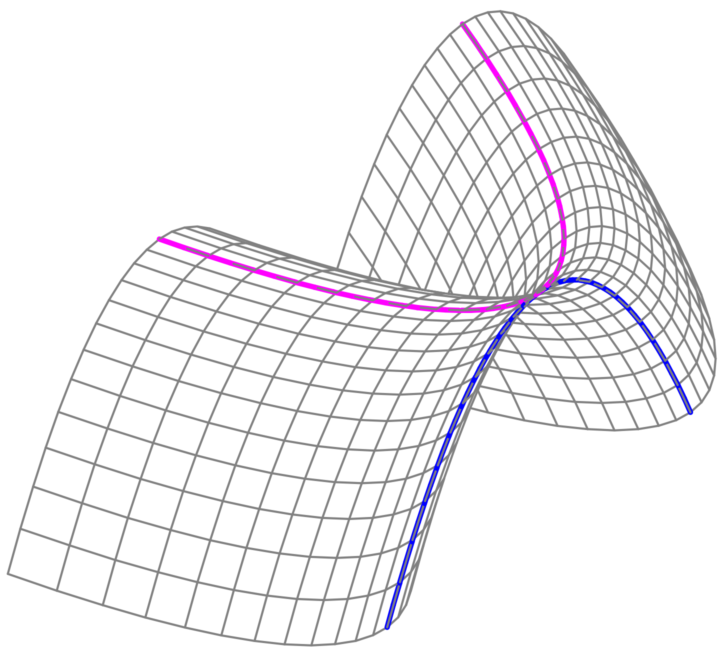
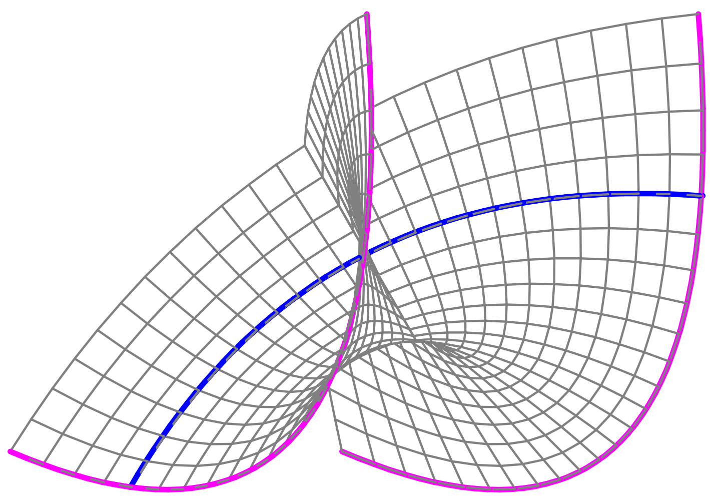
On Generalized LN-Surfaces in 4-Space

Martin Peternell and Boris Odehnal

Vienna University of Technology,

<martin@geometrie.tuwien.ac.at>





Motivation

A surface $Q \in \mathbb{R}^4$ is called *quadratically parameterizeable* if it admits a parameterization $\mathbf{q}(u,v) = (q_1, \ldots, q_4)(u,v)$, where q_i are quadratic polynomials. Q is the projection of a Veronese V_2^2.

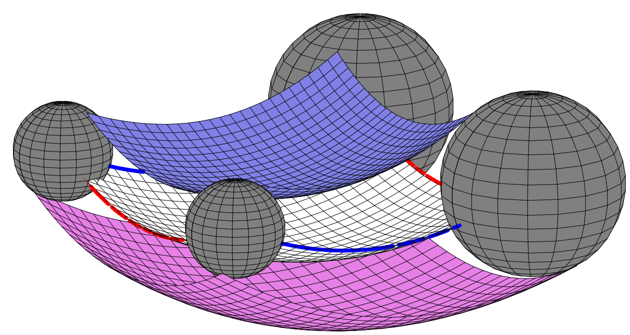
The tangent planes T of Q are spanned by *linear* vector fields $\mathbf{q}_u(u,v)$ and $\mathbf{q}_v(u,v)$.

We want to determine a class of surfaces in $\mathbb{R}^4$ which

- generalize quadratically parameterizeable surfaces Q concerning the structure of their tangent planes.

Consequence:

- The corresponding two-par. families of spheres in $\mathbb{R}^3$ have envelopes which admit rational parameterizations.



Rational surfaces with linear normal vector fields

Jüttler, 1998; Jüttler and Sampoli, 2000.

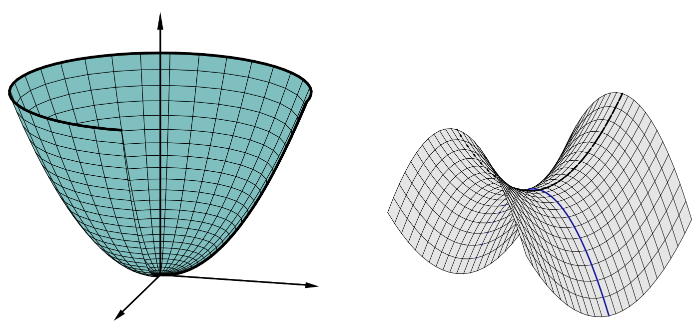
- A rational surface Φ in $\mathbb{R}^3$ is called *LN-surface* if its tangent planes admit the representation

$$T(u,v) : ux + vy + z = f(u,v)$$

  with a rational function $f(u,v)$.

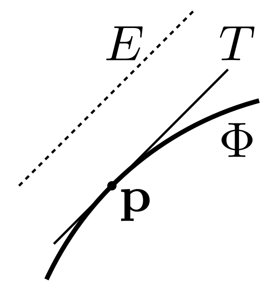
- Parameterization $\mathbf{p}(u,v) = (f_u, f_v, f - uf_u - vf_v)(u,v)$.
- Normal vector field: $\mathbf{n}(u,v) = (u, v, 1)$.

Example 1 *For quadratic f one obtains a paraboloid $\mathbf{p}(u,v)$ with z-parallel axis.*



Geometric Properties of LN surfaces

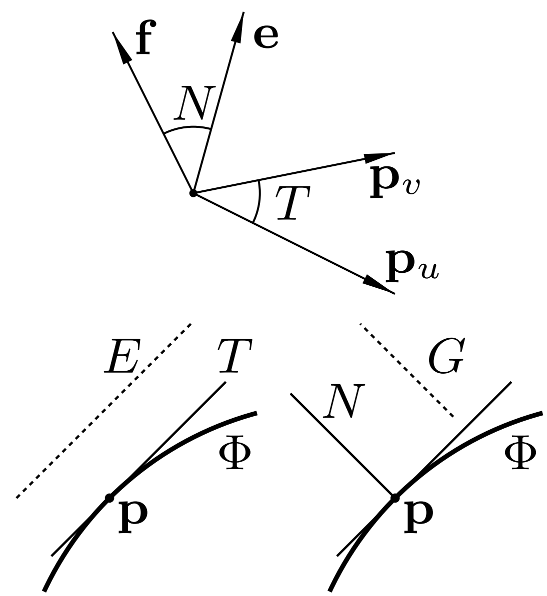
- For almost all planes E in $\mathbb{R}^3$ there exists a *unique* tangent plane T of Φ with $T \parallel E$. Exceptional normal vectors $\mathbf{n} = (n_1, n_2, 0)$.
- The set of tangent planes of an LN-surface is a *graph of a rational function*: $(u, v, f(u, v))$.
- The ideal plane $\omega = \mathbb{R}(1, 0, 0, 0)$ is an $n-1$-fold plane of Φ.



Generalized LN-surfaces in $\mathbb{R}^4$

Definition 1 *A rational two-dimensional surface Φ in $\mathbb{R}^4$ is called generalized LN-surface if for all 3-spaces $E \in \mathbb{R}^4$ the surface parameters u and v can be expressed in terms of rational functions depending on the coefficients e_i of E.*

- Tangent plane $T : \mathbf{p} + \lambda \mathbf{p}_u + \mu \mathbf{p}_v$
- Normal plane $N : \mathbf{p} + \lambda \mathbf{e} + \mu \mathbf{f}$ with $\mathbf{e} \perp \mathbf{p}_u$, $\mathbf{e} \perp \mathbf{p}_v$ and $\mathbf{f} \perp \mathbf{p}_u$, $\mathbf{f} \perp \mathbf{p}_v$.
- For almost all 3-spaces E in $\mathbb{R}^4$ there exists a unique tangent plane T of Φ with $T \parallel E$.
- For almost all lines G in $\mathbb{R}^4$ there exists a unique normal plane N of Φ with $G \parallel N$.



Construction of the surfaces

- A surface Φ in $\mathbb{R}^4$ is considered as envelope of its tangent planes.
- The tangent planes $T(u,v)$ of Φ are intersections of 3-spaces

$$\begin{aligned} E(u,v) : e_1x_1 + \ldots + e_4x_4 = \quad & \mathbf{e}^T\mathbf{x} \quad = a(u,v), \\ F(u,v) : f_1x_1 + \ldots + f_4x_4 = \quad & \mathbf{f}^T\mathbf{x} \quad = b(u,v). \end{aligned} \tag{1}$$

- The planes $T = E \cap F$ have an envelope $\Leftrightarrow$ the overdet. system

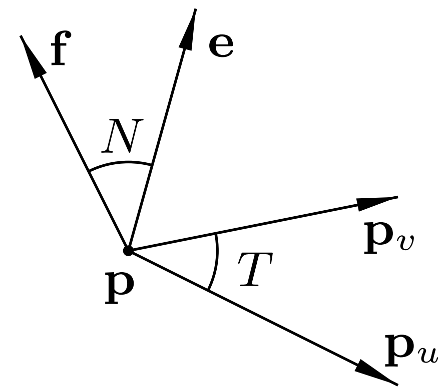
$$\begin{aligned} &E : \mathbf{e}^T\mathbf{x} = a, \quad E_u : \mathbf{e}_u^T\mathbf{x} = a_u, \quad E_v : \mathbf{e}_v^T\mathbf{x} = a_v, \\ &F : \mathbf{f}^T\mathbf{x} = b, \quad F_u : \mathbf{f}_u^T\mathbf{x} = b_u, \quad F_v : \mathbf{f}_v^T\mathbf{x} = b_v. \end{aligned} \tag{2}$$

  has a solution $\mathbf{p}(u,v)$.

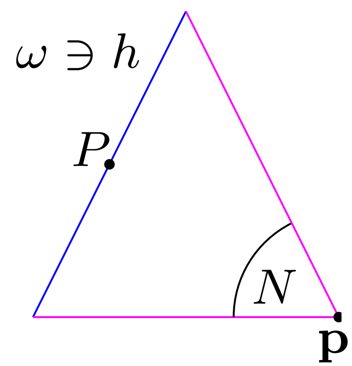
- Determine all suitable *normal vector fields* $\mathbf{e}(u,v)$ and $\mathbf{f}(u,v)$ of generalized LN-surfaces ('unique normal plane').
- Explicit parameterizations are found almost *without integration*.

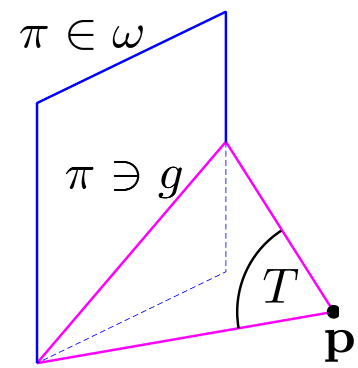
Some geometric properties of these surfaces

- $\mathbb{P}^4$ is the projective extension of $\mathbb{R}^4$, $\omega = \mathbb{P}^4 \setminus \mathbb{R}^4$ is the ideal plane.

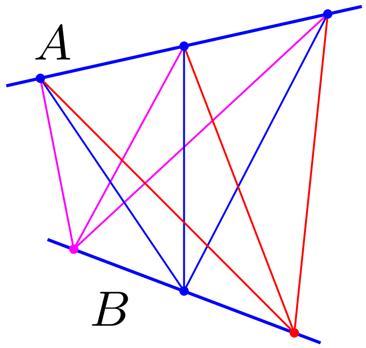


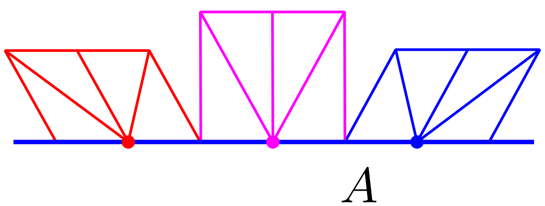
- $\mathcal{G}$ is the 2-par. family of ideal lines $g = T \cap \omega$ of $\Phi' s$ tangent planes.
- $\mathcal{H}$ is the 2-par. family of ideal lines $h = N \cap \omega$ of $\Phi' s$ normal planes.
- For almost all points $P \in \omega$ there is a unique line $h \in \mathcal{H}$ with $P \in h$.
- For almost all planes $\pi \in \omega$ there is a unique line $g \in \mathcal{G}$ with $g \in \pi$.

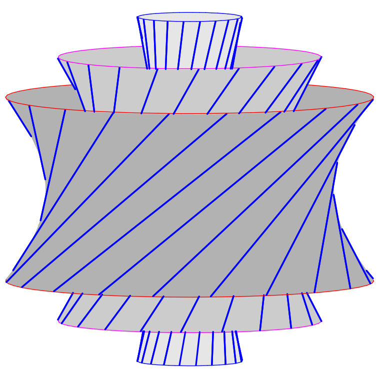




Special two-parameter families of lines







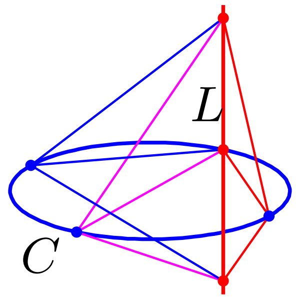
According to R. Sturm (1893) there exist the following rational families of lines $\mathcal{H}$, sending a unique line $h \in \mathcal{H}$ through a generic point $P \in \mathbb{P}^3$.

Type 1 – hyperbolic linear line congruence:
Lines meeting A and B.

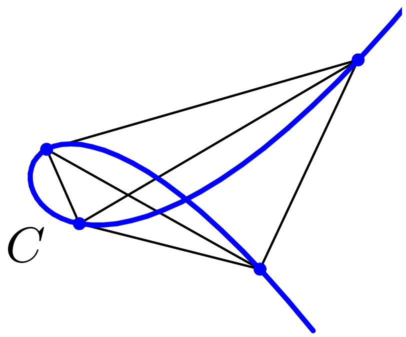
Type 2 – parabolic linear line congruence:
Pencils of lines in planes through the *axis* A and with vertices on A.

Type 3 – elliptic linear line congruence:
Lines meeting two conjugate complex lines A and $\bar{A}$, or generating lines of a family of hyperboloids of revolution.

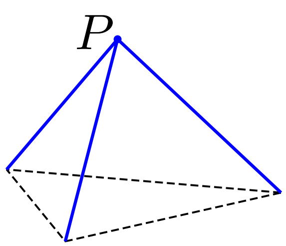
Special two-parameter families of lines, cont.



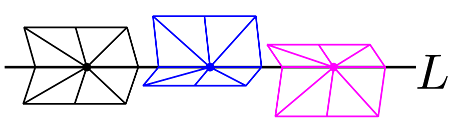
Type 4 – $(1, n)$- congruence of the first kind:
C is a degree n curve, L is a line meeting C in $n-1$ points. The family $\mathcal{H}$ comprises the lines intersecting both C and L. E.g.: lines meeting a conic C and a line L.



Type 5 – chordal variety of a cubic: Chords of a spatial cubic C.



Type 6 – star of lines: Lines through a fixed point P.

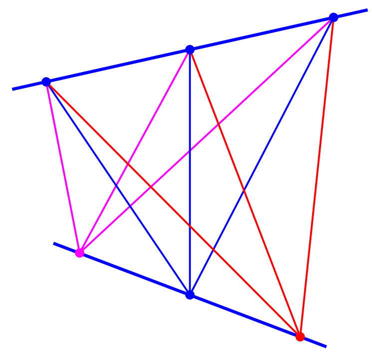


Type 7 – $(1, n)$- congruence of the second kind:
Pencils of lines with vertices $X \in L$ which lie in planes $\varepsilon \supset L$. $X \to \varepsilon$ is a rational $(1, n)$-correspondence. Points X correspond to n planes. Planes ε correspond to a single point.

Surfaces of Type 1

- $\mathcal{G}$ and $\mathcal{H}$ are hyperbolic linear line congruences
- Normal form for the surface construction:

$$\mathbf{e}(u) = (1, 0, u, 0), \text{ and } \mathbf{f}(v) = (0, 1, 0, v).$$



- Rational functions $a(u, v)$ and $b(u, v)$ satisfy

$$a_v = 0, \text{ and } b_u = 0.$$

- A rational parameterization of Φ reads

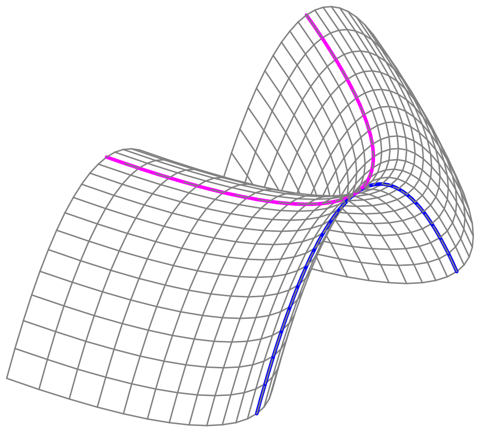
$$\begin{aligned} \mathbf{p}(u, v) &= (a - ua_u, b - vb_v, a_u, b_v) \\ &= (a - ua_u, 0, a_u, 0) + (0, b - vb_v, 0, b_v) . \end{aligned}$$

Theorem 2 *A generalized LN-surface Φ in $\mathbb{R}^4$ of type 1 is a translational surface $\mathbf{p}(u, v) = \mathbf{c}(u) + \mathbf{d}(v)$ with planar LN-curves as profile curves, and vice versa.*

Surfaces of Type 1, cont.

- The tangent planes T of Φ are spanned by $\mathbf{s} = (-u, 0, 1, 0)$ and $\mathbf{t} = (0, -v, 0, 1)$.
- For any vector $\mathbf{z} = (z_1, z_2, z_3, z_4)$, the equations $\mathbf{z}^T\mathbf{s} = 0$ and $\mathbf{z}^T\mathbf{t} = 0$ result in

$$u = \frac{z_3}{z_1}, v = \frac{z_4}{z_2}.$$



For $a(u) = 1/2u^2$ and $b(v) = 1/2v^2$ one obtains $\mathbf{p} = (-1/2u^2, -1/2v^2, u, v)$. The Fig. shows the projection $\mathbf{q} = 1/2(-u^2, -v^2, u+v)$ in $\mathbb{R}^3$.

Surfaces of Type 2

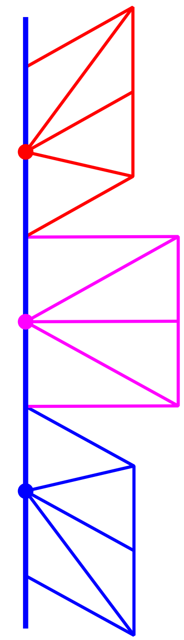
- $\mathcal{G}$ and $\mathcal{H}$ are parabolic linear line congruences
- Normal form for the surface construction:

$$\mathbf{e}(u) = (1, 0, u, 0), \text{ and } \mathbf{f}(u, v) = (0, -1, v, u).$$

- Rational functions $a(u, v)$ and $b(u, v)$ satisfy

$$a_v = 0, \text{ and } a_u - b_v = 0.$$

- Since $b(u, v) = va_u + \lambda(u)$, a rational parameterization of Φ reads



$$\begin{aligned}\mathbf{p}(u, v) &= \mathbf{c}(u) + \tilde{v}\mathbf{d}(u) \\ &= (a - ua_u, \lambda, a_u, 0) + \tilde{v}(0, u, 0, 1), \text{ with } \tilde{v} = va_{uu} - \lambda.\end{aligned}$$

Theorem 3 *A generalized LN-surface Φ in $\mathbb{R}^4$ of type 2 is a ruled surface with a rational curve C on a cylinder over an LN-curve as directrix curve and a linear direction vector field, and vice versa.*

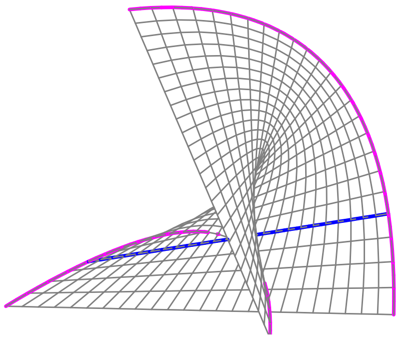
Surfaces of Type 2, cont.

- The tangent planes of Φ are spanned by $\mathbf{s} = (-u, v, 1, 0)$ and $\mathbf{t} = (0, u, 0, 1)$.
- For any vector $\mathbf{z} = (z_1, z_2, z_3, z_4)$ the equations $\mathbf{z}^T\mathbf{s} = 0$ and $\mathbf{z}^T\mathbf{t} = 0$ result in the rational expressions

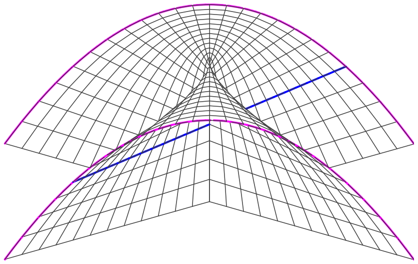
$$u = -\frac{z_4}{z_2}, v = -\frac{z_1 z_4 + z_2 z_3}{z_2^2}.$$

- For $a(u) = 1/2u^2$ one obtains $\mathbf{f}(u, v) = (-1/2u^2, uv, u, v)$.

Cayley surface and



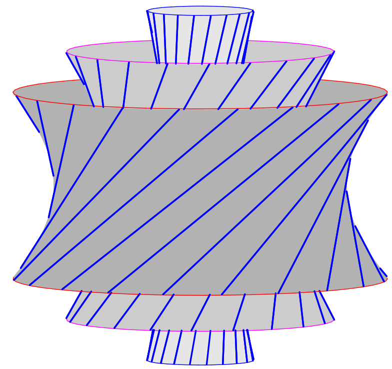
Plücker conoid



The Figures show the *Cayley-surface* $\mathbf{p}(u, v) = (-1/2u^2 + v, uv, u)$ and the *Whitney umbrella* or *Plücker conoid* $\mathbf{q}(u, v) = (-1/2u^2, uv, v)$.

Surfaces of Type 3

- $\mathcal{G}$ and $\mathcal{H}$ are elliptic linear line congruences
- Normal form for the surface construction:



$$\mathbf{e}(u,v) = (1,0,-u,v), \text{ and } \mathbf{f}(u,v) = (0,-1,v,u),$$

- Rational functions $a(u,v)$ and $b(u,v)$ satisfy

$$a_u = -b_v, \text{ and } a_v = b_u.$$

- A rational parameterization of Φ reads

$$\mathbf{p}(u,v) = (a - ua_u - va_v, -b - va_u + ua_v, -a_u, a_v).$$

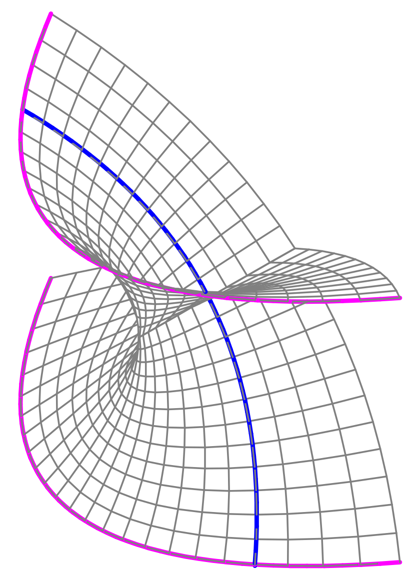
- Rational function $f(z)$ with $z = u + iv$, $f = b + ia$, $f_z = a_v + ia_u$
- $\mathbf{p}(u,v) = \frac{1}{2}(\mathbf{c}(z) + \overline{\mathbf{c}(z)})$, with $\mathbf{c}(z) = (i(zf_z - f), (zf_z - f), if_z, f_z)$.

Theorem 4 *A generalized LN-surface Φ in $\mathbb{R}^4$ of type 3 is a translational surface $\mathbf{p}(z) = 1/2(\mathbf{c}(z) + \overline{\mathbf{c}(z)})$ with a pair of planar conjugate complex LN-curves as profile curves, and vice versa.*

Surfaces of Type 3, cont.

- The tangent planes of Φ are spanned by $\mathbf{s} = (u, v, 1, 0)$ and $\mathbf{t} = (-v, u, 0, 1)$.
- For any vector $\mathbf{z} = (z_1, z_2, z_3, z_4)$ the equations $\mathbf{z}^T\mathbf{s} = 0$ and $\mathbf{z}^T\mathbf{t} = 0$ result in

$$\begin{pmatrix} z_1 & z_2 \\ z_2 & -z_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -z_3 \\ -z_4 \end{pmatrix}.$$



- Surfaces of type 3 are Euclidean minimal surfaces and $\mathbf{p}(u, v)$ is an isotermal parameterization.

For $a = 1/2(v^2 - u^2)$ and $b = uv$ one obtains $\mathbf{p} = (1/2(u^2 - v^2), uv, u, v)$. The projection $\mathbf{q} = (1/2(u^2 - v^2), uv, v)$ is shown in the figure.

Surfaces of Type 4

- $\mathcal{H}$ is a line congruence of type (1,n)
- Normal form for the surface construction for type(1,2):

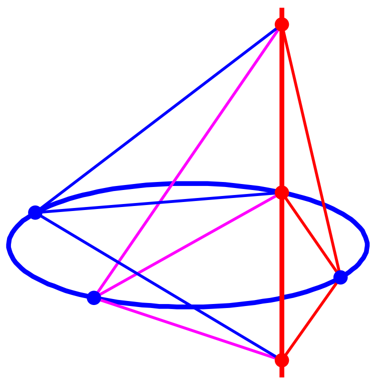
$$\mathbf{e}(u) = (-1, 0, 0, u) \text{ and } \mathbf{f}(v) = (0, u, -v, v^2),$$

- Rational functions $a(u, v)$ and $b(u, v)$ satisfy

$$a_v = 0, \text{ and } ub_u + vb_v - b - v^2 a_u = 0.$$

- $a(u)$ and $b(u, v) = uh(u/v) + v^2/ua(u)$.
- A rational parameterization of Φ reads

$$\mathbf{p}(u, v) = (ua_u - a, b_u, 2va_u - b_v, a_u).$$

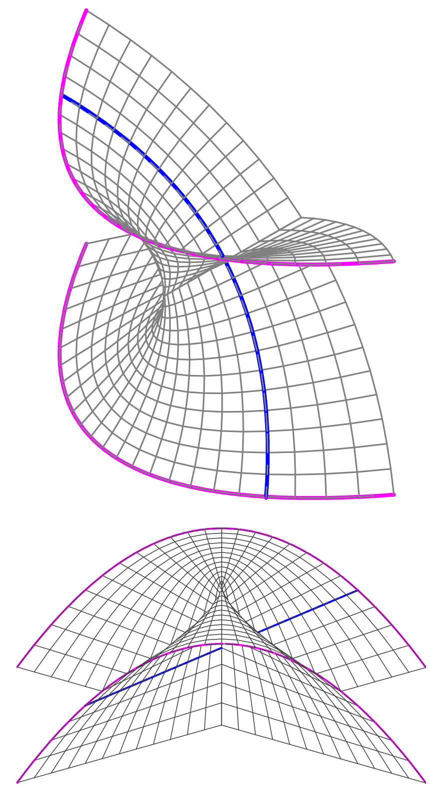


Surfaces of Type 4, cont.

- The tangent planes are spanned by vectors $\mathbf{s} = (u, 0, v, 1)$ and $\mathbf{t} = (0, v, u, 0)$.
- For any vector $\mathbf{z} = (z_1, z_2, z_3, z_4)$ the equations $\mathbf{z}^T\mathbf{s} = 0$ and $\mathbf{z}^T\mathbf{t} = 0$ result in

$$u = \frac{z_2 z_4}{z_3^2 - z_1 z_2}, \quad v = \frac{-z_3 z_4}{z_3^2 - z_1 z_2}.$$

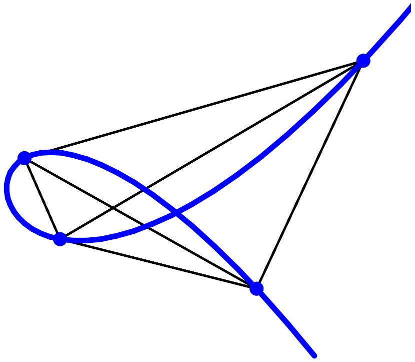
- For $a = 1/2u^2$ and $b = 1/2uv^2$ one obtains $\mathbf{p} = (1/2u^2, 1/2v^2, uv, u)$. Projections are shown in the Figure.



Surfaces of Type 5

- $\mathcal{H}$ is the chordal variety of a cubic.
- Normal form for the surface construction:

$$\mathbf{e} = (u, 0, -1, -u^2 + v), \text{ and } \mathbf{f} = (-v, 1, 0, uv).$$



- Rational functions $a(u, v)$ and $b(u, v)$ satisfy

$$b_u = va_v, \text{ and } b_v = -a_u - ua_v.$$

- $a_{uu} = -ua_{uv} - 2a_v - va_{vv}$ and $b = \int va_v du$ $- \int(\int(va_{vv} + a_v)du + a_u + ua_v)dv + C$.
- A rational parameterization of Φ reads

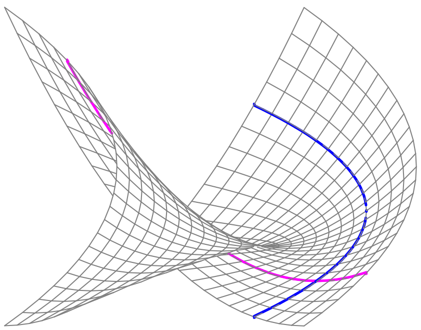
$$\mathbf{p}(u, v) = (ua_v - b_v, b - vb_v, -a + b_u - ub_v, a_v).$$

Surfaces of Type 5, cont.

- The tangent planes are spanned by $\mathbf{s} = (u, 0, v, 1)$ and $\mathbf{t} = (1, v, u, 0)$.
- For any vector $\mathbf{z} = (z_1, z_2, z_3, z_4)$ the equations $\mathbf{z}^T\mathbf{s} = 0$ and $\mathbf{z}^T\mathbf{t} = 0$ result in

$$\begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -z_4 \\ -z_1 \end{pmatrix}.$$

- For $a = -1/2u^3 + uv$ and $b = 1/2u^2v - 1/2v^2$ one obtains $\mathbf{p} = (1/2u^2 + v, 1/2v^2, uv, u)$. The projection $\mathbf{q} = (1/2u^2 + v, 1/2v^2, uv)$ is shown in the figure.



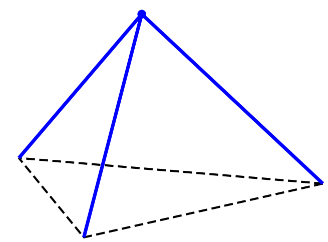
Surfaces of Type 6

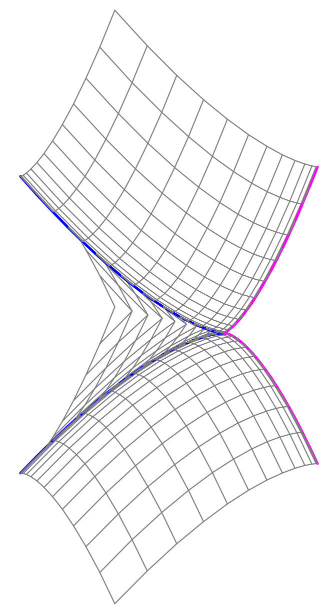
- $\mathcal{H}$ is a star of lines.
- Normal form: $\mathbf{e}(u,v) = (1,u,v,0)$ and $\mathbf{f} = (0,0,0,1)$.
- Rational function $a(u,v)$ and $b = \text{const}$.
- A rational parameterization of Φ reads

$$\mathbf{p}(u,v) = (a - ua_u - va_v, a_u, a_v, b)\,.$$

- Φ's tangent planes T are spanned by $\mathbf{s} = (-u,1,0,0)$ and $\mathbf{t} = (-v,0,1,0)$.
- For any vector $\mathbf{z} = (z_1, z_2, z_3, z_4)$ the equations $\mathbf{z}^T\mathbf{s} = 0$ and $\mathbf{z}^T\mathbf{t} = 0$ result in

$$u = \frac{z_2}{z_1}, v = \frac{z_3}{z_1}.$$





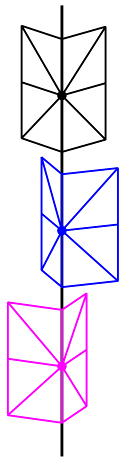
$a(u,v) = u^3 + v^3$.

Theorem 5 *A generalized LN-surface Φ in $\mathbb{R}^4$ of type 6 is an LN-surface in a hyperplane of $\mathbb{R}^4$.*

Surfaces of Type 7

- $\mathcal{H}$ is a $(1, n)$-congruence of the second kind.
- Normal form for the surface construction:

  $\mathbf{e}(u) = (1-u^2, 0, 0, 2u)$ and $\mathbf{f}(u, v) = (0, 1, u, v)$.

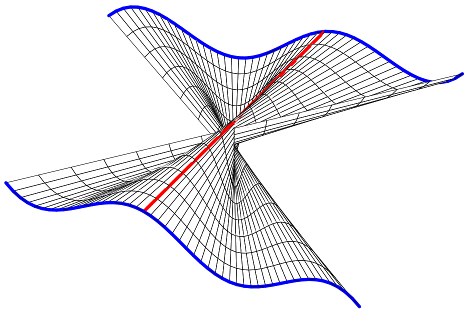


- Rational functions $a(u, v)$ and $b(u, v)$ satisfy

  $a_v = 0$ and $2b_v(1 + u^2) = a_u(1 - u^2) + 2ua$. $\Rightarrow$

- $a = a(u)$ and

$$b(u, v) = \frac{v}{2(1 + u^2)}(a_u(1 - u^2) + 2ua) + \lambda(u).$$



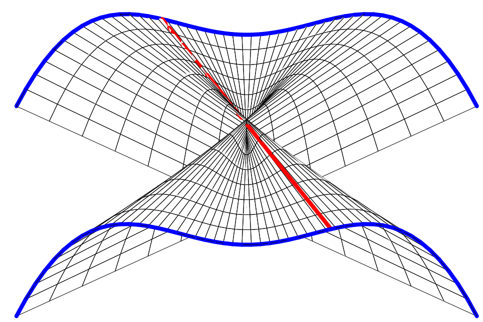
- A rational parameterization of Φ reads

$$\mathbf{p}(u, v) = \left(\frac{a - ua_u}{1 + u^2}, b - ub_u - vb_v, b_u, b_v\right).$$

Surfaces of Type 7, cont.

- Φ's tangent planes T are spanned by $\mathbf{s} = (0, -u, 1, 0)$ and $\mathbf{t} = (2u/(u^2-1), -v, 0, 1)$.
- For any vector $\mathbf{z} = (z_1, z_2, z_3, z_4)$ the equations $\mathbf{z}^T\mathbf{s} = 0$ and $\mathbf{z}^T\mathbf{t} = 0$ result in

$$u = \frac{z_3}{z_2} \text{ and } v = \frac{2z_1z_2z_3 + z_4(z_3^2 - z_2^2)}{z_2(z_3^2 - z_2^2)}.$$



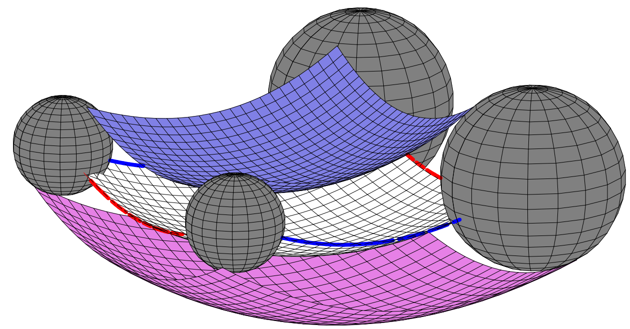
- Since $b = vb_v + \lambda$ is linear in v, surfaces Φ are ruled surfaces. By letting $\tilde{v} = b_u$ we find

$$\mathbf{p}(u,v) = \left(\frac{a - ua_u}{1+u^2}, \lambda, 0, \frac{a_u(1-u^2) + 2ua}{2(1+u^2)}\right) + \tilde{v}\,(0, -u, 1, 0)\,.$$

Theorem 6 *A generalized LN-surface Φ in $\mathbb{R}^4$ of type 7 is a rational ruled surface with linear direction vector field, and vice versa.*

Some consequences

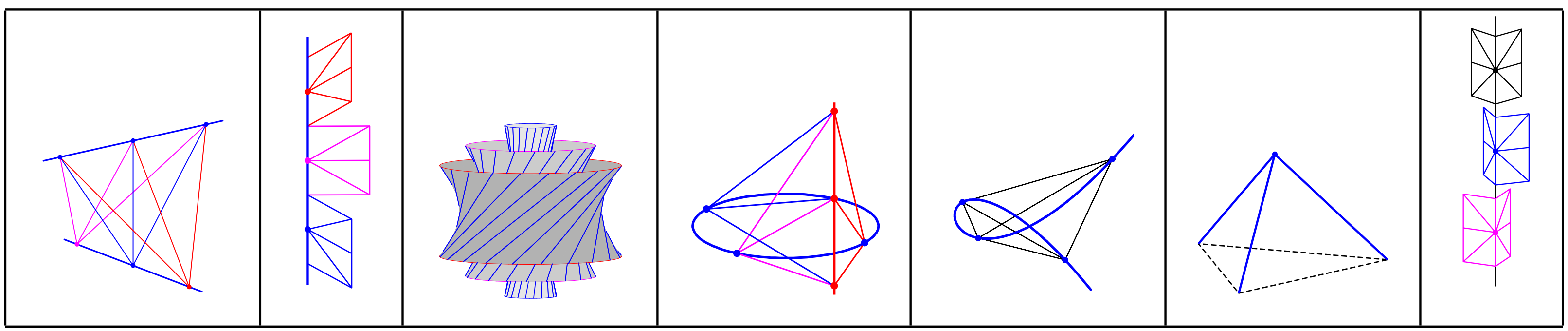
- Theses surfaces in $\mathbb{R}^4$ have the property that the corresponding two-par. families of spheres in $\mathbb{R}^3$ have envelopes which admit rational parameterizations.



- These surfaces share many nice properties with LN-surfaces in $\mathbb{R}^3$.
- The convolution (=envelope w.r.t. translational motions) of two generalized LN-surfaces is a rational LN-hyper-surface in $\mathbb{R}^4$.
- Parallel projections of these surfaces are LN-surfaces in $\mathbb{R}^3$.

Summary

- We have presented a special class of 2-dim. surfaces Φ in $\mathbb{R}^4$ which generalizes surfaces with linear normal vector fields of $\mathbb{R}^3$.
- Seven affine-invariant types of surfaces, corresponding to special types of line-congruences in $\mathbb{P}^3$.
- Most types are given in an integral-free representation.



Thank you for your attention.