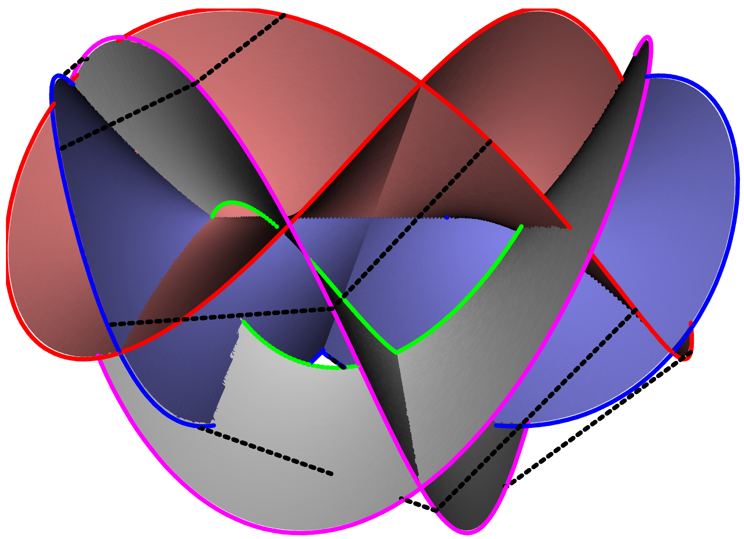
Rational Two-Parameter Families of Spheres and Rational Offset Surfaces

Martin Peternell

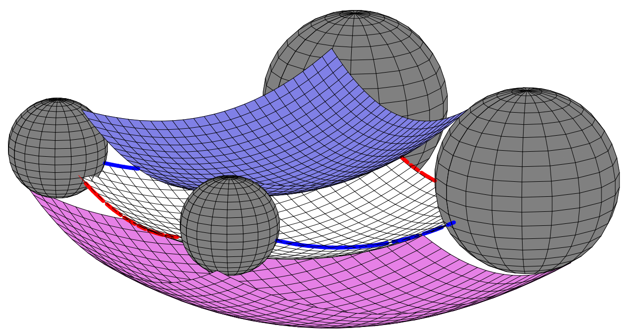
Vienna University of Technology,

<martin@geometrie.tuwien.ac.at>



Problem description

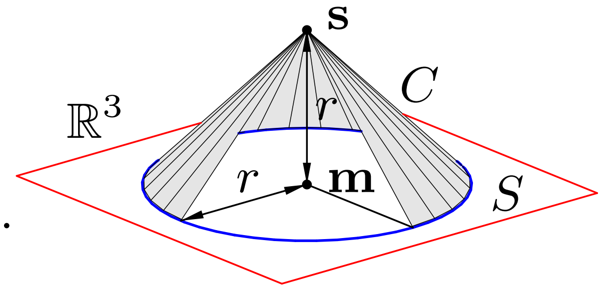
- Let $S(u, v)$ be a rational two-parameter family of spheres in $\mathbb{R}^3$.
- Conditions for rational envelope Φ of $S(u, v)$.
- Geometric characterization.
- Relation to rational offset surfaces.
- Explicit parameterizations.
- Known and new examples.



$\mathbb{R}^4$ as model of the space of spheres

- Cyclographic mapping: Points $\mathbf{s}$ in $\mathbb{R}^4$ $\rightarrow$ oriented spheres S in $\mathbb{R}^3$

$$\gamma : \mathbb{R}^4 \rightarrow \mathcal{S},$$
$$\mathbf{s} = (\mathbf{m}, r) \mapsto \gamma(\mathbf{s}) = S : (\mathbf{x} - \mathbf{m})^2 = r^2.$$



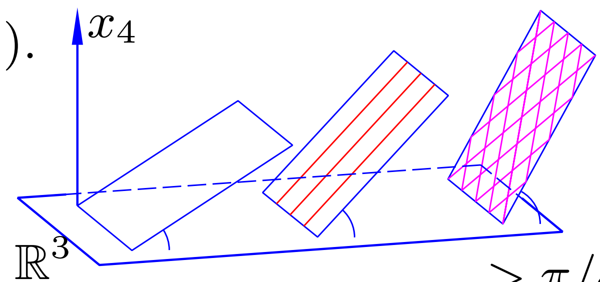
- Minkowski-space $\mathbb{R}^4$:

$$\langle \mathbf{x}, \mathbf{x} \rangle := \mathbf{x}^T D\mathbf{x}, \text{ with } D = \mathsf{diag}(1, 1, 1, -1).$$

- Light cone C with vertex $\mathbf{s}$:

$$C : \langle \mathbf{x} - \mathbf{s}, \mathbf{x} - \mathbf{s} \rangle = 0.$$

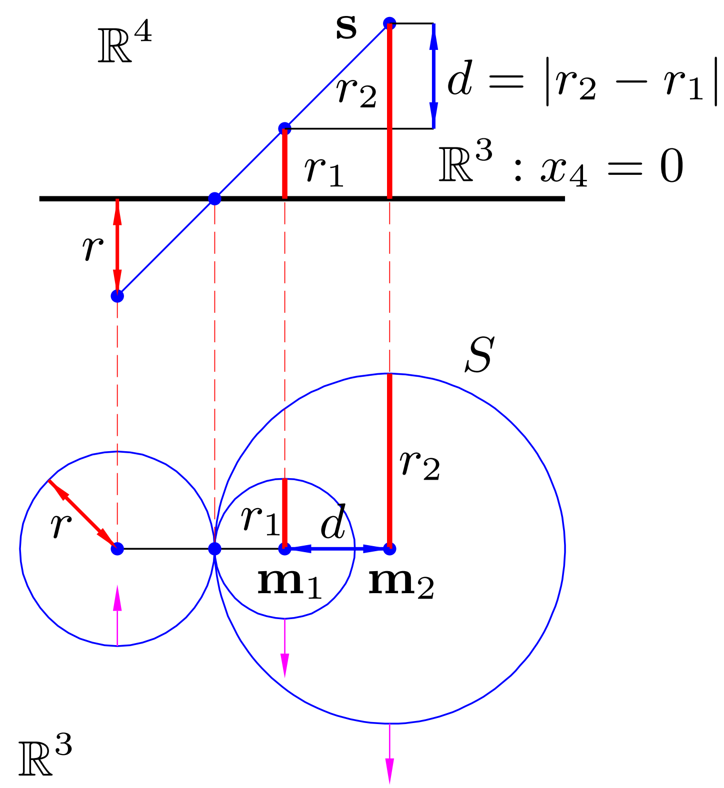
- $C \cap \mathbb{R}^3$ is the non-oriented sphere S.



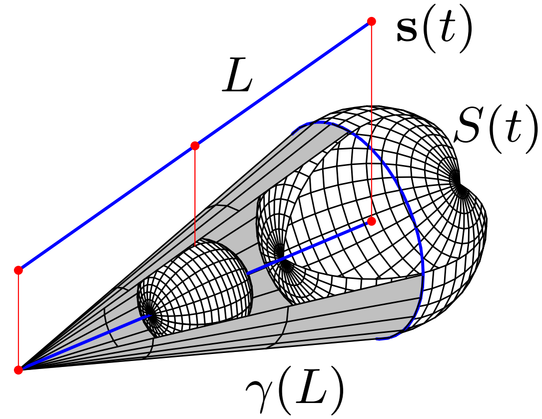
Touching spheres

Corollary 1 *The oriented spheres S_1 and S_2 are in oriented contact,*
$\iff \langle \mathbf{s}_2 - \mathbf{s}_1, \mathbf{s}_2 - \mathbf{s}_1 \rangle = 0.$

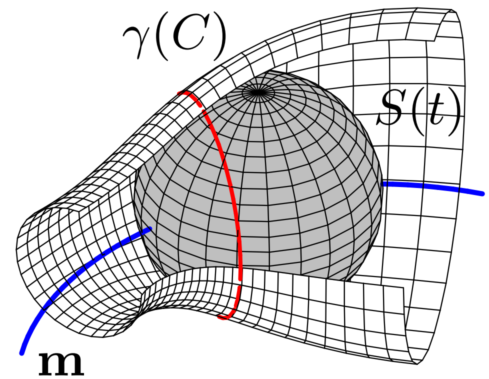
- $\mathbf{s}_2 - \mathbf{s}_1$ ... isotropic vector
- Sphere X touches sphere $S \Longrightarrow$
  $X \in$ light cone $C : \langle \mathbf{x} - \mathbf{s}, \mathbf{x} - \mathbf{s} \rangle = 0.$



One-parameter families of spheres



- A curve $C : \mathbf{c}(t)$ in $\mathbb{R}^4$ corresponds to a one-parameter family of spheres.
- Real envelope $\gamma(C) \iff \langle \dot{\mathbf{c}}, \dot{\mathbf{c}} \rangle \geq 0$.
- Line $L \mapsto \gamma(L)$ is a cone of revolution.
- Curve $C \mapsto \gamma(C)$ is a canal surface.



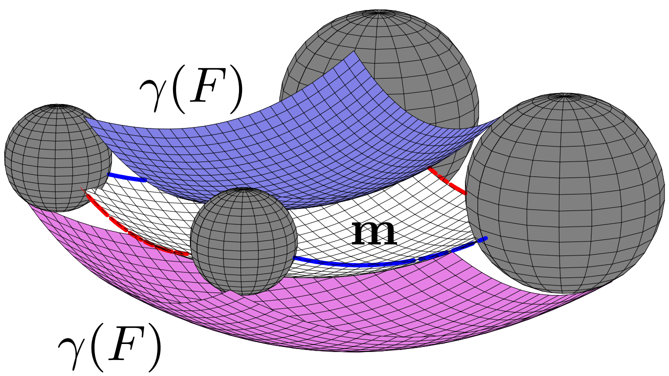
Two-parameter families of spheres

A surface $F : \mathbf{f}(u,v)$ in $\mathbb{R}^4$ corresponds to a two-par. family of spheres.

- The envelope $\gamma(F)$ is real $\iff$

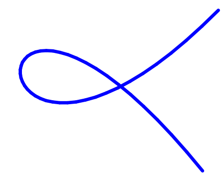
  $\langle \lambda \mathbf{f}_u + \mu \mathbf{f}_v, \lambda \mathbf{f}_u + \mu \mathbf{f}_v \rangle \geq 0,$ for all λ, μ.

- The envelope $\gamma(F)$ consists of two sheets of surfaces locally.

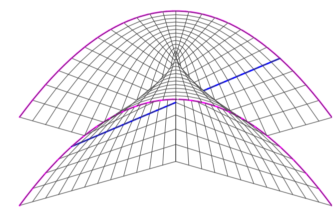


Known results

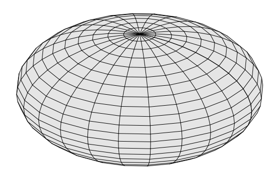
- Rational curve C in $\mathbb{R}^4$: $\gamma(F)$ is a rational canal surface in $\mathbb{R}^3$. [Lü, Pottmann, H.'96; Pet., Pottmann'96, Krasauskas'07]

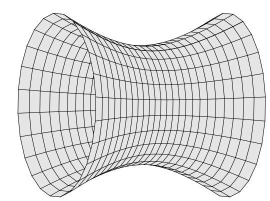


- Rational ruled surface $F \in \mathbb{R}^4$: $\gamma(F)$ is the envelope of cones of revolution. [Pottmann, Lü, Ravani'96, Pet.'97]

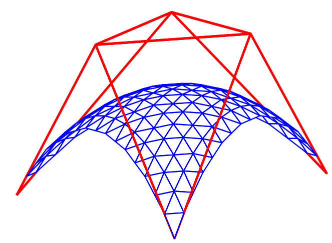


- 2-dim. quadric F in $\mathbb{R}^4$. [Lü'96; Pet.'97]





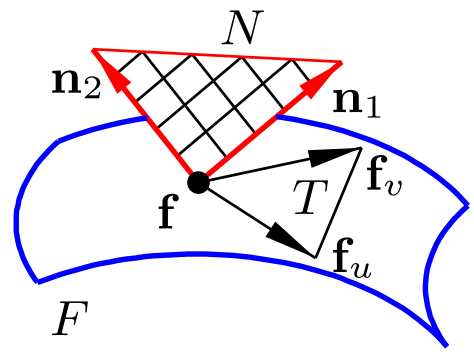
- Quadratically parameterized surface F in $\mathbb{R}^4$, (quadratic Bézier surface). [Pet., Odehnal, Sampoli'08]



- MOS-surfaces. [Kosinka, Jüttler'07]

Isotropic normal vector fields

- Rational surface $F : \mathbf{f}(u, v)$ in$\mathbb{R}^4$.
- Tangent plane T is Euclidean $\Longleftrightarrow$ normal plane N is pseudo-Euclidean.
- *Isotropic normal vectors* $\mathbf{n}_1$ and $\mathbf{n}_2 \in N$.



- MOS-surfaces [Kosinka, Jüttler '07]: The envelope $\gamma(F)$ admits rational parameterizations $\Longleftrightarrow$

$$A(\mathbf{f}) := \langle \mathbf{f}_u, \mathbf{f}_u \rangle \langle \mathbf{f}_v, \mathbf{f}_v \rangle - \langle \mathbf{f}_u, \mathbf{f}_v \rangle^2 = \sigma(u, v)^2 \qquad (1)$$

  is the complete square of a rational function $\sigma(u, v)$.

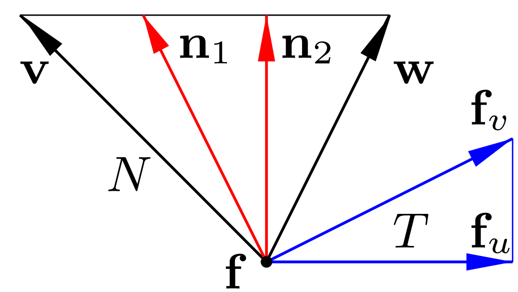
- We give geometric interpretations of condition (1) and explicit parameterizations of MOS-surfaces.

Rational isotropic normal vector fields

Corollary 2 *A rational surface* $\mathbf{f}(u,v)$ *in* $\mathbb{R}^4$ *is an MOS-surface* ($A(\mathbf{f}) = \sigma(u,v)^2$) $\Longleftrightarrow$ $\mathbf{f}(u,v)$ *possesses a rational isotropic normal vector field* $\mathbf{n}(u,v)$.

- Tangent plane T: $\mathbf{f} + \lambda \mathbf{f}_u + \mu \mathbf{f}_v$. Normal plane N: $\mathbf{f} + \lambda \mathbf{v} + \mu \mathbf{w}$.
- $\mathbf{n} = \mathbf{v} + t\mathbf{w}$ is isotropic, $\Longleftrightarrow$

$$\langle \mathbf{v} + t\mathbf{w}, \mathbf{v} + t\mathbf{w} \rangle = 0. \qquad (2)$$



- (2) has a *rational* solution, $\Longleftrightarrow$

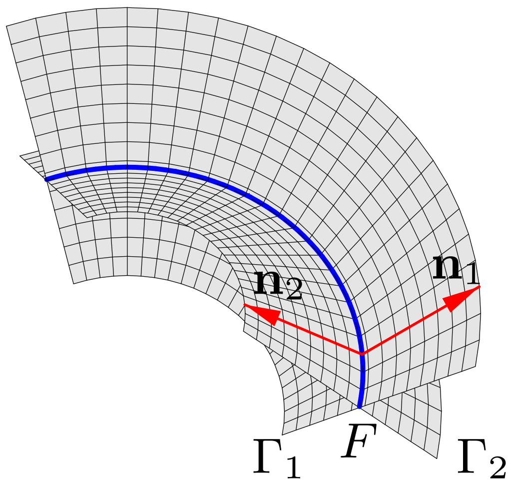
$$\begin{aligned} d &= \langle \mathbf{v}, \mathbf{w} \rangle^2 - \langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{w}, \mathbf{w} \rangle \\ &= \rho^2 (\langle \mathbf{f}_u, \mathbf{f}_u \rangle \langle \mathbf{f}_v, \mathbf{f}_v \rangle - \langle \mathbf{f}_u, \mathbf{f}_v \rangle^2) \\ &= \rho^2 A(\mathbf{f}) = \rho^2 \sigma(u,v)^2. \end{aligned}$$

Isotropic hypersurfaces in $\mathbb{R}^4$

- Rational 2-surface $F \in \mathbb{R}^4$ parameterized by $\mathbf{f}(u, v)$.
- Isotropic normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$.
- Isotropic hyper-surfaces Γ_1 nd Γ_2 through F are parameterized by

$$\mathbf{g}_i(u, v, w) = \mathbf{f}(u, v) + w\mathbf{n}_i(u, v), \; i = 1, 2.$$

- Rational $\mathbf{n}_i \Longrightarrow$ rational Γ_i.

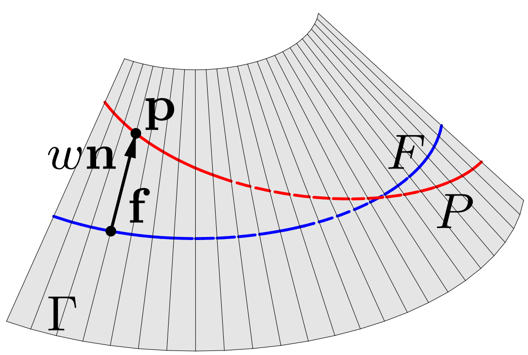


Corollary 3 *A rational surface* $\mathbf{f}(u, v)$ *in* $\mathbb{R}^4$ *is an MOS-surface* $\Longleftrightarrow$ $\mathbf{f}(u, v)$ *possesses a rational isotropic hyper-surface* $\mathbf{G}$.

Rational isotropic hypersurfaces in $\mathbb{R}^4$

Theorem 4 *Rational isotropic hyper-surface* Γ *through* F*, with rational isotropic normal vectors* $\mathbf{n} \Longrightarrow$

$\mathbf{n}(u, v)$ *is a rational isotropic normal vector field of any rational 2-dim. sub-variety* P *with* $\mathbf{p}(u, v) = \mathbf{f}(u, v) + w(u, v)\mathbf{n}(u, v)$*, with a rational* $w(u, v)$*.*

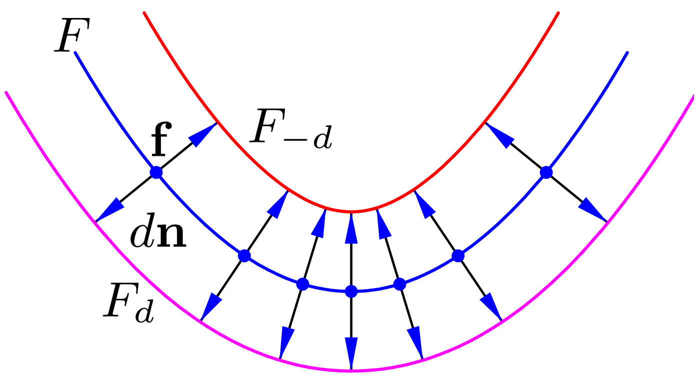


PN-surfaces – Rational offset surfaces

Definition 5 *A rational surface Φ in $\mathbb{R}^3$ is called PN-surface if it possesses a rational unit normal vector field $\mathbf{n}(u, v)$ corresponding to $\mathbf{q}(u, v)$.*

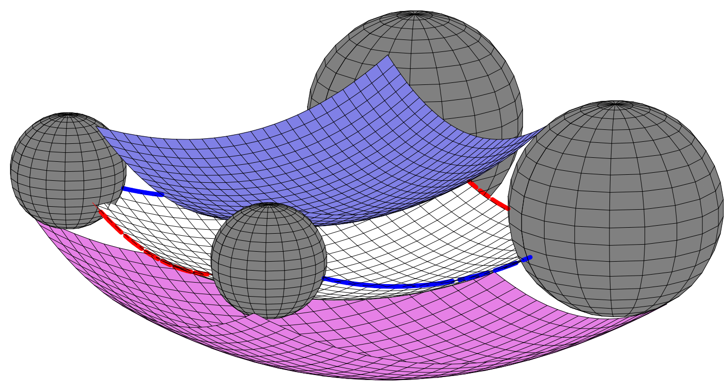
The offset surface Φ_d of Φ at oriented distance d admits a rational parameterization $\mathbf{q}_d(u, v) = \mathbf{q}(u, v) + d\mathbf{n}(u, v)$.

[Pottmann'95]



Rational envelopes of spheres and PN-surfaces

Corollary 6 *A rational surface $\Phi \in \mathbb{R}^3$ is a PN-surface $\iff$ Φ is the intersection of a rational isotropic hypersurface $\Gamma \subset \mathbb{R}^4$ with $\mathbb{R}^3 : x_4 = 0$.*



Theorem 7 *Let $S(u,v)$ be a rational two-parameter family of spheres corresponding to the MOS-surface F $\iff$ the envelope surface $\gamma(F)$ is a PN-surface.*

Explicit Representations of MOS-surfaces

- Generalized stereographic projection [Dietz, Hoschek,Jüttler'93] $\Longrightarrow$ universal parameterization $(A/D, B/D, C/D)$ of S^2, with

  $$A = 2(ac+bd),\ B = 2(bc-ad),\ C = a^2+b^2-c^2-d^2,\ D = a^2+b^2+c^2+d^2,$$

  with polynomials a,b,c and d in u and v.

- Tangent planes of a PN-surface Φ in $\mathbb{R}^3$:

  $$T(u, v) : Ax + By + Cz = h, \text{ with rational } h.$$

- A point representation $\mathbf{q}(u, v)$ of Φ follows by $T \cap T_u \cap T_v$.

Explicit Representations 2

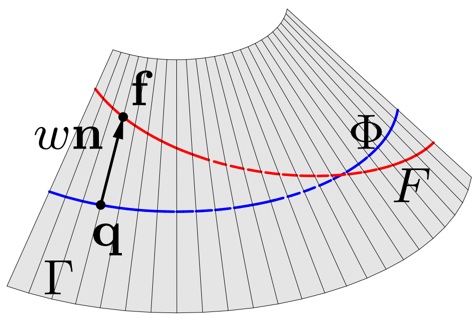
$$\mathbf{q} = \frac{1}{E}\begin{pmatrix} B(C_u h_v - C_v h_u) + C(h_u B_v - h_v B_u) + h(B_u C_v - B_v C_u) \\ C(A_u h_v - A_v h_u) + A(h_u C_v - h_v C_u) + h(C_u A_v - C_v A_u) \\ A(B_u h_v - B_v h_u) + B(h_u A_v - h_v A_u) + h(A_u B_v - A_v B_u) \end{pmatrix},$$

with $E = A(B_u C_v - B_v C_u) + B(C_u A_v - A_u C_v) + C(A_u B_v - A_v B_u)$.

- Rat. isotropic hyper-surface Γ through Φ:

$$\mathbf{g}(u,v,w) = (\mathbf{q}(u,v),0) + w\mathbf{n}(u,v),$$

with $\mathbf{n} = (A,B,C,D)$.



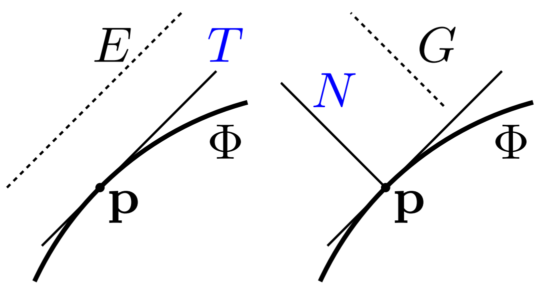
- MOS-surfaces:
  Rational 'radius function' $w(u,v)$

$$\mathbf{f}(u,v) = (\mathbf{q}(u,v),0) + w(u,v)\mathbf{n}(u,v).$$

New Example: Generalized LN-surfaces

Definition 8 *A rational two-dimensional surface* Φ *in* $\mathbb{R}^4$ *is called generalized LN-surface if it admits a parameterization* $\mathbf{p}(u,v)$ *in a way that* Φ*'s tangent planes are spanned by vectors* $\mathbf{s}(u,v)$ *and* $\mathbf{t}(u,v)$ *which are linear in the surface parameters* u *and* v*.*

- For almost all 3-spaces E in $\mathbb{R}^4$ there exists a unique tangent plane T of Φ with $T \parallel E$.
- For almost all lines G in $\mathbb{R}^4$ there exists a unique normal plane N of Φ with $G \parallel N$.



Construction of the surfaces in $\mathbb{R}^4$

- The tangent planes $T(u,v)$ of $\Phi \subset \mathbb{R}^4$ are intersections of 3-spaces

$$E(u,v) : \mathbf{e}(u,v)^T \mathbf{x} = a(u,v), F(u,v) : \mathbf{f}(u,v)^T \mathbf{x} = b(u,v).$$

- $T(u,v) = E(u,v) \cap F(u,v)$ possesses an envelope $\Longleftrightarrow$ it exists a solution of the overdetermined system

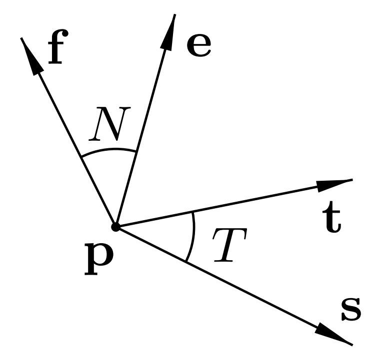
$$\begin{array}{lll} E : \mathbf{e}^T \mathbf{x} = a, & E_u : \mathbf{e}_u^T \mathbf{x} = a_u, & E_v : \mathbf{e}_v^T \mathbf{x} = a_v, \\ F : \mathbf{f}^T \mathbf{x} = b, & F_u : \mathbf{f}_u^T \mathbf{x} = b_u, & F_v : \mathbf{f}_v^T \mathbf{x} = b_v. \end{array}$$

  has a solution $\mathbf{p}(u,v)$.

- For any vector $\mathbf{w} \in \mathbb{R}^4$, the equations

$$\langle \mathbf{s}, \mathbf{w} \rangle = 0, \langle \mathbf{t}, \mathbf{w} \rangle = 0$$

  lead to rational expressions for u and v.



Surfaces of Type 1

- Normal form for the surface construction:

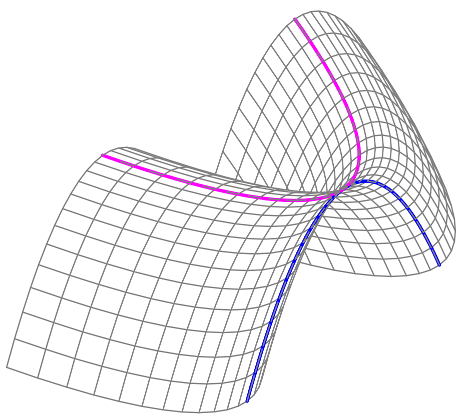
$$\mathbf{e}(u) = (1, 0, u, 0), \mathbf{f}(v) = (0, 1, 0, v), \text{ and } a_v = 0, b_u = 0.$$

- A generalized LN-surface Φ in $\mathbb{R}^4$ of type 1 is a *translational surface* $\mathbf{p}(u, v) = \mathbf{c}(u) + \mathbf{d}(v)$ with planar LN-curves as profile curves,

$$\mathbf{p}(u, v) = (a - ua_u, 0, a_u, 0) + (0, b - vb_v, 0, b_v)\,.$$

- The tangent planes T of Φ are spanned by $\mathbf{s} = (-u, 0, 1, 0)$ and $\mathbf{t} = (0, -v, 0, 1)$.
- For any vector $\mathbf{w} = (w_1, w_2, w_3, w_4)$, the equations $\mathbf{w}^T\mathbf{s} = 0$ and $\mathbf{w}^T\mathbf{t} = 0$ result in

$$u = \frac{w_3}{w_1}, v = -\frac{w_4}{w_2}.$$



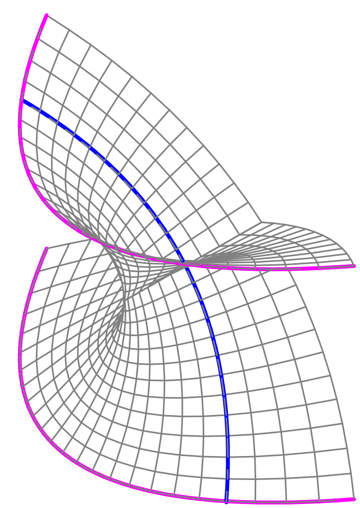
Surfaces of Type 2

- Normal form for the surface construction:

$$\mathbf{e}(u,v) = (1,0,-u,v), \mathbf{f}(u,v) = (0,-1,v,u), \text{ and } a_u = -b_v, a_v = b_u.$$

- A generalized LN-surface Φ in $\mathbb{R}^4$ of type 2 is a translational surface $\mathbf{p}(z) = 1/2(\mathbf{c}(z) + \overline{\mathbf{c}(z)})$ with a pair of planar conjugate complex LN-curves as profile curves, with $f = b + ia$, $z = u + iv$ and $\mathbf{c}(z) = (i(zf_z - f), (zf_z - f), if_z, f_z)$.

- Φ is a Euclidean minimal surface.



- A rational parameterization of Φ reads

$$\mathbf{p}(u,v) = (a - ua_u - va_v, -b - va_u + ua_v, -a_u, a_v)\,.$$

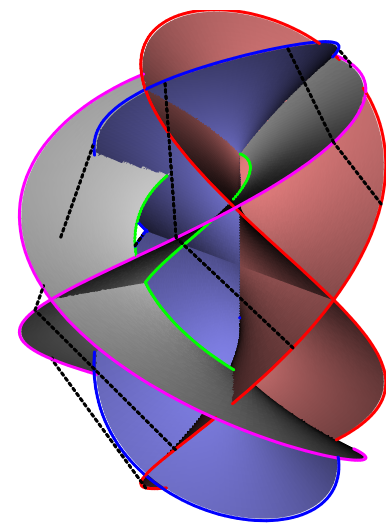
- The tangent planes T of F are spanned by $\mathbf{s} = (u,v,1,0)$ and $\mathbf{t} = (-v,u,0,1)$.

Envelopes corresponding to Type 2

- The tangent planes T of Φ are spanned by $\mathbf{s} = (u, v, 1, 0)$ and $\mathbf{t} = (-v, u, 0, 1)$.
- For any vector $\mathbf{w} = (w_1, w_2, w_3, w_4)$, the equations $\mathbf{w}^T\mathbf{s} = 0$ and $\mathbf{w}^T\mathbf{t} = 0$ result in

$$\begin{pmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -w_3 \\ w_4 \end{pmatrix}.$$

- With $\mathbf{w} = (2s, 2t, 1 - s^2 - t^2, 1 + s^2 + t^2)$ one finally obtains a rational parameterization of the envelope surface of the corresponding 2-par. family of spheres.



Surfaces of Type 3

- Normal form for the surface construction:

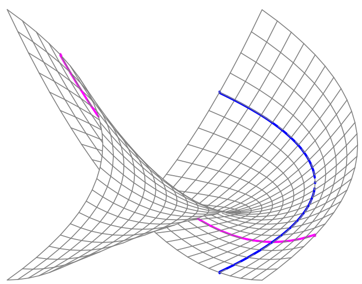
  $\mathbf{e} = (u, 0, -1, -u^2+v), \mathbf{f} = (-v, 1, 0, uv),$ and $b_u = va_v, b_v = -a_u - ua_v.$

- $a_{uu} = -ua_{uv} - 2a_v - va_{vv}$ and $b = \int va_v du$
  $- \int (\int (va_{vv} + a_v)du + a_u + ua_v)dv + C.$

- A rational parameterization of Φ reads

$$\mathbf{p}(u, v) = (ua_v - b_v, b - vb_v, -a + b_u - ub_v, a_v)\,.$$

Choosing polynomials $a(u, v) = -1/2u^3 + uv$ and $b(u, v) = 1/2u^2v - 1/2v^2$ we obtain a quadratically parameterized surface by $\mathbf{p}(u, v) = (1/2u^2 + v, 1/2v^2, uv, u)$.



Summary

- We have presented a geometric characterization of the class of surfaces in $\mathbb{R}^4$ which correspond to two-parameter families of spheres whose envelope surfaces admit rational parameterizations.
- New example: Generalized LN-surfaces in $\mathbb{R}^4$.

Thank you for your attention.