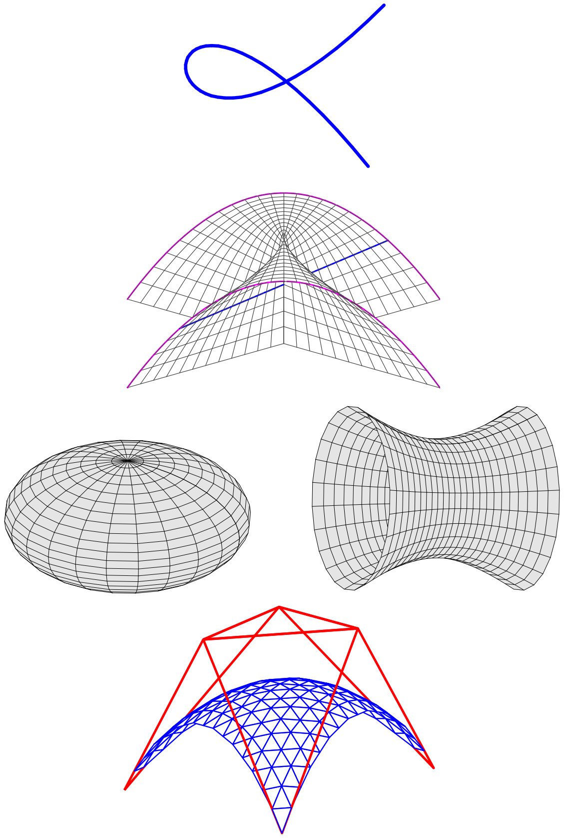
- A surface $F : \mathbf{f}(u, v)$ in $\mathbb{R}^4$ corresponds to a two-par. family of spheres.
- The envelope is real iff $\langle l\mathbf{f}_u + \mu\mathbf{f}_v, l\mathbf{f}_u + \mu\mathbf{f}_v\rangle \geq 0$, for all λ, μ.
- The envelope $\gamma(F)$ consists of two sheets of surfaces locally.



Known results

The following objects in $\mathbb{R}^4$ imply envelopes in $\mathbb{R}^3$ which admit rational parameterizations:

- Rational curve C in $\mathbb{R}^4$: $\gamma(F)$ is a rational canal surface in $\mathbb{R}^3$. [Lü, Pottmann, H.'96; Pet., Pottmann'96, Krasauskas'07]
- Rational ruled surface $F \in \mathbb{R}^4$: $\gamma(F)$ is the envelope of cones of revolution. [Pottmann, Lü, Ravani'96, Pet.'97].
- 2-dim. quadric F in $\mathbb{R}^4$. [Lü'96; Pet.'97]
- Quadratically parameterized surface F in $\mathbb{R}^4$, (quadratic Bézier-surface). [Pet., Odehnal, Sampoli'08]
- MOS-surfaces. [Kosinka, Jüttler'07]



Envelope construction

- Given a two-par. family of spheres $S(u,v) : (\mathbf{x}-\mathbf{m})^2 = r^2$, their envelope Φ is obtained as solution of

$$S : (\mathbf{x}-\mathbf{m})^2 - r^2 = 0,$$
$$S_u : (\mathbf{x}-\mathbf{m})^T\mathbf{m}_u + rr_u = 0,\ S_v : (\mathbf{x}-\mathbf{m})^T\mathbf{m}_v + rr_v = 0.$$

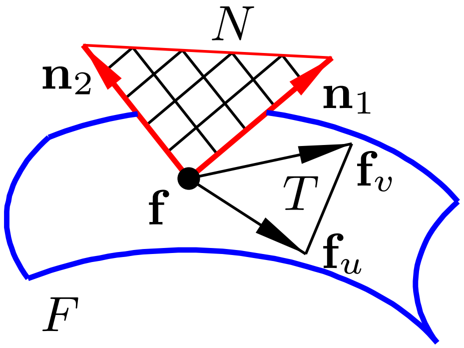
- Given a 2-surface $\mathbf{f}(u,v)$ in $\mathbb{R}^4$. The envelope Ψ of the light cones $C(u,v)$ with vertices $\mathbf{f}(u,v)$ is computed as solution of

$$C : \langle \mathbf{x}-\mathbf{f}, \mathbf{x}-\mathbf{f}\rangle = 0,\ C_u : \langle \mathbf{x}-\mathbf{f}, \mathbf{f}_u\rangle = 0,\ C_v : \langle \mathbf{x}-\mathbf{f}, \mathbf{f}_v\rangle = 0,$$

- $S = C \cap \mathbb{R}^3$ and $\Phi = \Psi \cap \mathbb{R}^3$.

Isotropic normal vector fields

- Let $F \in \mathbb{R}^4$ be a rational 2-surface, point $\mathbf{f} \in F$.
- Tangent plane T is Euclidean $\iff$ normal plane N is pseudo-Euclidean. N and is spanned by $\mathbf{f}$ and two *isotropic normal vectors* $\mathbf{n}_1$ and $\mathbf{n}_2$.



- The envelope $\gamma(F)$ admits rational parameterizations iff

$$A(\mathbf{f}) := \langle \mathbf{f}_u, \mathbf{f}_u \rangle \langle \mathbf{f}_v, \mathbf{f}_v \rangle - \langle \mathbf{f}_u, \mathbf{f}_v \rangle^2 = \sigma(u,v)^2 \qquad (1)$$

  is the perfect square of a rational function σ.

  MOS-condition from [Kosinka, Jüttler '07].

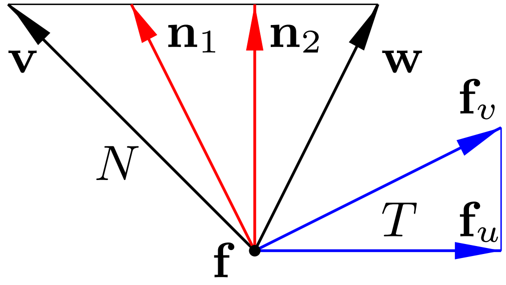
- We give geometric interpretations of condition (1).

Rational isotropic normal vector fields

Theorem 1 *A rationally parameterized surface* $\mathbf{f}(u,v)$ *in* $\mathbb{R}^4$ *possesses a rational isotropic normal vector field* $\mathbf{n}(u,v)$ *iff* $A(\mathbf{f}) = \sigma(u,v)^2$, *with a rational function* $\sigma(u,v)$.

- Tangent plane T is spanned by $\mathbf{f}_u$ and $\mathbf{f}_v$.
- Normal plane N is spanned by $\mathbf{v}$ and $\mathbf{w}$.
- $\mathbf{n} = \mathbf{v} + t\mathbf{w}$ is isotropic, $\Longleftrightarrow$

$$\langle \mathbf{v} + t\mathbf{w}, \mathbf{v} + t\mathbf{w} \rangle = 0. \qquad (2)$$



- (2) has a *rational* solution, $\Longleftrightarrow$

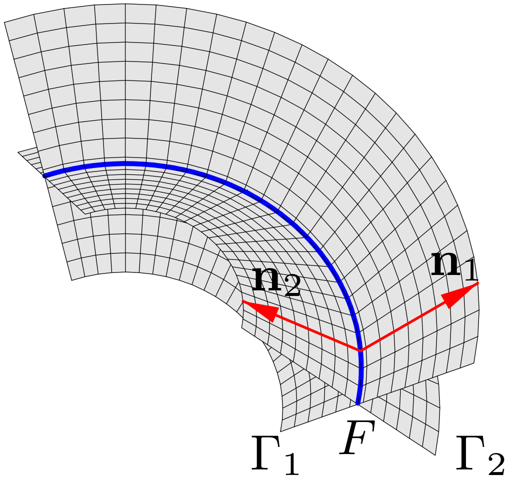
$$\begin{aligned} d &= \langle \mathbf{v}, \mathbf{w} \rangle^2 - \langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{w}, \mathbf{w} \rangle \\ &= \rho^2 (\langle \mathbf{f}_u, \mathbf{f}_u \rangle \langle \mathbf{f}_v, \mathbf{f}_v \rangle - \langle \mathbf{f}_u, \mathbf{f}_v \rangle^2) \\ &= \rho^2 A(\mathbf{f}) = \rho^2 \sigma(u,v)^2. \end{aligned}$$

Isotropic hypersurfaces in $\mathbb{R}^4$

- 2-surface $F \in \mathbb{R}^4$ parameterized by $\mathbf{f}(u, v)$.
- Isotropic normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$.
- Two isotropic hyper-surfaces Γ_1 nd Γ_2 passing through F. Parameterizations of Γ_i are obtained by

$$\mathbf{g}_i(u, v, w) = \mathbf{f}(u, v) + w\mathbf{n}_i(u, v),\ i = 1, 2.$$

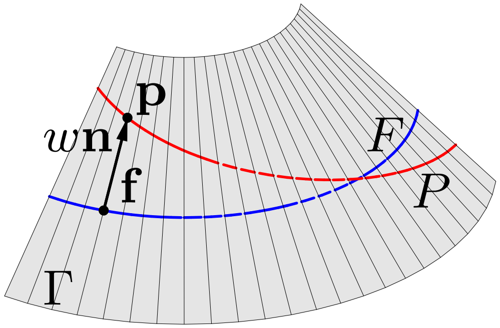
- If $\mathbf{n}_i$ are rational vector fields, Γ_i are rational isotropic hyper-surfaces.



Rational isotropic hypersurfaces in $\mathbb{R}^4$

Theorem 2 *Let P be a rational 2-dim. sub-variety of the rational isotropic hyper-surface Γ through F, with rat. isotropic normal vectors $\mathbf{n}$. Then $\mathbf{n}(u,v)$ is a rational isotropic normal vector field of $\mathbf{p}(u,v) = \mathbf{f}(u,v) + w(u,v)\mathbf{n}(u,v)$, with a rational $w(u,v)$.*

- $\langle \mathbf{f}_u, \mathbf{n} \rangle = 0$ and $\langle \mathbf{f}_v, \mathbf{n} \rangle = 0$.
- $\langle \mathbf{n}, \mathbf{n} \rangle = 0 \Rightarrow \langle \mathbf{n}, \mathbf{n}_u \rangle = 0$ and $\langle \mathbf{n}, \mathbf{n}_v \rangle = 0$.
- Partial derivatives $\mathbf{p}_u, \mathbf{p}_v$,



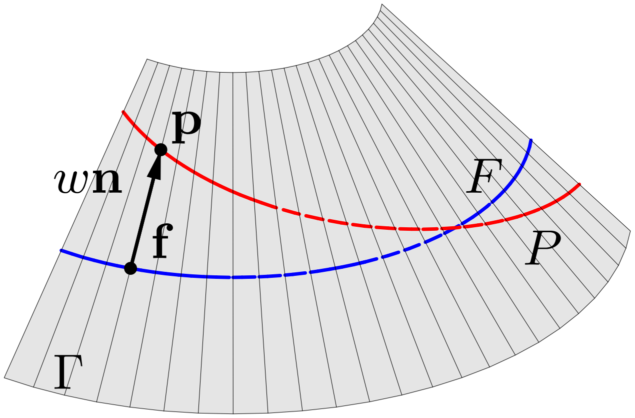
$$\mathbf{p}_u \quad = \mathbf{f}_u + w_u \mathbf{n} + w \mathbf{n}_u .$$

$$\langle \mathbf{p}_u, \mathbf{n} \rangle = \langle \mathbf{f}_u, \mathbf{n} \rangle + w_u \langle \mathbf{n}, \mathbf{n} \rangle + w \langle \mathbf{n}_u, \mathbf{n} \rangle = 0.$$

$\Rightarrow$ $\mathbf{n}(u,v)$ is a rational isotropic normal vector field of $\mathbf{p}(u,v)$, and $A(\mathbf{p}) = \tau(u,v)^2$, with rational $\tau(u,v)$. ☐

Condition for rational envelopes

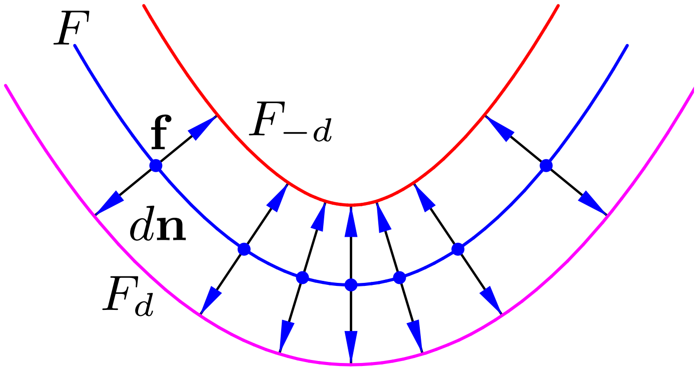
Theorem 3 *The envelope surface* $\Phi \in \mathbb{R}^3$ *of the 2-par. family of spheres* $S(u, v)$ *corresponding to the surface* F *in* $\mathbb{R}^4$ *admits a rational parameterization* $\mathbf{q}(u, v)$ *iff the isotropic hyper-surface* Γ *through* F *admits rational parameterizations.*



PN-surfaces

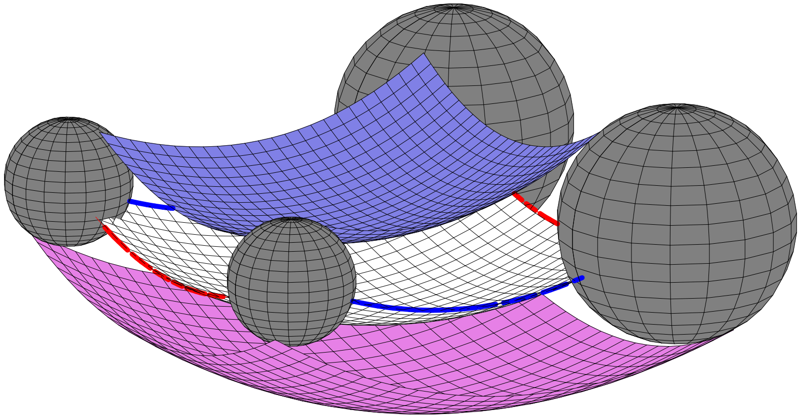
- A surface F in $\mathbb{R}^3$ is called PN–surface or if it possesses a rational unit normal vector field $\mathbf{n}(u,v)$ corresponding to a rational parameterization $\mathbf{f}(u,v)$.
- The offset surface F_d of F at oriented distance d admits a rational parameterization $\mathbf{f}_d(u,v) = \mathbf{f}(u,v) + d\mathbf{n}(u,v)$.

[Pottmann'95]



Rational envelopes and PN-surfaces

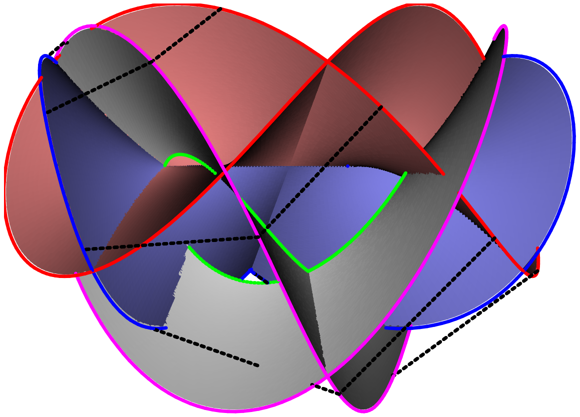
Corollary 4 *A rational surface* $\Phi \in \mathbb{R}^3$ *is a PN-surface iff it is the intersection of a rational isotropic hypersurface* Γ *with* $x_4 = 0$.



Theorem 5 *Let* $S(u, v)$ *be a rational two-parameter family of spheres* $S(u, v)$ *whose envelope surface* Φ *admits rational parameterizations. Then* Φ *is a PN-surface.*

Explicit Representations

Theorem 6 *Let F be a rational surface in $\mathbb{R}^4$ whose corresponding 2-par. family of spheres $S(u,v)$ admits rational parameterizations. Based on universal parameterizations of the unit sphere S^2, rational parameterizations $\mathbf{f}(u,v)$ of F can be constructed explicitly.*



Universal Parameterizations 1

- Choose polynomials a,b,c and d in u and v.
- Generalized stereographic projection [Dietz, Hoschek,Jüttler'93] yields a universal parameterization $\mathbf{m} = (A/D, B/D, C/D)$ of S^2, with

  $$A = 2(ac{+}bd),\ B = 2(bc{-}ad),\ C = a^2{+}b^2{-}c^2{-}d^2,\ D = a^2{+}b^2{+}c^2{+}d^2.$$

  Universal parameterizations: [Krasauskas'96,'02]
- The tangent planes of a PN-surface Φ in $\mathbb{R}^3$ can be represented by $T(u,v) : Ax + By + Cz = h$, with rational $h(u,v)$.
- A point representation $\mathbf{q}(u,v)$ of Φ follows by $T \cap T_u \cap T_v$.

Universal Parameterizations 2

$$\mathbf{q}=\frac{1}{E}\begin{pmatrix} B(C_uh_v-C_vh_u)+C(h_uB_v-h_vB_u)+h(B_uC_v-B_vC_u) \\ C(A_uh_v-A_vh_u)+A(h_uC_v-h_vC_u)+h(C_uA_v-C_vA_u) \\ A(B_uh_v-B_vh_u)+B(h_uA_v-h_vA_u)+h(A_uB_v-A_vB_u) \end{pmatrix},$$

with $E=A(B_uC_v-B_vC_u)+B(C_uA_v-A_uC_v)+C(A_uB_v-A_vB_u)$.

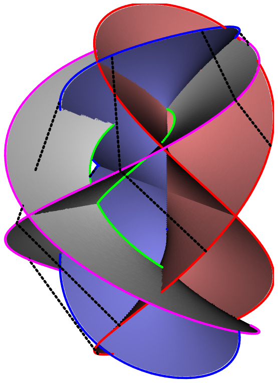
- $\mathbf{n}=(A,B,C,D)$ is a rat. isotropic normal vector field of $\mathbf{q}\in\mathbb{R}^4$.
- Rat. isotropic hyper-surface Γ through Φ:

$$\mathbf{g}(u,v,w)=(\mathbf{q}(u,v),0)+w\mathbf{n}(u,v).$$

- With a rational 'radius function' $w(u,v)$ we get

$$\mathbf{f}(u,v)=(\mathbf{q}(u,v),0)+w(u,v)\mathbf{n}(u,v).$$

Conclusion



We have presented

- results about PN–surfaces and
- their relations to rational envelopes of two-par. families of spheres.

Thank you for your attention!