# Designing Rational Surfaces with Rational Offsets

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**Abstract.** Classical Laguerre geometry provides an efficient approach to the design of rational surfaces with rational offsets. The geometric transformation which describes the change from the isotropic model of 3–dimensional Euclidean Laguerre space to the standard model, maps an arbitrary rational surface onto a rational surface all whose offsets are rational. It is shown how to apply this concept to surface modeling. In particular, we present a new surface design scheme which uses triangular patches on parabolic Dupin cyclides.

## §1. Introduction

Offset curves and surfaces arise in various applications including NC milling, path planning for rapid prototyping, font design and geometric tolerancing. It is well–known that rational curves or surfaces do in general not possess rational offsets. In order to comply with current industry standards, offset curves or surfaces therefore have to be approximated in rational B–spline form [6,11].

A different approach to offsets takes its origin in the work of R. Farouki and T. Sakkalis [9] who proposed to use only those curves in the design process that do possess rational offsets. Farouki and Sakkalis introduced the so–called *Pythagorean-hodograph (PH) curves*, which are polynomial curves $x(t) = (x_1(t), x_2(t))$ with polynomial parametric speed $\sigma(t) = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$. Among other remarkable properties, these curves possess rational offsets $x_d(t) = x(t) + dn(t)$, where $n(t)$ is a field of unit normal vectors of the progenitor curve $x(t)$. Recent work on PH curves and their generalizations to the full class of rational curves with rational offsets showed that these curves are well suited for practical design purposes [1,2,7,8,12,19,20].

Using the dual representation, it has been shown in [19] that all rational surfaces with rational offsets (briefly called *PH surfaces* in the sequel) can be described explicitly. Based on the dual Bézier or B–spline representation, one can construct PH

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surfaces from the dual representation of spherical patches. Furthermore, unexpected results on offsets of special classes of surfaces could be derived [13,14,15,18,21].

We will now present a new geometric approach to PH surfaces based on Laguerre geometry. After a brief introduction into this classical type of geometry, we describe a geometric transformation that maps any rational surface onto a PH surface. Any PH surface may be obtained in this way. Graphs of polynomial or rational bivariate spline functions are mapped onto patchworks of PH surfaces. As an example, paraboloids are mapped onto parabolic Dupin cyclides and any modeling scheme with bivariate quadratic splines is transformed into a surface design scheme with cyclide patches. This new application of cyclides in free–form surface design offers more flexibility than existing techniques (see [10,16,23,24,25,26] and the references therein).

## §2. A Brief Introduction to Laguerre Geometry

The fundamental elements of Laguerre geometry in Euclidean $\mathbb{R}^3$ are *oriented planes* and *cycles*. A cycle is an oriented sphere or a point (sphere with radius 0). The orientation is determined by a unit normal vector field or equivalently by a signed radius in the case of the sphere. If no ambiguity can arise, oriented planes will simply be called 'planes' in the sequel.

The basic relation in Laguerre geometry is that of *oriented contact of cycle and plane*. An oriented sphere and a plane are in oriented contact, if they are tangent and the unit normals coincide in the point of contact. For a point and a plane, oriented contact equals incidence.

Laguerre geometry studies properties which are invariant under Laguerre transformations. A *Laguerre transformation* consists of two bijective maps, one in the set of cycles and the other one in the set of planes. Additionally, a Laguerre transformation preserves oriented contact and non–contact between cycles and planes. A simple example of a Laguerre transformation is a *dilatation* which adds a constant $d \neq 0$ to the signed radius of each cycle and leaves its midpoint unchanged. Note that this map does not preserve points. Considering a surface as envelope of its oriented tangent planes, a dilatation maps the surface onto its offset at distance $d$. This already indicates the advantage of using Laguerre geometry in connection with offsets.

Using cartesian coordinates $x_i$ in $\mathbb{R}^3$, each plane $e$ is defined by a linear equation $e_0 + e_1x_1 + e_2x_2 + e_3x_3 = 0$. The coefficients $e_i$ are called plane coordinates of $e$. The vector $(e_1, e_2, e_3)$ defines the orientation of the plane and is always assumed to be normalized, $e_1^2+e_2^2+e_3^2 = 1$. Embedding $\mathbb{R}^3$ in $\mathbb{R}^4$ as hyperplane $x_4 = 0$, we map each plane $e$ to the hyperplane $E = \zeta(e)$, defined by the homogeneous coordinates

$$E = (e_0, e_1, e_2, e_3, 1). \tag{2.1}$$

Note that the euclidean angle of $E$ and $\mathbb{R}^3$ equals $\pi/4$. A sphere with midpoint $m$ and signed radius $r$ is represented by the point $(m_1, m_2, m_3, r)$ in $\mathbb{R}^4$. This four–dimensional model of Laguerre geometry is called *cyclographic model*.

If more emphasis is on planes rather than cycles, one might be interested in a model where planes appear as points. This can easily be done by applying a duality $\delta : \mathbb{R}^{4*} \to \mathbb{R}^4$ (Blaschke map),

$$\delta(E) = (1, e_1, e_2, e_3, e_0). \tag{2.2}$$

The set of hyperplanes $E$ is mapped to points contained in a hypercylinder $\Delta \subset \mathbb{R}^4$ with equation $\Delta : x_1^2 + x_2^2 + x_3^2 = 1$. In this *Blaschke model*, Laguerre transformations are seen as projective maps which preserve $\Delta$.

Finally, we obtain an affine space (appropriately extended) as model of Laguerre space. For that, let $w$ be the generator line of $\Delta$ containing the point $W(0,0,1,0)$. Furthermore, let $\bar{\mathbb{R}}^3$ be the hyperplane $x_3 = 0$ in $\mathbb{R}^4$; we use coordinate functions $y_1 = x_1, y_2 = x_2, y_3 = x_4$ in it. Applying the 'stereographic' projection $\sigma : \Delta - w \to \bar{\mathbb{R}}^3$ with center $W$ to the cylindrical model, we get a map from planes $e$ with unit normal $\neq (0,0,1)$ to points in $\bar{\mathbb{R}}^3$ via

$$\sigma \circ \delta \circ \zeta(e) = \frac{1}{1 - e_3}(e_1, e_2, e_0). \tag{2.3}$$

Interpreting cycles as sets of tangent planes, we may state an important well-known result.

**Lemma 2.1.** *The set of tangent planes of a cycle $\Sigma$ is mapped with $\sigma \circ \delta \circ \zeta$ to the set of points of a paraboloid of revolution or a plane $\Psi$ satisfying*

$$\Psi : 2y_3 + (y_1^2 + y_2^2)(r + m_3) + 2y_1 m_1 + 2y_2 m_2 + r - m_3 = 0. \tag{2.4}$$

This can be proved by straightforward calculation.

So far, planes with unit normal $(0,0,1)$ do not have an image point in $\bar{\mathbb{R}}^3$. One therefore forms the so-called isotropic conformal closure $I^3 := \bar{\mathbb{R}}^3 \cup \mathbb{R}$ of $\bar{\mathbb{R}}^3$ and an extended map

$$\Lambda := \bar{\sigma} \circ \delta \circ \zeta,$$

which maps the plane $(e_0, 0, 0, 1)$ onto the real number $e_0$. To fix the problem of missing images of exceptional planes in Lemma 2.1, we have to extend the paraboloids $\Psi$ by $r + m_3$, which equals 0 for a plane $\Psi$. The resulting model of Laguerre space, where planes are represented as points and cycles appear as paraboloids or planes, is called *isotropic model*. The transformation $\Lambda$ describes the change from the standard model to the isotropic model.

In $I^3$, $y_3$–parallel lines and planes (called *isotropic lines and planes*) need a special treatment. The points of isotropic lines represent parallel planes. Non-isotropic lines as well as ellipses, whose projection onto $y_3 = 0$ are circles, and parabolas with isotropic axis are called isotropic Möbius circles. They may be obtained as intersection of 2 surfaces (2.4) and therefore are the $\Lambda$–image of the common tangent planes of 2 cycles, i.e. the planes of a pencil or the tangent planes of a cone of revolution. This also shows that isotropic planes (as well as isotropic cylinders through the other isotropic Möbius circles) represent planes parallel to the tangent planes of a cone of revolution, that may degenerate to a pencil.

In the isotropic model, Laguerre transformations are seen as so-called isotropic Möbius transformations. We just mention two special cases. For a translation in the standard model, represented in plane coordinates by $(e_0, \dots, e_3) \mapsto (e_0 + ae_1 + be_2 + ce_3, e_1, e_2, e_3)$, we obtain in $I^3$

$$(y_1, y_2, y_3) \mapsto (y_1, y_2, y_3 + ay_1 + by_2 + \frac{c}{2}(y_1^2 + y_2^2 - 1)). \tag{2.5}$$

A dilatation $(e_0, \dots, e_3) \mapsto (e_0 + d, e_1, e_2, e_3)$ appears as isotropic Möbius transformation

$$(y_1, y_2, y_3) \mapsto (y_1, y_2, y_3 + \frac{d}{2}(y_1^2 + y_2^2 + 1)). \tag{2.6}$$

For more details on Laguerre geometry, we refer the reader to Benz [3], Blaschke [4] and Cecil [5].

## §3. A Laguerre Geometric Approach to PH Surfaces

A surface $\Phi$ in $\mathbb{R}^3$ is a PH surface iff it possesses a rational parametric representation $s(u, v) = (s_1(u, v), s_2(u, v), s_3(u, v))$ such that the field of unit normal vectors $n(u, v)$ is rational in $u$ and $v$. Then, all its offsets $s_d = s + dn$ are rational.

We now view $\Phi$ as set of its oriented tangent planes. They are oriented with $n$ and their plane coordinates are rational functions,

$$(e_0, \dots, e_3) = (-n_1 s_1 - n_2 s_2 - n_3 s_3, n_1, n_2, n_3). \tag{3.1}$$

Applying the transformation $\delta \circ \zeta$, we obtain a rational 2–surface in the Blaschke–cylinder $\Delta$. Conversely, if a rational 2–surface in $\Delta$ is given, $(\delta \circ \zeta)^{-1}$ maps it onto a PH surface $\Phi$. Laguerre transformations appear as projective maps in the Blaschke model and projective maps preserve the rationality of a surface. This proves the following result.

**Theorem 3.1.** *Laguerre transformations map PH surfaces (viewed as sets of oriented tangent planes) onto PH surfaces.*

The stereographic projection $\sigma$ defines a bijective map between rational surfaces in $\Delta$ and rational surfaces in the isotropic model $I^3$, which gives us a simple construction of PH surfaces.

**Theorem 3.2.** *The geometric transformation $\Lambda^{-1}$, which describes the change from the isotropic model of Laguerre space to the standard model, maps arbitrary* *rational surfaces onto PH surfaces. All non–developable PH surfaces may be obtained in this way.*

Note that surfaces $\Phi$ with a one–parameter set of tangent planes (*developable PH surfaces*) correspond to *rational curves* $\Psi \subset I^3$.

A further degeneracy should be pointed out. An isotropic line is mapped via $\Lambda^{-1}$ onto a pencil of parallel planes. Hence, a rational cylinder $\Psi$ with isotropic generators belongs to a 2–parameter set of planes that touch a rational curve at infinity. Moreover, if the rational surface $\Psi$ possesses a real curve along which it is touched by isotropic lines (occluding contour for $y_3$–parallel projection), the corresponding surface $\Phi$ contains a real curve at infinity.

Let us assume that $\Psi$ is the graph surface of a bivariate piecewise rational function defined over a triangulation in the $y_1, y_2$–plane. Then $\Phi$ is composed of triangular PH patches without points at infinity and it possesses an injective Gauss map $s(u,v) \mapsto n(u,v)$. If $\Phi$ is regular and $C^2$, it does not possess a change in the sign of Gaussian curvature. Note that parabolic surface points of $\Phi$ correspond to singularities in the Gauss map and therefore to singularities in the surface $\Psi$.

Conversely, suppose that $\Psi$ is regular, then $\Phi$ is regular as a set of planes, but not necessarily as a point set. Note also that Laguerre geometry cannot distinguish between a surface and its offsets (those surfaces are related by Laguerre transformations!). But clearly, the offsets of a regular surface need not be regular.

Let $\Psi_1, \Psi_2$ be two rational surfaces that meet at a point $p$ with a common non–isotropic tangent plane $\tau$. Then $\Phi_1, \Phi_2$ are PH surfaces meeting with common tangent plane $\Lambda^{-1}(p)$ at a common point $x \in \mathbb{R}^3$ (the point of tangency between $\Lambda^{-1}(p)$ and the cycle $\Lambda^{-1}(\tau)$). Thus, two adjacent rational patches $\Psi_1, \Psi_2$ on a $G^1$ surface are mapped onto two PH patches $\Phi_1, \Phi_2$ meeting with common tangent planes along a common curve. This joint can either be smooth or like an edge of regression (see Fig. 3). If it is smooth and $\Psi_1, \Psi_2$ meet with $G^k$ continuity, also $\Phi_1, \Phi_2$ form a $G^k$ joint.

## §4. Surface Design with Cyclide Patches

### 4.1 Parabolic Dupin cyclides as $\Lambda^{-1}$ images of paraboloids

We summarize some properties of parabolic Dupin cyclides which are important for surface design with cyclide patches. Let $\mathbb{R}^3$ be Euclidean 3-space, $\mathbb{P}^3$ its projective extension. $\mathbb{P}^3 - \mathbb{R}^3 = \Omega$ is the *ideal plane* at infinity. Using homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$, points in $\Omega$ are characterized by $x_0 = 0$.

A parabolic Dupin cyclide $\Phi$ is a special rational surface of algebraic order 3. Using an appropriate coordinate system, a homogeneous representation is

$$\phi(u,v) = (1 + u^2 + v^2, au + (a-b)uv^2, bv + (b-a)u^2v, au^2 + bv^2). \qquad (4.1)$$

An implicit equation corresponding to (4.1) is

$$x_0(bx_1^2 + ax_2^2 + (a+b)x_3^2) - x_3(abx_0^2 + x_1^2 + x_2^2 + x_3^2) = 0. \tag{4.2}$$

The set of tangent planes of $\Phi$ has the parametrization

$$\psi(u,v) = (-(au^2 + bv^2), 2u, 2v, u^2 + v^2 - 1). \tag{4.3}$$

The intersection of a parabolic Dupin cyclide $\Phi$ with the plane at infinity $\Omega : x_0 = 0$ consists of the absolute conic $j$, which is determined by $x_1^2 + x_2^2 + x_3^2 = 0$ and a real line $g : x_3 = 0$. Further, $\Phi$ contains two real lines $e, f \subset \mathbb{R}^3$ which are perpendicular to each other and determined by the equations $e : x_1 = 0, x_3 - ax_0 = 0$ and $f : x_2 = 0, x_3 - bx_0 = 0$.

For a plane $\varepsilon$ passing through $e$ or $f$, the intersection $\varepsilon \cap \Phi$ is a circle. These circles (parameter lines in (4.1)) are principal curvature lines. Along each of the lines $e, f$ the tangent plane is constant, thus $e, f$ are called *parabolic lines.*

There are three types of parabolic Dupin cyclides, determined by the choice of the constants $a, b$. We speak of a *horn cyclide* for $ab > 0$ (Fig.1) or a *ring cyclide* for $ab < 0$. The limit case ($a = 0$ or $b = 0$) is sometimes called *thorn cyclide.* The special choice $a = b$ yields a sphere. Offsetting needs not preserve the type of the cyclide, hence these types are not distinguishable within Laguerre geometry.

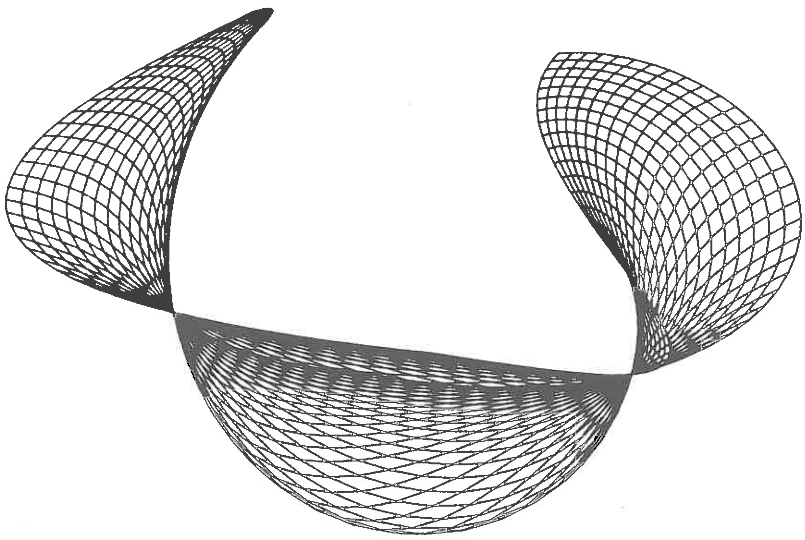

**Fig. 1.** A rational bicubic patch on a parabolic horn cyclide.

Parabolic Dupin cyclides form an eight parameter family of surfaces in $\mathbb{R}^3$. Let $g \in \Omega$ be the real line $x_0 = 0$, $x_3 = 0$. The family of parabolic Dupin cyclides passing through $g$ has six real parameters, and will be denoted by $G$.

Using a linear parameter transformation, each parabolic Dupin cyclide $\Phi \in G$ admits a dual parametrization of the form

$$\psi(u,v) = (q(u,v), 2u, 2v, u^2 + v^2 - 1). \tag{4.4}$$

where $q(u,v)$ is a quadratic polynomial. $\Lambda$ maps tangent planes of $\Phi \in G$ to points of the graph of a quadratic polynomial, denoted by $\Psi$. With $u = y_1$ and $v = y_2$ the equation of $\Psi$ is

$$y_3 = \frac{1}{2} q(y_1, y_2).$$

**Theorem 4.1.** *The $\Lambda^{-1}$ image of the graph $\Psi$ of a quadratic function is a parabolic Dupin cyclide $\Phi$ or a sphere.*

Substituting $u = t$ and $v = \lambda t + \mu$ in (4.3), the parametrization defines a quadratic cone $\Gamma(\lambda, \mu)$. The same substitution applied to (4.1), we obtain in general cubic curves $c(\lambda, \mu)$ and $\Gamma(\lambda, \mu)$ is tangent to $\Phi$ at points of $c$. The intersection $c \cap \Omega$ consists of one real point in $g$ and two conjugate complex points in $j$. Thus $c$ is called *cubic circle*. Let $k = \Gamma \cap \Omega$. Then $k$ is tangent to $g$ and $j$ in the points $c \cap \Omega$. This implies that $\Gamma(\lambda, \mu)$ is a cone of revolution, and leads us to the following

**Theorem 4.2.** *For a parabolic Dupin cyclide $\Phi$ there exists a two parameter family of cones of revolution each of which is tangent to the surface along a rational cubic or along a circle and a line.*

These properties can also be derived in $I^3$. But note that $\Lambda(\Phi)$ depends on the stereographic projection $\sigma$. By our special choice of the center, only parabolic Dupin cyclides $\Phi \in G$ are mapped to quadratic graph surfaces. Then, $\Lambda$–images of cones $\Gamma(\lambda, \mu)$ are non–isotropic lines or parabolas. If $\Phi \notin G$, $\Lambda(\Phi)$ is a rational surface of order 4. $\Lambda(\Gamma)$ can be any type of isotropic Möbius circle.

**Remark 4.1.** Along a cubic $c$ the surface normals form a constant angle with the axis of $\Gamma(\lambda, \mu)$, i.e. $c$ is an *isophote* on the cyclide. More general isophotes correspond to circles in the parameter plane. Then, (4.1) shows that the general isophotes on parabolic cyclides are rational sextics. They are good candidates for boundaries of trimmed cyclide patches, since one may apply a circular spline algorithm to generate the boundary in the parameter plane.

A quadratic Bézier patch on the paraboloid $\Psi$ over a triangle in the $y_1, y_2$–plane yields via $\Lambda^{-1}$ a dual quadratic Bézier triangle on a cyclide $\Phi$ with control planes $u_{ijk}$. To perform the conversion from the dual to the point representation, we run one step of the deCasteljau algorithm. This results in 3 planes that intersect in the desired surface point,

$$x(r,s,t) = (ru_{200} + su_{110} + tu_{101}) \times (ru_{110} + su_{020} + tu_{011}) \times (ru_{101} + su_{011} + tu_{002}),$$

where $r, s, t$ are the barycentric coordinates and $a \times b \times c$ denotes the exterior product of 3 vectors in $\mathbb{R}^4$. Hence, in the standard point representation the patch is representable as a *rational cubic Bézier triangle* on the cyclide $\Phi$, whose control points follow immediately from the formula above. Analogously, parallelogram domains in the parameter plane yield *rational bicubic cyclide patches* (Fig. 1).

**Remark 4.2.** Consider the lines $e$ and $f$ on the cyclide (4.1) as axes of a *hyperbolic net* consisting of all lines that intersect both $e$ and $f$. Then pick a point $\phi(u,v) \notin e, f$ on the surface, pass the unique line of the net through it and intersect it with the plane $x_3 - cx_0 = 0$. One gets the point

$$(x_1/x_0, x_2/x_0) = ((a-c)u, (b-c)v).$$

We may view the image plane as parameter plane and this shows that the map from the surface to the parameter plane is a net projection composed with an affine map (similarity for $c = (a+b)/2$). This standard way to parameterize a cubic surface (see also [26]) gives a better understanding of the geometry of the parametrization. It is now quite obvious that (4.1), $\mathbb{R}^2 \to \Phi$, does not reach the regular points on the parabolic lines $e, f$. We can generate rational curves and surface patches on the cyclide by projecting rational curves and surfaces with the net projection onto it. The image of a straight line, which does not intersect $e, f$, is a hyperbola of the form $uv + \alpha u + \beta v + \gamma = 0$ in the parameter plane and a rational quartic on the surface. We would like to mention that a detailed investigation of rational curves and surface patches on cyclides is currently performed by C. Mäurer [17] (for special patches, see also [26]).

**Remark 4.3.** *The transformation* $\Lambda$ *preserves principal curvature lines in the following sense.* Take a principal curvature line $c$ on a surface $\Phi$ and consider the developable surface $\Gamma$ touching $\Phi$ along $c$. Then $\Lambda$ maps the planes of $\Gamma$ onto the points of an *isotropic principal curvature line* of the image surface $\Psi$ of $\Phi$. Isotropic principal curvature directions at a surface point are conjugate directions, whose orthogonal projections onto the plane $y_3 = 0$ are perpendicular. This result may be proved by considering an osculating paraboloid (second order Taylor approximant) in $I^3$, which corresponds to an osculating parabolic cyclide in the standard model.

As an example, consider in $I^3$ the rational translational surface

$$\Psi(u,v) = (f_1(u), g_1(v), f_2(u) + g_2(v)),$$

with arbitrary rational functions $f_1, f_2, g_1, g_2$. Clearly, the parameter lines are isotropic curvature lines on $\Psi$. Thus, the transformation $\Lambda^{-1}$ into the standard model yields a *rational surface with rational offsets and rational principal curvature lines.* The principal curvature lines are isophotes and therefore *planar* (see [4], p. 279). Any parameter rectangle $[a,b] \times [c,d]$ corresponds to a *rational principal patch* in the sense of R.R. Martin [16]. It is an interesting subject for future research to use Laguerre geometry for generating an even larger class of rational principal PH patches, in particular to study those of low degree. Applications in NC milling mainly motivated the study of PH surfaces. Since curvature lines have been proposed as good cutter paths, principal PH patches are also interesting from a practical point of view.

## 4.2 Surface Modeling

**Proposition 4.3.** *If two parabolic Dupin cyclides $\Phi_1, \Phi_2$ share a rational cubic $c$ and the tangent cone of revolution $\Gamma$ along it, then they possess the same line at infinity.*

**Proof:** The intersection of $\Phi_1$ and $\Phi_2$ is an algebraic curve of order 9, which contains the absolute conic $j$ and the doubly counted cubic $c$. Therefore, the remaining component of the intersection is a line $g$. The complete intersection of a parabolic cyclide and the ideal plane $\Omega$ is the union of $j$ and a line. By our asssumption, $\Phi_1$ and $\Phi_2$ possess a common tangent plane ($\neq \Omega$) at the real point at infinity $U$ of $c$. Hence, the curves at infinity of the two cyclides share a common tangent at $U$ and this must be the common ideal line $g$. ■

$\Phi_1$ and $\Phi_2$ belong to the six parameter family $G$ of all parabolic Dupin cyclides passing through the line at infinity $g$. Hence, if we would like to construct a $G^1$ surface composed of parabolic Dupin cyclide patches, such that the patches meet smoothly along segments of cubic circles, we have to use patches on cyclides in the same family $G$. In the isotropic model (with an appropriate center of the stereographic projection), such a surface is the graph of a piecewise quadratic function. This suggests the following modeling technique.

Let $(A_i, \alpha_i)$ be coinciding vertices and oriented planes, considered as data elements in $\mathbb{R}^3$. The unit normals of $\alpha_i$ are denoted by $a_i$. The data must stem from a smooth surface with a constant sign of Gaussian curvature. We describe the construction of a $G^1$ surface, interpolating the given data. The surface is composed of rational cubic triangular patches in parabolic Dupin cyclides.

First, a coordinate transformation of the data $(A_i, \alpha_i)$ may be necessary. As the 'northpole' $W$ is the center of the stereographic projection, the points $a_i$ in the unit sphere $S^2$ should be situated suitably, i.e. the Gaussian image of the interpolating surface should not contain $W$. This means, for example, that we are not able to model closed convex surfaces. Note that $W$ determines the familiy $G$ of cyclides we are using; they all contain the line at infinity $x_0 = 0, x_3 = 0$.

$\Lambda$ maps the elements $(A_i, \alpha_i)$ to elements $(B_i, \beta_i)$, where $B_i = \Lambda(\alpha_i)$ is a point in $I^3$. Interpreting $A_i$ as cycle, $\Lambda$ maps it to a paraboloid of revolution or plane (2.4). Then $\beta_i$ is the tangent plane to $\Lambda(A_i)$ at $B_i$.

Let $b_i$ be the normal projection of $B_i$ onto $y_3 = 0$. We triangulate the vertices $b_i$, and then interpolate the data $(B_i, \beta_i)$ with the graph of a piecewise quadratic function. For that, Powell–Sabin elements [11,22] can be used. Although the isotropic model depends on the choice of the origin, our result will be independent of it. This follows from (2.5), which shows that a translation in design space will 'add' a quadratic function to the surface $\Psi$ in $I^3$. Translational invariance is now achieved by the quadratic precision of Powell–Sabin interpolants. Similarly, (2.6) shows that the method is *invariant under offsetting*: offsetting the data and

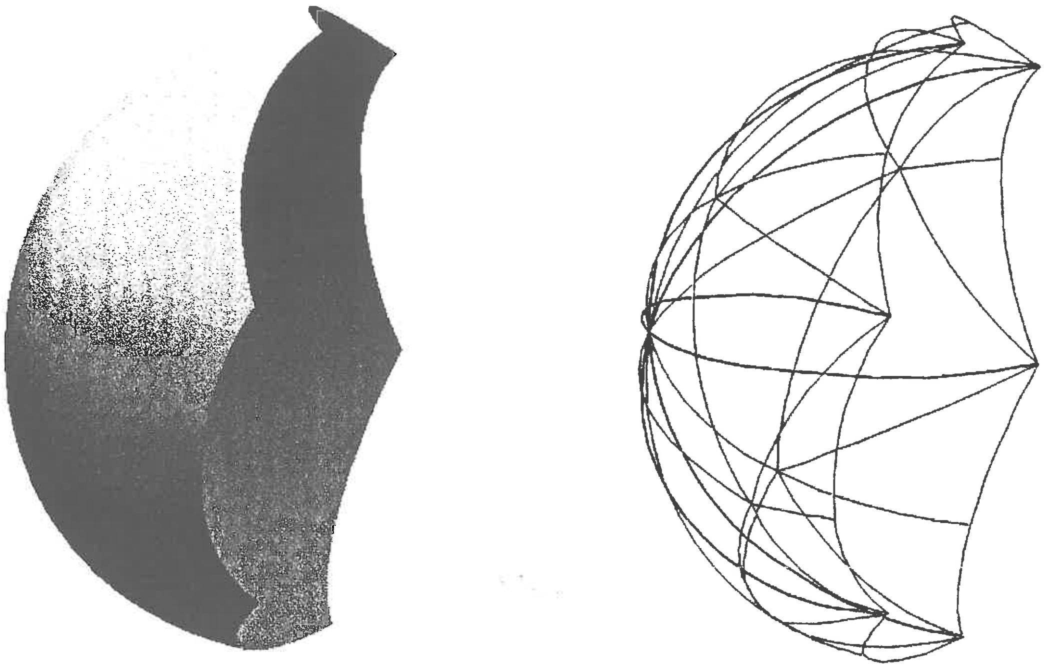

**Fig. 2.** Convex surface modelled with parabolic Dupin cyclide patches.

then constructing the interpolating surface yields the offset to the interpolant of the original data.

An example of the modeling technique based on Powell–Sabin 6–splits is shown in Figure 2. Here, the patches meet with $G^1$ continuity. As noted earlier, it is possible that although the patches share boundary curves with common tangent planes, the transition is not smooth, but forms a rigde (see Figure 3).

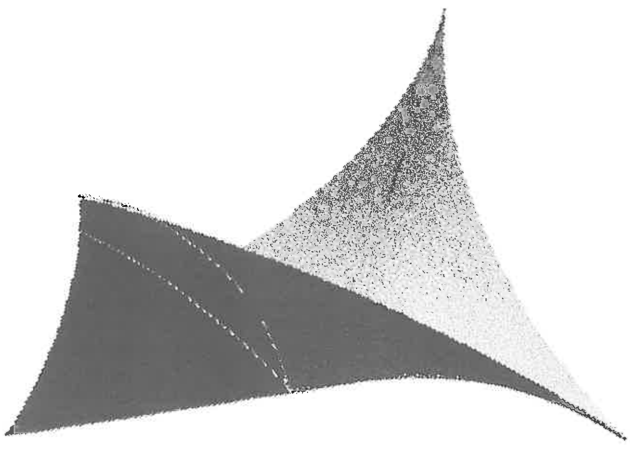

**Fig. 3.** The singularity problem.

Our modeling technique offers more flexibility than existing design schemes based on Dupin cyclides, although we use just cyclides of order 3. Of course, there are still the restrictions mentioned above. Some of them may be eliminated in future research. For example, the freedoms in constructing the microtriangles of a Powell–Sabin element may be used to apply a variational technique with an energy functional that serves to regularize the patches. However, smooth surfaces with vanishing Gaussian curvature along curves other than straight lines can never be modeled with parabolic Dupin cyclides.

In summary, the paper shows that appropriate geometric transformations can help to solve advanced geometric design problems with known approximation techniques. This has not yet been fully explored and seems to be a promising direction for future research.

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