

The Relation Between Offset and Conchoid Constructions

Martin Peternell^a, Lukas Gotthart^a, J. Rafael Sendra^b, Juana Sendra^c

^a*Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Vienna, Austria*

^b*Dpto. de Física y Matemáticas, Universidad de Alcalá, E-28871 Alcalá de Henares, Madrid, Spain*

^c*Dpto. Matemática Aplicada a la I.T. de Telecomunicación. Research Center on Software Technologies and Multimedia Systems for Sustainability (CITSEM), UPM, Spain*

Abstract

The one-sided offset surface F_d of a given surface F is, roughly speaking, obtained by shifting the tangent planes of F in direction of its oriented normal vector. The conchoid surface G_d of a given surface G is roughly speaking obtained by increasing the distance of G to a fixed reference point O by d . Whereas the offset operation is well known and implemented in most CAD-software systems, the conchoid operation is less known, although already mentioned by the ancient Greeks, and recently studied by some authors.

These two operations are algebraic and create new objects from given input objects. There is a surprisingly simple relation between the offset and the conchoid operation. As derived there exists a rational bijective quadratic map which transforms a given surface F and its offset surfaces F_d to a surface G and its conchoidal surfaces G_d , and vice versa. Geometric properties of this map are studied and illustrated at hand of some examples. Furthermore, rational universal parameterizations for offsets and conchoid surfaces are provided.

Keywords: offset surfaces, conchoid surfaces, pedal surface, inverse pedal surface, Darboux and Dupin cyclide.

1. Introduction

There is a large variety of contributions dealing with the geometry of offsets constructions discussing different aspects. Since we are focusing on parametrization problems we mention here only some of them, see for instance [2], [4], [5], [7], [9], [13], [17], [23] and references on the topic in [22]. Conchoidal constructions, although not so extensively studied, have been recently addressed by different authors too, see for instance [1], [11], [12], [18], [19], [20], [21]. Both geometric constructions were already utilized in the past (Leibnitz studied parallel curves and ancient Greeks used conchoids), and nowadays are used in practical applications (see e.g. [4], [5], [7], [8] and introduction of [18]).

Contribution: Considering an algebraic irreducible surface F and its continuous family of one-sided offset surfaces F_d , for $d \in \mathbb{R}$, there exists a birational quadratic map α so that the surface $G_d = \alpha(F_d)$ is the one-sided conchoid surface of $G = \alpha(F)$ for $d \in \mathbb{R}$. The inverse map realizes the correspondence between a family of one-sided conchoid surfaces G_d and a family of one-sided offset surfaces F_d . Since α is a birational map, rationality is preserved in both directions. All geometric properties and results, which are known for one family can be transformed to properties and results for the other family. To derive this correspondence, one-sided offset surfaces are considered

as envelopes of tangent planes. In addition, we introduce α for implicitly defined algebraic surfaces, and thus the results extend to the two-sided offsets and conchoids. Throughout this paper, we present the results for surfaces but they are valid for hypersurfaces, in particular for plane algebraic curves.

Sections 1 and 2 introduce to representations of offset surfaces and conchoid surfaces. Section 3 discusses the foot-point map realizing the correspondence between offsets and conchoids. Further, Sections 4 and 5 present elementary examples and with offsets and conchoids of ruled surfaces and quadrics more advanced ones. Finally the conclusion also outlines the relations to bisector surfaces. All computations and figures are carried out with the mathematical software Maple.

Notation.: The scalar product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ is denoted as $\mathbf{x} \cdot \mathbf{y}$. Points in the Euclidean space \mathbb{R}^3 are identified with their coordinate vectors $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ with respect to a chosen coordinate system. The projective extension of the Euclidean space \mathbb{R}^3 is denoted as \mathbb{P}^3 . Points X in projective space \mathbb{P}^3 are identified with their homogeneous coordinate vectors $X = (x_0, x_1, x_2, x_3)\mathbb{R} = (x_0 : x_1 : x_2 : x_3)$. Let $\omega = \mathbb{P}^3 \setminus \mathbb{R}^3$ be the plane at infinity in \mathbb{P}^3 , that is the plane defined by $x_0 = 0$. Ideal points, that are points at infinity, are represented by $(0, \mathbf{x})\mathbb{R}$, with $\mathbf{x} \in \mathbb{R}^3$. For points $X \notin \omega$, thus $x_0 \neq 0$, the relation between Cartesian and homogeneous coordinates is given by

$$x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}.$$

Planes U in \mathbb{P}^3 are the zero-set of a linear equation $U : u_0x_0 + \dots + u_3x_3 = 0$. The coefficients (u_0, \dots, u_3) are the homogeneous coordinates of U . Identifying U with these coordinates, we write $U = \mathbb{R}(u_0, u_1, u_2, u_3)$. The ideal plane ω is represented by $\mathbb{R}(1, 0, 0, 0)$. To distinguish points from planes, we use $(x_0, \dots, x_3)\mathbb{R}$ for points and $\mathbb{R}(u_0, \dots, u_3)$ for planes. In this way a plane $U \subset \mathbb{P}^3$ is identified with a point $U = \mathbb{R}(u_0, u_1, u_2, u_3) \in \mathbb{P}^{3*}$, where \mathbb{P}^{3*} denotes the dual space associated to \mathbb{P}^3 .

1.1. Offset construction

Consider an irreducible algebraic surface $F \subset \mathbb{R}^3$ given by a (non necessarily rational) parametrization $\mathbf{f}(u, v)$. Let $\mathbf{n}(u, v)$ be the unit normal vector field of $\mathbf{f}(u, v)$. Then the one-sided offset surface F_d of F at oriented distance $d \in \mathbb{R}$ is given by the parametrization

$$\mathbf{f}_d(u, v) = \mathbf{f}(u, v) + d\mathbf{n}(u, v), \text{ with } \|\mathbf{n}\| = 1. \quad (1)$$

Since \mathbf{n} is normalized, its partial derivatives \mathbf{n}_u and \mathbf{n}_v are orthogonal to \mathbf{n} . This implies that the tangent planes of the offset surface F_d are parallel to the tangent planes of F at corresponding points $\mathbf{f}(u, v)$ and $\mathbf{f}_d(u, v)$.

Often the offset surface of the surface F is defined as the envelope of a family of spheres of radius d , centered at the base surface F . Let us denote this offset as $O_d(F)$. This definition obviously differs from the definition given by (1). Let us analyze the relation between these two concepts. In [17] it is shown that $O_d(F)$ is algebraic and that it has at most two irreducible components. If $O_d(F)$ is reducible then the two components are F_d and F_{-d} , the one-sided offsets at distances d and $-d$. If $O_d(F)$ is irreducible, then $O_d(F)$ is the Zariski closure of F_d and also of F_{-d} . In any case, $O_d(F) = F_d \cup F_{-d}$. But it should be noted that, if $O_d(F)$ is irreducible, then F_d and F_{-d} are not algebraic surfaces in the strict sense, since they represent the exterior and interior analytic components of the two-sided offset $O_d(F)$. Focusing on rational surfaces, we have the following definition.

Definition 1. *A rational surface F is called rational offset surface if F admits a rational parametrization $\mathbf{f}(u, v)$ with rational unit normal vector field $\mathbf{n}(u, v)$ of $\mathbf{f}(u, v)$.*

Note that if F is a rational offset surface, then both F_d and F_{-d} admit a rational representation (1). Furthermore,

- if $O_d(F)$ is irreducible then $O_d(F)$ is rational iff F_d admits a rational representation (1), and analogously for F_{-d} ;
- if $O_d(F)$ is reducible then all components of $O_d(F)$ are rational iff F_d and F_{-d} admit a rational representation (1).

1.2. Conchoidal construction

Consider, as above, an irreducible algebraic surface $G \subset \mathbb{R}^3$, $d \in \mathbb{R}$, and a fixed reference point O . Without loss of generality we assume that O is the origin of a Cartesian coordinate system. Let G be represented by a polar (not necessarily rational) representation $\mathbf{g}(u, v) = r(u, v)\mathbf{s}(u, v)$, with $\|\mathbf{s}(u, v)\| = 1$. We call $\mathbf{s}(u, v)$ the spherical part of the parameterization $\mathbf{g}(u, v)$ and $r(u, v)$ the radius function. In this situation, the one-sided conchoid surface G_d of G is obtained by increasing the radius $r(u, v)$ by d and thus G_d admits the polar representation

$$\mathbf{g}_d(u, v) = (r(u, v) + d)\mathbf{s}(u, v). \quad (2)$$

In [1], [18], [19], the conchoidal variety is introduced considering both analytic components G_d and G_{-d} . More precisely, the conchoid surface $C_d(G)$ of G with respect to O at distance d is the (Zariski closure) set of points Q in the line OP at distance d of a moving point $P \in G$. That is the Zariski closure of the set

$$C_d(G) = \{Q \in OP \text{ with } P \in G, \text{ and } \overline{QP} = d\}. \quad (3)$$

As in the case of offset surfaces in Section 1.1 the two given definitions (2) and (3) for conchoids differ, but are clearly related. The conchoid surface $C_d(G)$ has at most two irreducible components (see [18], [19]). If $C_d(G)$ is reducible then the two components are $C_d(G)$ and $C_{-d}(G)$, the one-sided conchoids at distances d and $-d$. If $C_d(G)$ is irreducible, then $C_d(G)$ is the Zariski closure of G_d , and also of G_{-d} . In any case, $C_d(G) = G_d \cup G_{-d}$. But it should be noted that, if $C_d(G)$ is irreducible, then G_d and G_{-d} are not algebraic surfaces in the strict sense, since they represent the exterior and interior analytic components of the two-sided conchoid $C_d(G)$. To deal with rational surfaces G and their conchoid surfaces G_d , we define the following.

Definition 2. A surface G is called rational conchoid surface with respect to the reference point O if G admits a rational polar representation $r(u, v)\mathbf{s}(u, v)$, with a rational radius function $r(u, v)$ and a rational parametrization $\mathbf{s}(u, v)$ of the unit sphere S^2 .

Note that if G is a rational conchoid surface, G_d admits the rational representation (2). Furthermore, an analogous remark as on the rationality of the offset done above, is valid also for conchoids.

2. Representation of Offset and Conchoid surfaces

This section introduces the representations of the offset and conchoid surfaces. While conchoid surfaces are represented as point sets, offset surfaces are represented as envelopes of tangent planes. Performing in this way, we realize that their representations (9) and (11) are closely related.

2.1. Dual representation of offset surfaces

Consider an irreducible algebraic surface $F \subset \mathbb{R}^3$, and its offset surfaces $F_d \subset \mathbb{R}^3$. Let $\mathbf{f}(u, v)$ be an affine parameterization of F , and its offsets are thus represented by (1). Dealing with offset surfaces in this context, it is preferable to consider F as envelope of its tangent planes

$$E(u, v) : \mathbf{n}(u, v) \cdot \mathbf{x} = e(u, v), \text{ with } e = \mathbf{f} \cdot \mathbf{n}, \quad (4)$$

where $\mathbf{n}(u, v)$ denotes a unit normal vector field of $\mathbf{f}(u, v)$. The function $e(u, v)$ represents the oriented distance of the origin O from the planes $E(u, v)$. Consequently the offset surfaces F_d of F have tangent planes

$$E_d(u, v) : \mathbf{n}(u, v) \cdot \mathbf{x} = e(u, v) + d. \quad (5)$$

Let $\bar{F} \subset \mathbb{P}^3$ be the projective surface corresponding to F . The tangent planes E of \bar{F} determine a surface $\bar{F}^* \subset \mathbb{P}^{3*}$, called the **dual surface** of \bar{F} . While a parameterization of \bar{F} in terms of homogeneous point coordinates is $(1, \mathbf{f}(u, v))\mathbb{R}$, a parameterization of \bar{F}^* and its offset surfaces \bar{F}_d^* reads

$$E = \mathbb{R}(-e(u, v), \mathbf{n}(u, v)), \text{ and } E_d = \mathbb{R}(-e(u, v) - d, \mathbf{n}(u, v)),$$

where we use same symbols for the planes E , E_d and their homogeneous coordinate vectors. The homogeneous coordinates of planes are the coefficients of the defining linear polynomial.

Given a two parameter family of planes $E(u, v) : \mathbf{n}(u, v) \cdot \mathbf{x} = e(u, v)$, an affine parameterization $\mathbf{f}(u, v)$ of the envelope surface F is computed as solution of the system of linear equations

$$\{\mathbf{n}(u, v) \cdot \mathbf{x} = e(u, v), \mathbf{n}_u(u, v) \cdot \mathbf{x} = e_u(u, v), \mathbf{n}_v(u, v) \cdot \mathbf{x} = e_v(u, v)\} \quad (6)$$

where X_u, X_v denote the partial derivatives of $X(u, v)$ with respect to u and v . Concerning the envelope surface F of planes $E(u, v)$ there exist some degenerate cases, depending on the rank of the coefficient matrix $(\mathbf{n}, \mathbf{n}_u, \mathbf{n}_v)$. If \mathbf{n} and e are constant, F is a plane. If \mathbf{n} and e are functions of one variable only, \mathbf{n} represents a curve in the unit sphere S^2 , and consequently the surface F is developable. Otherwise if $\mathbf{n}(u, v)$ represents a surface in S^2 and F is typically a non-developable surface in \mathbb{R}^3 . But even if $\mathbf{n}(u, v)$ is two-dimensional, it might happen that the envelope F of the planes E is a single point, see Section 4.2, or that the planes E are tangent planes of a curve, see Section 5.3.2.

Let $\bar{F} \subset \mathbb{P}^3$ be an irreducible algebraic surface, defined as zero-set of a homogeneous polynomial $\bar{F}(x_0, x_1, x_2, x_3)$. To obtain a more compact notation, we use the notation $\bar{F}(x_0, \mathbf{x})$, with $\mathbf{x} = (x_1, x_2, x_3)$. Likewise, the dual surface $\bar{F}^* \subset \mathbb{P}^{3*}$ is the zero-set of a homogeneous polynomial $\bar{F}^*(u_0, \mathbf{u})$, with $\mathbf{u} = (u_1, u_2, u_3)$. Since the tangent planes of $\bar{F}(x_0, \mathbf{x}) = 0$ are represented by partial derivatives \bar{F}_{x_i} of \bar{F} with respect to x_i , the homogeneous plane coordinates of \bar{F} are

$$(u_0 : u_1 : u_2 : u_3) = (\bar{F}_{x_0}, \dots, \bar{F}_{x_3})(x_0, \dots, x_3) = \nabla(\bar{F})(x_0, \mathbf{x}), \quad (7)$$

evaluated at points $X = (x_0 : \dots : x_3) \in \bar{F}$. The implicit equation $\bar{F}^*(u_0, \mathbf{u})$ of the dual surface \bar{F}^* can be computed by eliminating x_0, \mathbf{x} in the algebraic system $\{\bar{F}(x_0, \mathbf{x}) = 0, (u_0, \mathbf{u}) = \nabla(\bar{F})(x_0, \mathbf{x})\}$.

Let us assume that \bar{F} is of class n , which means that the dual surface \bar{F}^* has degree n . The class of \bar{F} expresses the algebraically counted number of tangent planes passing through a generic line in \mathbb{P}^3 , whereas the degree of a surface \bar{F} counts the intersection points with a generic line. Let \bar{F}^* be expressed as

$$\bar{F}^*(u_0, \mathbf{u}) = u_0^n f_0 + u_0^{n-1} f_1(\mathbf{u}) + \dots + u_0^{n-j} f_j(\mathbf{u}) + \dots + u_0 f_{n-1}(\mathbf{u}) + f_n(\mathbf{u}), \quad (8)$$

where $f_j(\mathbf{u})$ are homogeneous polynomials of degree j in \mathbf{u} . The plane at infinity $\omega : x_0 = 0$ is tangent to \bar{F} , or equivalently $\omega \in \bar{F}^*$, if $f_0 = 0$. Furthermore, ω is an r -fold plane of \bar{F}^* , exactly if $f_0 = \dots = f_{r-1} = 0$, but $f_r \neq 0$.

2.2. Parametric representation of offset and conchoid surfaces

Let \mathcal{P} be the point set of \mathbb{R}^3 and let \mathcal{E} be the set of planes in \mathbb{R}^3 . Considering the offset construction, an irreducible affine surface F is defined as envelope of its tangent planes (4). Let $S^2 : x^2 + y^2 + z^2 = 1$ be the unit sphere in \mathbb{R}^3 , then the tangent planes E of F are defined by the map

$$\begin{aligned} \varphi : S^2 \times \mathbb{R} &\rightarrow \mathcal{E} \\ (\mathbf{n}, e) &\mapsto E : \mathbf{n} \cdot \mathbf{x} = e, \text{ with } \mathbf{n} \in S^2, e \in \mathbb{R}. \end{aligned} \quad (9)$$

Any rational parametrization $\mathbf{n}(u, v)$ of S^2 and any rational radius function $e(u, v)$ define a rational offset surface F in the sense of Definition 1. Since the offset surface F_d is obtained as envelope of the planes (5), the offset map o^* with respect to the offset distance d is defined by

$$\begin{aligned} o^* : \mathcal{E} &\rightarrow \mathcal{E} \\ \varphi(\mathbf{n}, e) &\mapsto \varphi(\mathbf{n}, e + d). \end{aligned} \quad (10)$$

Considering the conchoid construction, an irreducible algebraic surface $G \subset \mathbb{R}^3$ is represented by a polar representation $\mathbf{g}(u, v) = r(u, v)\mathbf{s}(u, v)$, with $\|\mathbf{s}(u, v)\| = 1$. Thus G is defined by the map

$$\begin{aligned} \gamma : S^2 \times \mathbb{R} &\rightarrow \mathcal{P} \\ (\mathbf{s}, r) &\mapsto \mathbf{g} = r\mathbf{s}, \text{ with } \mathbf{s} \in S^2, r \in \mathbb{R}. \end{aligned} \quad (11)$$

Any rational parametrization $\mathbf{s}(u, v)$ of S^2 and any rational radius function $r(u, v)$ define a rational conchoid surface G in the sense of Definition 2. The conchoidal map c , specifying the relation between a surface G and its conchoid surface G_d , is defined as

$$\begin{aligned} c : \mathcal{P} &\rightarrow \mathcal{P}, \\ \gamma(\mathbf{s}, r) &\mapsto \gamma(\mathbf{s}, r + d). \end{aligned} \quad (12)$$

We note that φ as well as γ are considered as local parameterizations of the tangent planes or the points of surfaces in \mathbb{R}^3 . It is obvious that $\varphi(\mathbf{n}, e)$ and $\varphi(-\mathbf{n}, -e)$ define the same non-oriented plane in \mathcal{E} . Likewise $\gamma(\mathbf{s}, r)$ and $\gamma(-\mathbf{s}, -r)$ define the same point in \mathcal{P} . Thus, φ and γ are not injective maps, but any element in \mathcal{E} and \mathcal{P} , respectively, has two pre-images. If injectivity of the maps is an issue, one can identify antipodal points in $S^2 \times \mathbb{R}$. Another possibility to overcome this problem for the map φ is by defining oriented planes in \mathbb{R}^3 , where the orientation is determined by the oriented normal vector \mathbf{n} . When having introduced the one-sided offset F_d in (1), we have used an oriented normal vector field. But when dealing with algebraic objects, the offset surface $O_d(F)$ contains both, the inner and outer offset. The same holds for the conchoid surface $C_d(G)$ of an algebraic surface G . Since the construction in Section 3 ignores orientations, we consider planes in \mathbb{R}^3 as non-oriented. Additionally we notice that when parameterizing surfaces and constructing offsets or conchoids, the base surface is traced twice, in order to represent both components of the offset or conchoid.

2.3. Rational parameterizations of S^2

Since the focus is on rational families of offset surfaces and conchoidal surfaces we discuss universal rational parameterizations of S^2 . Following [3] we choose four arbitrary rational functions $a(u, v)$, $b(u, v)$, $c(u, v)$ and $d(u, v)$ without common factor. Let

$$A = 2(ac + bd), B = 2(bc - ad), C = a^2 + b^2 - c^2 - d^2, D = a^2 + b^2 + c^2 + d^2,$$

then $\mathbf{q}(u, v) = \frac{1}{D}(A, B, C)$ is a rational parametrization of the unit sphere S^2 . Thus $\varphi(\mathbf{q}(u, v), \rho(u, v))$ with a rational function $\rho(u, v)$ defines a rational parametrization of a rational offset surface and likewise $\gamma(\mathbf{q}(u, v), \rho(u, v))$ defines a rational parametrization of a rational conchoid surface. The similarities between representations (9) and (11) indicate that there is a close relation between the offset construction and the conchoid construction. These correspondences are studied in Section 3.

3. The foot-point map

According to Section 1.2, we consider the origin O in \mathbb{R}^3 as reference point for the conchoidal construction. We now introduce a map, that establishes the connection between offsets and conchoids. The foot-point map α with respect to O is defined as

$$\begin{aligned} \alpha : \mathcal{E} &\rightarrow \mathcal{P} \\ E : \mathbf{x} \cdot \mathbf{n} = e &\mapsto P = \alpha(E) = \frac{e}{\|\mathbf{n}\|^2} \mathbf{n}, \end{aligned} \quad (13)$$

and maps planes $E \in \mathcal{E}$ to points $P \in \mathcal{P}$ of \mathbb{R}^3 , see Figure 1(a). The map α is rational and bijective except for planes E passing through O . The inverse map α^{-1} equals the dual map α^* , and transforms points $P \neq O$ to planes $E = \alpha^{-1}(P)$ which have OP as normal and pass through P . It reads

$$\begin{aligned} \alpha^* : \mathcal{P} \setminus \{O\} &\rightarrow \mathcal{E} \\ P = \mathbf{p} &\mapsto \alpha^*(P) = E : \mathbf{x} \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{p}, \end{aligned} \quad (14)$$

The maps α and α^* are the basic ingredients to construct rational conchoid surfaces from rational offset surfaces and vice versa. If the reference point O is replaced by some other point Z , the foot-point map α_Z with respect to Z reads

$$\begin{aligned} \alpha_Z : \mathcal{E} &\rightarrow \mathcal{P} \\ E : \mathbf{x}^T \cdot \mathbf{n} = e &\mapsto P = \alpha_Z(E) = \mathbf{z} + \frac{e - \mathbf{z} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n}, \end{aligned}$$

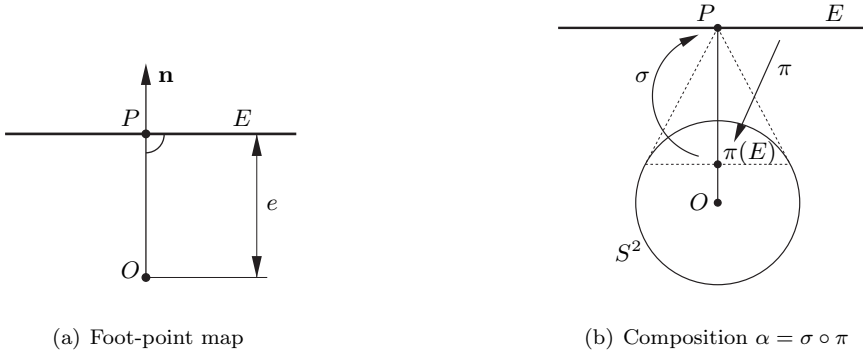


Figure 1: Geometric properties of the foot-point map α

The map (13) is a quadratic plane-to-point mapping. To get more insight to the correspondence between offsets and conchoids we provide representations of α and α^* in terms of homogeneous coordinates. Consider the projective extension \mathbb{P}^3 of \mathbb{R}^3 , and its dual space \mathbb{P}^{3*} . The points $\mathbb{R}(u_0, \dots, u_3) \in \mathbb{P}^{3*}$ are identified with the planes $U \subset \mathbb{P}^3$.

The map $\alpha : \mathcal{E} \rightarrow \mathcal{P}$ uniquely determines the map $\bar{\alpha} : \mathbb{P}^{3*} \rightarrow \mathbb{P}^3$, and for simplicity both maps are denoted by α . In terms of homogeneous coordinates it reads

$$\begin{aligned} \alpha : \mathbb{P}^{3*} &\rightarrow \mathbb{P}^3 \\ U = \mathbb{R}(u_0, \dots, u_3) &\mapsto X = (x_0, x_1, x_2, x_3)\mathbb{R} \\ &= (-(u_1^2 + u_2^2 + u_3^2), u_0 u_1, u_0 u_2, u_0 u_3)\mathbb{R}. \end{aligned} \quad (15)$$

The exceptional planes of α are the ideal plane $\omega : x_0 = 0$, and the tangent planes of the isotropic cone $\Delta : u_0 = 0, u_1^2 + u_2^2 + u_3^2 = 0$ with vertex at $O = (1, 0, 0, 0)\mathbb{R}$. The dual map $\alpha^* = \alpha^{-1}$ reads

$$\begin{aligned} \alpha^* : \mathbb{P}^3 &\rightarrow \mathbb{P}^{3*} \\ X = (x_0, \dots, x_3)\mathbb{R} &\mapsto U = \mathbb{R}(u_0, u_1, u_2, u_3), \\ &= \mathbb{R}(-(x_1^2 + x_2^2 + x_3^2), x_0x_1, x_0x_2, x_0x_3). \end{aligned} \quad (16)$$

The base-points of α^* are the origin $O = (1, 0, 0, 0)\mathbb{R}$ and all points of the absolute conic $j : x_0 = 0, x_1^2 + x_2^2 + x_3^2 = 0$, consisting of the circular points at infinity. Considering the isotropic cone Δ as point set, we have $j = \Delta \cap \omega$.

For the practical examples in Section 5 we use the fact that α and α^* can be represented as composition of an inversion and a polarity, see Figure 1(b). Let $\sigma : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ be the inversion at S^2 , and let $\pi : \mathbb{P}^{3*} \rightarrow \mathbb{P}^3$ be the polarity with respect to S^2 , and $\pi^* : \mathbb{P}^3 \rightarrow \mathbb{P}^{3*}$ its dual map. These maps satisfy $\sigma = \sigma^{-1}$ and $\pi \circ \pi^* = \text{id}$, and their coordinate representations are

$$\begin{aligned} \sigma : X = (x_0, \dots, x_3)\mathbb{R} &\mapsto \sigma(X) = (x_1^2 + x_2^2 + x_3^2, x_0x_1, x_0x_2, x_0x_3)\mathbb{R}, \\ \pi : U = \mathbb{R}(u_0, \dots, u_3) &\mapsto \pi(U) = X = (-u_0, u_1, u_2, u_3)\mathbb{R}, \\ \pi^* : X = (x_0, \dots, x_3)\mathbb{R} &\mapsto \pi^*(X) = U = \mathbb{R}(-x_0, x_1, x_2, x_3). \end{aligned} \quad (17)$$

Proposition 3. *The foot-point map α and its inverse map α^* from equations (15) and (16) are represented as compositions of the maps σ, π and π^* from equation (17) by*

$$\alpha = \sigma \circ \pi, \text{ and } \alpha^* = \pi^* \circ \sigma.$$

Given a rational unit vector field $\mathbf{n}(u, v) \subset S^2$ and a rational function $r(u, v)$, the map φ from (9) creates a rational offset surface F^* , and the map γ from (11) creates a rational conchoid surface G . Composing these maps with α , we have $\alpha \circ \varphi = \gamma$ and $\alpha^* \circ \gamma = \varphi$. The relations between α, φ and γ are displayed in the following diagrams.

$$\begin{array}{ccc} & S^2 \times \mathbb{R} & \\ \varphi \swarrow & & \searrow \gamma \\ \mathcal{E} & \xrightarrow{\alpha} & \mathcal{P} \end{array} \quad \begin{array}{ccc} & S^2 \times \mathbb{R} & \\ \varphi \swarrow & & \searrow \gamma \\ \mathcal{E} & \xleftarrow{\alpha^*} & \mathcal{P} \end{array} \quad (18)$$

Theorem 4. *Let $F^* \subset \mathcal{E}$ be a dual surface, and let o^* be the offset map (10), with $o^*(F^*) = F_d^*$, for arbitrary offset distance d . Let c be the conchoidal map (12), with $c(G) = G_d$. Then the family of surfaces F_d^* is mapped by $\alpha : \mathcal{E} \rightarrow \mathcal{P}$ to a family of surfaces $G_d = \alpha(F_d^*)$, being conchoid surfaces of G at distance d , with respect to the chosen reference point O of α . Likewise, α^* maps a family of conchoid surfaces G_d to a family of offset surfaces $F_d^* = \alpha^*(G_d)$. The following diagrams are commutative.*

$$\begin{array}{ccc} F^* & \xrightarrow{\alpha} & G \\ o^* \downarrow & & \downarrow c \\ F_d^* & \xrightarrow{\alpha} & G_d \end{array} \quad \begin{array}{ccc} F^* & \xleftarrow{\alpha^*} & G \\ o^* \downarrow & & \downarrow c \\ F_d^* & \xleftarrow{\alpha^*} & G_d \end{array} \quad (19)$$

Proposition 5. *Let F be an irreducible algebraic surface and $G = \alpha(F^*)$. Then, the offset surfaces F_d of F are birationally equivalent to the conchoid surfaces G_d of G .*

This statement is implied by the birationality of the map α . But other properties, as the degree, cannot be translated directly. There are relations between the degree of a dual surface F^* and

the degree of the image surface $G = \alpha(F^*)$, as outlined in the following. Let $\overline{F^*}$ be the zero-set of the irreducible polynomial (8). Applying α , which means inserting the representation (16) into $\overline{F^*}$ yields the polynomial

$$\begin{aligned} \tilde{G}(x_0, \mathbf{x}) &= (-\mathbf{x}^2)^n f_0 + (-\mathbf{x}^2)^{n-1} x_0 f_1(\mathbf{x}) + \dots + (-\mathbf{x}^2)^k x_0^{n-k} f_{n-k}(\mathbf{x}) + \dots + \\ &\quad (-\mathbf{x}^2) x_0^{n-1} f_{n-1}(\mathbf{x}) + x_0^n f_n(\mathbf{x}). \end{aligned} \quad (20)$$

According to the fact that α possesses exceptional elements, the polynomial \tilde{G} might have factors of the form x_0^r or $(\mathbf{x}^2)^k = (\mathbf{x} \cdot \mathbf{x})^k$. Since we are not interested in these factors, the pedal surface $\overline{G} = \alpha(\overline{F^*})$ is defined as the zero-set of the irreducible component of \tilde{G} . We analyze the degree of the image surface \overline{G} , depending on the position of $\overline{F^*}$ with respect to the exceptional planes of α .

If ω is an r -fold plane of $\overline{F^*}$, thus $f_0 = \dots = f_{r-1} = 0$, but $f_r \neq 0$, then the polynomial \tilde{G} contains the factor x_0^r . If f_n has the factor \mathbf{u}^2 , it also appears in \tilde{G} . More generally, consider the case where the polynomials f_n, \dots, f_{n-k} have a common factor. In detail, let

$$f_{n-j}(\mathbf{u}) = (\mathbf{u}^2)^{k-j} h_{n-2k+j}(\mathbf{u}), \text{ for } j = 0, \dots, k,$$

with $\gcd(h_{n-2k+j}, \mathbf{u}^2) = 1$. Thus the polynomial (20) has the factor $(\mathbf{x}^2)^k$, and the irreducible component G is of degree $2n - 2k$. Summarizing, the degree of \overline{G} is $2n - r - 2k$, where r is the multiplicity of ω and k is the multiplicity of the cone $u_0 = 0, \mathbf{u}^2 = 0$ for \overline{F} .

In a similar way one can deduce the defining equation of a surface $\overline{F^*} = \alpha^*(\overline{G})$, starting from a homogeneous polynomial defining \overline{G} . Exchanging F^* by G and dual coordinates u_i by x_i , one obtains the degree of the dual object $\overline{F^*}$. Hereby, one has to exchange also ω by O and the isotropic cone $\Delta \ni O$ by the conic $j \subset \omega$.

4. Elementary examples

To illustrate the relation between the offset and conchoid construction we present two elementary examples. The first discusses conchoid surfaces G_d of a plane G with respect to a reference point $O \notin G$ and the corresponding offset surfaces, which are offsets of a paraboloid of revolution F with O as focal point. The second example considers offsets of a sphere F and their corresponding conchoid surfaces, which are typically rotational surfaces with a Pascal curve as profile, and double point at O .

4.1. Conchoid surfaces of a plane

Consider the plane $G : z = 1$ and the reference point $O = (0, 0, 0)$. To compute the conchoid surfaces of G , one might use the trigonometric parameterization

$$\mathbf{n}(u, v) = (\cos u \cos v, \cos v \sin u, \sin v) \quad (21)$$

of S^2 . Then G and its conchoid surfaces G_d admit the trigonometric polar representations

$$\begin{aligned} \mathbf{g}(u, v) &= r(u, v) \mathbf{n}(u, v), \text{ and} \\ \mathbf{g}_d(u, v) &= (r(u, v) + d) \mathbf{n}(u, v), \text{ with } r(u, v) = \frac{1}{\sin v}. \end{aligned} \quad (22)$$

Applying the map α^* from (14) to these parameterizations gives dual parametric representations of F and F_d . The tangent planes of these surfaces read

$$E(u, v) : \mathbf{x} \cdot \mathbf{n}(u, v) = r(u, v), \text{ and } E_d(u, v) : \mathbf{x} \cdot \mathbf{n}(u, v) = r(u, v) + d.$$

The surface $F : x^2 + y^2 + 4z = 4$ is a paraboloid of revolution with focal point O and axis z , and F_d are its offset surfaces. To illustrate the projective version of α , we turn to the projective setting. Let $X = (x_0, \dots, x_3)$ and $U = (u_0, \dots, u_3)$, the surfaces $\overline{G} \subset \mathbb{P}^3$ and $\overline{F}^* \subset \mathbb{P}^{3*}$ are the zero-sets of the homogeneous polynomials

$$\overline{G}(X) = (x_3 - x_0) \mapsto \alpha^*(\overline{G}) = F^*(U) = u_1^2 + u_2^2 + u_3^2 + u_0 u_3,$$

To obtain the implicit equation of the conchoid surface \overline{G}_d , one has to eliminate the parameters u and v from (22). One obtains the polynomial of degree four,

$$\overline{G}_d(X) = d^2 x_0^2 x_3^2 - (x_1^2 + x_2^2 + x_3^2)(x_0 - x_3)^2.$$

Since the highest power of x_0 is two, the origin is a double point of \overline{G}_d . This tells us that $\alpha^*(\overline{G}_d)$ contains the factor u_0^2 . Additionally we may divide by $(u_1^2 + u_2^2 + u_3^2)$, such that finally the offset \overline{F}_d^* of the paraboloid \overline{F}^* is the zero-set of the polynomial

$$\overline{F}_d^*(U) = d^2 u_3^2 (u_1^2 + u_2^2 + u_3^2) - (u_1^2 + u_2^2 + u_3^2 + u_0 u_3)^2.$$

Since the highest power of u_0 is two, the plane at infinity $\omega = \mathbb{R}(1, 0, 0, 0)$ is a two-fold tangent plane of \overline{F}_d^* . Figure 2(c) provides an illustration of that example in the 2d-case.

4.2. Offsets of a sphere and corresponding conchoid surfaces

Consider the sphere $F : (x - m)^2 + y^2 + z^2 - R^2 = 0$. Its offset surfaces F_d are spheres with same center and Radius $R + d$. For a 2d-illustration see Figure 2(b). To establish the correspondence with a family of conchoid surfaces G_d , we have to consider the dual surfaces F_d^* , determined by the polynomial

$$\overline{F}_d^*(u_0, \mathbf{u}) = ((R + d)^2 - m^2)u_1^2 + (R + d)^2(u_2^2 + u_3^2) - 2mu_0u_1 - u_0^2.$$

Substituting $d = 0$ gives \overline{F}^* . The bi-rational map α maps surfaces \overline{F}_d^* to the family of conchoid surfaces \overline{G}_d , which are the zero-set of the polynomial of degree four,

$$\overline{G}_d(x_0, \mathbf{x}) = x_0^2(x_1^2((R + d)^2 - m^2) + (R + d)^2(x_2^2 + x_3^2)) + 2mx_0x_1(\mathbf{x}^2) - (\mathbf{x}^2)^2.$$

Since the highest power of x_0 is two, the origin $O = (1, 0, 0, 0)\mathbb{R}$ is a double point of \overline{G}_d . Substituting $d = 0$ gives \overline{G} . We notice that when letting $R = 0$ and $d = 0$, the sphere F degenerates to a (double traced) bundle of planes passing through the point $M = (m, 0, 0)$. The corresponding 2d-example is illustrated in Figure 2(a). Its defining polynomial is $\overline{F}^*(u_0, \mathbf{u}) = (u_0 + mu_1)$. The corresponding surface $\overline{G} = \alpha(\overline{F}^*)$ is the zero-set of the polynomial $G(x_0, \mathbf{x}) = \mathbf{x} \cdot \mathbf{x} - mx_0x_1$. Thus, \overline{G} is a sphere with diameter OM .

Dual rational parameterizations of F_d and rational polar representations of G_d may be derived as follows. Consider again the trigonometric parameterization (21) of S^2 . Tangent planes E_d of F_d and a parameterization \mathbf{g}_d are consequently given by

$$\begin{aligned} E_d(u, v) : \quad & \mathbf{x} \cdot \mathbf{n}(u, v) = r(u, v) + d, \\ \mathbf{g}_d(u, v) = \quad & (r(u, v) + d)\mathbf{n}(u, v), \text{ with } r(u, v) = m \cos u \cos v + R. \end{aligned}$$

5. Pedal surfaces and inverse pedal surfaces

Consider a surface F , its offsets F_d and a fixed reference point $O = (0, 0, 0)$. The surface $G_d = \alpha(F_d^*)$ is called **pedal surface** of F_d , and consists of the foot-points at the tangent planes of F_d with

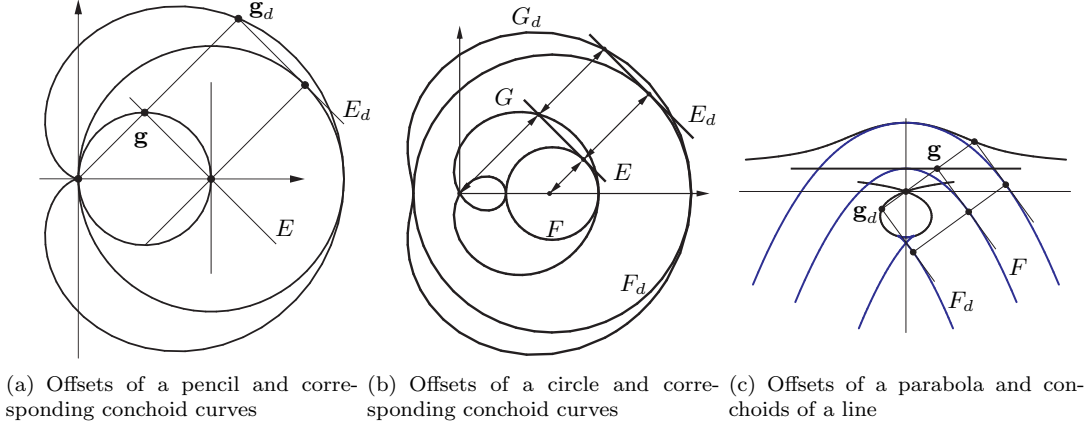


Figure 2: Relation between offsets and conchoids

respect to O . According to Theorem 4, the surfaces G_d are the conchoid surfaces of G with respect to O . Conversely, the surface $F_d^* = \alpha^*(G_d)$ is called **negative pedal surface** of G_d with respect to O . Since the operation applied is rather the inverse than the negative, we use the notion **inverse pedal surface**. Consider points $X \in G_d$, then the tangent planes of F_d are perpendicular to OX and contain X .

There exist several interesting examples, both for rational offset surfaces and for rational conchoid surfaces and their counterparts with respect to α and α^* . These maps apply to translate geometric properties between offset surfaces and conchoid surfaces and vice versa, as stated in Proposition 5. The following subsections discuss pedal surfaces and inverse pedal surfaces of ruled surfaces and quadrics.

5.1. Pedal surfaces of rational ruled surfaces

Consider a skew ruled surface $\overline{F} \subset \mathbb{P}^3$. Its tangent planes form the dual surface \overline{F}^* which is itself a skew ruled surface in \mathbb{P}^{3*} . Consider a fixed line $l \in F$, and let D be the foot-point of O on l . Actually we interpret l as pencil of planes, and not as point set. According to Thales' theorem, the pedal curve $\alpha(l)$ is a circle with diameter OD , in a plane perpendicular to l , see Figure 3(a). Consequently, the pedal surface $\overline{G} = \alpha(\overline{F}^*)$ contains a one-parameter family of circles in planes perpendicular to the lines l of F .

An implicit representation of a surface \overline{F} does not tell us if the surface is ruled or not, except for some special cases. Thus we derive a construction for $\overline{G} = \alpha(\overline{F}^*)$ in terms of a parameterization of \overline{F}^* . It is known, see for instance [9], that a rational ruled surface is a rational offset surface in the sense of Definition 1. But typically it makes some effort to construct a rational offset parameterization for \overline{F}^* . Thus we start to demonstrate the construction in terms of a general parameterization of a ruled surface.

Let $\mathbf{f}(u, v) = \mathbf{c}(u) + v\mathbf{e}(u)$ be an affine parameterization of F . The foot-points \mathbf{d} on the generating lines of F with respect to O have to satisfy $\mathbf{d} \cdot \mathbf{e} = 0$. This implies that $v = -(\mathbf{c} \cdot \mathbf{e})/(\mathbf{e} \cdot \mathbf{e})$ is the parameter value of \mathbf{d} and we have

$$\mathbf{d}(u) = \mathbf{c}(u) - \frac{\mathbf{c}(u) \cdot \mathbf{e}(u)}{\mathbf{e}(u) \cdot \mathbf{e}(u)} \mathbf{e}(u). \quad (23)$$

We exchange the directrix curve \mathbf{c} by \mathbf{d} . The circles $\alpha(l(u))$ have centers at $\mathbf{m}(u) = \frac{1}{2}\mathbf{d}(u)$, and their carrier planes $\varepsilon(u)$ are orthogonal to l , thus $\varepsilon(u) : \mathbf{x} \cdot \mathbf{e}(u) = 0$.

To find a parameterization of the pedal surface G , we need to parameterize the family of circles $\alpha(l(u))$. For that we need an orthonormal frame in $\varepsilon(u)$. Consider two orthonormal vectors $\mathbf{a}(u), \mathbf{b}(u)$, such that $\{\mathbf{e}/\|\mathbf{e}\|, \mathbf{a}, \mathbf{b}\}$ is an orthonormal basis in \mathbb{R}^3 . Obviously these vectors are solutions of $\mathbf{e}(u) \cdot \mathbf{x} = 0$. Since the radius of $\alpha(l(u))$ is given by $r(u) = \|\mathbf{d}\|/2$, an affine parameterization of G reads

$$\mathbf{g}(u, v) = \frac{1}{2} (\mathbf{d}(u) + \|\mathbf{d}(u)\|\mathbf{a}(u) \cos v + \|\mathbf{d}(u)\|\mathbf{b}(u) \sin v).$$

Typically this is a non-rational parameterization even for rational input surfaces, since the radius $r(u)$ and also $\mathbf{a}(u)$ and $\mathbf{b}(u)$ are typically non-rational. But on the other hand, G is a rational surface, since it carries a rational family of circles given by the equations

$$\|\mathbf{x} - \mathbf{m}(u)\|^2 - r(u)^2 = 0, \text{ and } \mathbf{x} \cdot \mathbf{e}(u) = 0.$$

Rational parameterizations of a rational one-parameter family of conics can be computed explicitly, see for instance [16]. Nevertheless, to represent the whole family of conchoid surfaces G_d by a rational parameterization, we have to start with a representation of the tangent planes of F , involving a rational unit normal vector. Computing the partial derivatives of $\mathbf{f}(u, v) = \mathbf{c}(u) + v\mathbf{e}(u)$ gives $\mathbf{f}_u = \dot{\mathbf{c}}(u) + v\dot{\mathbf{e}}(u)$ and $\mathbf{f}_v = \mathbf{e}(u)$. Thus the normal $\mathbf{n}(u, v)$ reads

$$\mathbf{n}(u, v) = \mathbf{n}_1(u) + v\mathbf{n}_2(u), \text{ with } \mathbf{n}_1(u) = \dot{\mathbf{c}}(u) \times \mathbf{e}(u), \text{ and } \mathbf{n}_2(u) = \dot{\mathbf{e}}(u) \times \mathbf{e}(u).$$

In case that $\mathbf{n}_1(u)$ and $\mathbf{n}_2(u)$ are linearly dependent for all u , F is a developable ruled surface, which can be considered as the envelope of its one-parameter family of tangent planes. In this case, the pedal 'surface' $\alpha(F^*)$ degenerates to a curve, and therefore we exclude this case.

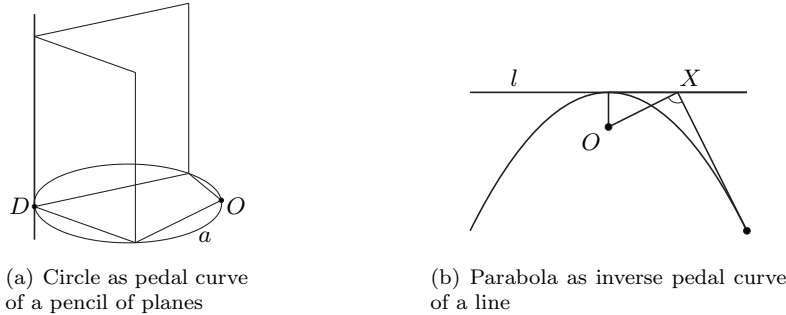


Figure 3: Pedal and inverse pedal curve of a line

Let \mathbf{n}_1 and \mathbf{n}_2 be linearly independent, which is equivalent to $\det(\dot{\mathbf{c}}, \mathbf{e}, \dot{\mathbf{e}}) \neq 0$, since $\mathbf{e}(u)$ is assumed to be a non-zero direction vector field. In order to represent the family of offset surfaces F_d by a rational parameterization, we have to construct a rational unit normal vector field of F . To obtain this, we require $\|\mathbf{n}(u, v)\|^2 = w^2$, with some function $w(u, v)$ to be determined. To simplify the practical computation, we exchange the directrix curve $\mathbf{c}(u)$ by the striction curve $\mathbf{s}(u) = \mathbf{c}(u) + v_s(u)\mathbf{e}(u)$, having the property that the normal vector \mathbf{n}_s at \mathbf{s} is orthogonal to $\mathbf{n}_2 = \dot{\mathbf{e}} \times \mathbf{e}$. This condition determines the parameter

$$v_s(u) = -\frac{(\dot{\mathbf{c}} \times \mathbf{e}) \cdot (\dot{\mathbf{e}} \times \mathbf{e})}{(\dot{\mathbf{e}} \times \mathbf{e})^2}.$$

Representing F with \mathbf{s} as directrix curve by $\mathbf{f}(u, v) = \mathbf{s}(u) + v\mathbf{e}(u)$ yields that $\mathbf{n} = \mathbf{n}_s + v\mathbf{n}_2$, with $\mathbf{n}_s \perp \mathbf{n}_2$. The condition $\|\mathbf{n}\|^2 = w^2$ turns into $\mathbf{n}_s^2 + v^2\mathbf{n}_2^2 = w^2$, where we consider v and w as affine coordinates in \mathbb{R}^2 . By the substitution $v = y_2/y_1$ and $w = y_0/y_1$, this quadratic equation becomes

$$a(u) : a_1(u)y_1^2 + a_2(u)y_2^2 - y_0^2 = 0, \text{ with } a_1(u) = \mathbf{n}_s(u)^2, \text{ and } a_2(u) = \mathbf{n}_2(u)^2. \quad (24)$$

Equation (24) defines a **real rational** family of conics in the projective plane \mathbb{P}^2 , with coordinates y_0, y_1, y_2 . There exists a parameterization $\mathbf{y}(u, t) = (y_0, y_1, y_2)(u, t)$, satisfying equation (24) identically, in a way that $\mathbf{y}(u_0, t)$ is a parameterization of the conic $a(u_0)$, for any fixed $u_0 \in \mathbb{R}$, see for instance [16]. Consequently one has constructed a parameterization

$$\mathbf{f}(u, t) = \mathbf{s}(u) + \frac{y_2(u, t)}{y_1(u, t)} \mathbf{e}(u), \text{ with } \|\mathbf{n}(u, t)\| = \frac{y_0(u, t)}{y_1(u, t)}. \quad (25)$$

Now, the norm of the normal vector $\mathbf{n}(u, t)$ is a rational function in the surface parameters u and t . We represent the ruled surface F and its offset surfaces F_d as envelopes of their tangent planes

$$\begin{aligned} E(u, t) &: \mathbf{x} \cdot \mathbf{n}(u, t) = \mathbf{f}(u, t) \cdot \mathbf{n}(u, t), \\ E_d(u, t) &: \mathbf{x} \cdot \mathbf{n}(u, t) = \mathbf{f}(u, t) \cdot \mathbf{n}(u, t) + d\|\mathbf{n}(u, t)\|. \end{aligned}$$

In other words, $E_d = \mathbb{R}(-\mathbf{f} \cdot \mathbf{n} - d\|\mathbf{n}\|, \mathbf{n})$ is a dual rational parameterization of the family of offset surfaces F_d^* of the given ruled surface F , and analogous to equation (5), up to the normalization of \mathbf{n} . Applying the foot-point map α yields a rational parameterization of the corresponding family of rational conchoid surfaces $G_d = \alpha(F_d^*)$. In terms of homogeneous coordinates we have

$$E_d = \mathbb{R}(-\mathbf{f} \cdot \mathbf{n} - d\|\mathbf{n}\|, \mathbf{n}) \mapsto \alpha(E_d) = (\mathbf{n}^2, (\mathbf{f} \cdot \mathbf{n} + d\|\mathbf{n}\|)\mathbf{n})\mathbb{R}.$$

The according affine rational polar representation of G_d , with $\mathbf{f}(u, t)$ from equation (25) and $\mathbf{n}(u, t)$ its corresponding normal vector field, reads

$$\mathbf{g}_d(u, t) = \left(\frac{\mathbf{f} \cdot \mathbf{n}}{\|\mathbf{n}\|} + d \right) \frac{\mathbf{n}}{\|\mathbf{n}\|}.$$

Proposition 6. *The pedal surfaces G_d of the family of offset surfaces F_d of non-developable rational ruled surfaces F admit rational polar representations with respect to any chosen reference point O . The pedal surface G of F is generated by a rational family of circles in planes through O and perpendicular to F 's generating lines l . The conchoid surface G_d of G is generated by the planar conchoid curves of that circles lying on G .*

5.2. Inverse pedal surfaces of rational ruled surfaces

Consider a rational ruled surface G as set of points, and a fixed reference point $O = (0, 0, 0)$. Applying α^* yields a two parameter family of tangent planes $F^* = \alpha^*(G)$, whose envelope F is the inverse pedal surface of G . While the ruled surface F in Section 5.1 has been assumed to be non-developable, since it has been considered as two-parameter family of tangent planes, the ruled surface G in this current section may be a developable surface as well, since it is considered as two-parameter family of points. Thus, G might be a tangent surface of a curve, a cylinder, a cone or even a plane. The latter case makes sense only if $O \notin G$.

Let $l \subset G$ be a generating line, and consider points $X \in l$. The one-parameter family of planes $\alpha^*(X)$ through X , whose normal vector is OX , envelope a parabolic cylinder P , with O as focal point of the cross section parabola p in the plane connecting O and l , see Figure 3(b). The vertex of the parabola p is obviously the foot-point of O on l . Consequently, the inverse pedal surface F of a ruled surface G is the envelope of a one parameter family of parabolic cylinders. The orthogonal cross sections of these parabolic cylinders with planes through O have O as common focal point.

Consider a general rational ruled surface parameterization $\mathbf{g}(u, v) = \mathbf{d}(u) + v\mathbf{e}(u)$ of G , where $\mathbf{d}(u)$ is the foot-point curve with respect to O , compare equation (23). The vertices of above mentioned parabolas are $\mathbf{d}(u)$. The tangent planes of F are $E(u, v) : (\mathbf{x} - \mathbf{g}) \cdot \mathbf{g} = 0$, but since \mathbf{g} is typically

not a rational polar representation, the normal vector of $E(u, v)$ does not have rational norm. In detail we have

$$\begin{aligned} E(u, v) &: \mathbf{x} \cdot (\mathbf{d}(u) + v\mathbf{e}(u)) = \mathbf{d}(u)^2 + v^2\mathbf{e}(u)^2, \\ E_u(u, v) &: \mathbf{x} \cdot (\dot{\mathbf{d}}(u) + v\dot{\mathbf{e}}(u)) = 2(\mathbf{d}(u) \cdot \dot{\mathbf{d}}(u) + v^2\mathbf{e}(u) \cdot \dot{\mathbf{e}}(u)), \\ E_v(u, v) &: \mathbf{x} \cdot \mathbf{e}(u) = 2v\mathbf{e}(u)^2. \end{aligned} \tag{26}$$

The solution $\mathbf{f}(u, v)$ of (26) is a rational representation of the inverse pedal surface F of the ruled surface G . Although F is a rational offset surface in the sense of Definition 1, the unit normal vector field $\mathbf{n}(u, v)$, corresponding to $\mathbf{f}(u, v)$ is typically non-rational. Nevertheless, the intersection $E \cap E_v$ gives the generating lines of the one-parameter family of parabolic cylinders $P(u)$. These cylinders $P(u)$ admit the parameterization

$$\mathbf{q}(u, v, \lambda) = \mathbf{p}(u, v) + \lambda\mathbf{a}(u),$$

with cross-section parabolas \mathbf{p} and \mathbf{a} as direction vectors of its generating lines, given by

$$\mathbf{p}(u, v) = \left(1 - \frac{v^2\mathbf{e}^2}{\mathbf{d}^2}\right) \mathbf{d} + 2v\mathbf{e}, \text{ and } \mathbf{a}(u) = \mathbf{d}(u) \times \mathbf{e}(u).$$

The intersection of $\mathbf{q}(u, v, \lambda)$ with planes $E_u(u, v)$ from (26) determines λ as a rational function in u and v , and finally gives the parameterization $\mathbf{f}(u, v)$ of F . The v -lines of $\mathbf{f}(u, v)$, that are the characteristic curves of the family of cylinders $P(u)$, are rational curves of degree at most three.

The development of a parameterization $\mathbf{f}(u, v)$ of F with the property that its unit normal vectors are rational, is more involved. To construct such a parameterization, one needs to start with a rational parameterization $\mathbf{g}(u, v)$ of G , whose norm $\|\mathbf{g}(u, v)\|$ is a rational function in u and v . We do not give the detailed construction here but refer to [11]. To outline the method, one studies the squared norm

$$\|\mathbf{g}(u, v)\|^2 = \mathbf{d}(u)^2 + v^2\mathbf{e}(u)^2 = w^2, \tag{27}$$

which shall be the square of a rational function $w(u, v)$. Substituting $v = y_1/y_2$ and $w = y_0/y_2$, equation (27) defines a rational one-parameter family of conics

$$a(u) : y_2^2\mathbf{d}(u)^2 + y_1^2\mathbf{e}(u)^2 - y_0^2 = 0. \tag{28}$$

As mentioned in Section 5.1 and proved in [16], these objects have rational parameterizations. More precisely, one can construct rational functions $y_i(u, t)$, satisfying (28) identically, so that the t -lines of $(y_0, y_1, y_2)(u, t)$ are the conics $a(u)$. Considering the reparameterization $v = y_1/y_2$, one obtains the parameterization

$$\mathbf{g}(u, t) = \mathbf{d}(u) + \frac{y_1(u, t)}{y_2(u, t)}\mathbf{e}(u), \text{ with } \|\mathbf{g}(u, t)\| = \frac{y_0(u, t)}{y_2(u, t)}.$$

Using this specific parameterization of G instead of $\mathbf{g} = \mathbf{d} + v\mathbf{e}$, the family of tangent planes $E(u, t) : (\mathbf{x} - \mathbf{g}(u, t)) \cdot \mathbf{g}(u, t) = 0$ possesses a normal vector $\mathbf{g}(u, t)$ with rational length $\|\mathbf{g}(u, t)\|$. Consequently the intersection point $\mathbf{f}(u, t)$ of the planes $E(u, t) \cap E_u(u, t) \cap E_t(u, t)$ is a rational parameterization of F , whose corresponding unit normal vector $\mathbf{n}(u, t) = \mathbf{g}(u, t)/\|\mathbf{g}(u, t)\|$ is rational.

Proposition 7. *The inverse pedal surfaces $F = \alpha^*(G)$ of rational ruled surfaces G admit rational parameterizations $\mathbf{f}(u, t)$, whose corresponding unit normal vector $\mathbf{n}(t, u)$ is rational, too. The family of rational conchoid surfaces G_d of the rational ruled surface G is mapped by α^* to a family of offset surfaces F_d of F . While F is the envelope of a rational one-parameter family of parabolic cylinders $P(u)$ with cross section parabolas $p(u)$, the offset surfaces F_d are the envelopes of the offset cylinders $P_d(u)$, whose cross sections are offsets of the parabolas $p(u)$.*

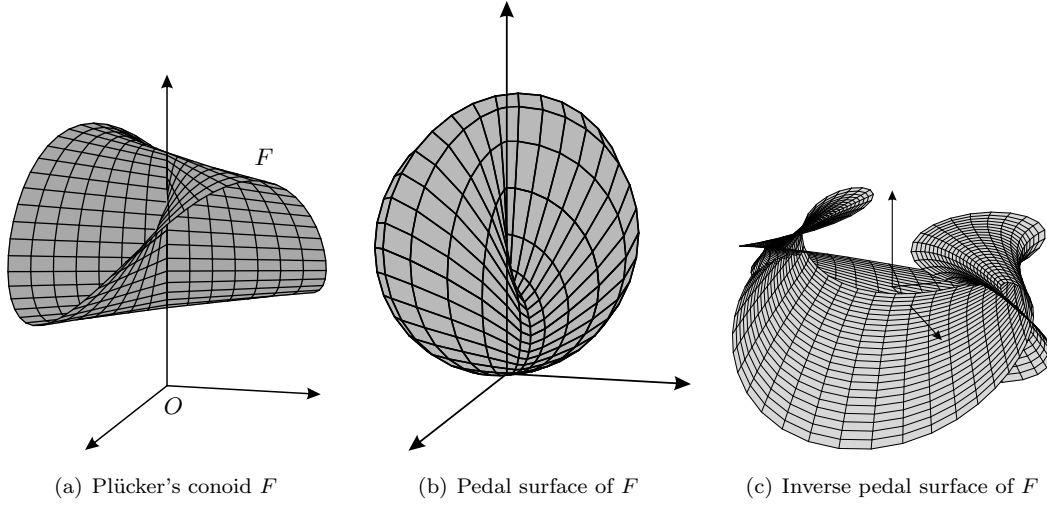


Figure 4: Pedal surface and inverse pedal surface of Plücker's conoid F with respect to the origin O

Example. We consider Plücker's conoid F , which is a ruled surface of degree three, projectively equivalent to Whitney's umbrella. A possible parametrization of F reads

$$\mathbf{f}(r, u) = (r \cos u, r \sin u, \sin 2u) = \mathbf{c}(u) + r\mathbf{e}(u), \quad (29)$$

with directrix curve $\mathbf{c}(u) = (0, 0, \sin 2u)$ and direction vectors $\mathbf{e}(u) = (\cos u, \sin u, 0)$. Considering $X = (x_0, \dots, x_3)$ as homogeneous coordinates in \mathbb{P}^3 , the surface F is the zero-set of the polynomial

$$\overline{F}(X) = x_3(x_1^2 + x_2^2) - 2x_0x_1x_2. \quad (30)$$

We observe that the line $x_1 = x_2 = 0$ is a double line of \overline{F} . Note that for the computation of the equations and parameterizations we choose O as symmetry point on the z -axis, whereas the illustrations in Figure 4 have been generated for a different reference point on the z -axis. The symmetric position implying (30) gives more compact equations for offsets, pedal surfaces and inverse pedal surfaces of F .

To compute the pedal surface G of F , we have to compute the tangent planes and the dual representation of F . From (29) it follows that the tangent planes E and E_d of F and its offset surface F_d are represented by the homogeneous plane coordinates

$$\begin{aligned} E(r, u) &= \mathbb{R}(r \sin 2u, -2 \sin u \cos 2u, 2 \cos u \cos 2u, -r), \\ E_d(r, u) &= \mathbb{R}(r \sin 2u - d\sqrt{\mathbf{n} \cdot \mathbf{n}}, -2 \sin u \cos 2u, 2 \cos u \cos 2u, -r), \end{aligned} \quad (31)$$

where $\mathbf{n} = (-2 \sin u \cos 2u, 2 \cos u \cos 2u, -r)$. Note that $E(r, u)$ is not a rational offset parameterization of F . Either by eliminating λ, r and u from the equations $\lambda u_i = E_i$, and $\lambda u_i = E_{d_i}$, $i = 0, \dots, 3$, or by applying the method outlined at the end of Section 2.1, one obtains implicit dual representations of F and its offset surface F_d by

$$\begin{aligned} \overline{F}^*(U) &= u_0(u_1^2 + u_2^2) - 2u_1u_2u_3, \text{ with } U = (u_0, \dots, u_3), \\ \overline{F}_d^*(U) &= d^2(u_1^2 + u_2^2)^2(u_1^2 + u_2^2 + u_3^2) - (u_0(u_1^2 + u_2^2) - 2u_1u_2u_3)^2 \\ &= d^2(u_1^2 + u_2^2)^2(u_1^2 + u_2^2 + u_3^2) - \overline{F}^*(U)^2. \end{aligned}$$

This tells that Plücker's conoid is not only of degree three but also of class three, and its offsets are of class six. To represent F by a parameterization so that its normal vector has rational norm, one considers the equation $\|\mathbf{n}(r, u)\|^2 = 4 \cos^2 2u + r^2 = w^2$. Its right hand side is a family of

conics in the plane \mathbb{R}^2 with coordinates r and w . These conics are parameterized by

$$r(u, t) = \frac{2 \cos 2u \cos t}{\sin t}, \text{ and } w(u, t) = \|\mathbf{n}\| = \frac{2 \cos 2u}{\sin t}.$$

Thus, the surface F admits a parameterization $\mathbf{f}(r(u, t), u)$, whose unit normal vector is $\mathbf{n}(u, t) = (-\sin u \sin t, \cos u \sin t, \cos t)$. That reparameterization implies that the tangent planes (31) of F and its offsets F_d are represented by

$$E_d(u, t) = \mathbb{R}(-\cos t \sin 2u - d, -\sin u \sin t, \cos u \sin t, \cos t).$$

With help of the transformation α and its inverse α^* , the defining polynomials of the pedal surfaces \bar{G} and \bar{G}_d follow by

$$\begin{aligned} \bar{G}(X) &= 2x_0x_1x_2x_3 + (x_1^2 + x_2^2)(x_1^2 + x_2^2 + x_3^2), \\ \bar{G}_d(X) &= d^2x_0^2(x_1^2 + x_2^2)^2(x_1^2 + x_2^2 + x_3^2) - \bar{G}(X)^2. \end{aligned}$$

Additionally we want to give the implicit representations of the conchoid surfaces of Plücker's conoid and their inverse pedal surfaces. Since Plücker's conoid and its conchoid surfaces are now considered as point sets, we use a different notation as above. Let Plücker's conoid A be the zero-set of the polynomial

$$\bar{A}(X) = x_3(x_1^2 + x_2^2) - 2x_0x_1x_2,$$

which is just the same as equation (30). Since A is a rational conchoid surface, there exists an affine parameterization $\mathbf{a}(u, v)$ with rational norm. This and the parameterization of its conchoid surfaces read

$$\begin{aligned} \mathbf{a}(u, v) &= \left(\frac{2 \sin u \cos v^2 \sin v}{\cos(u)}, \frac{2 \sin u \cos v \sin v^2}{\cos u}, 2 \cos v \sin v \right), \\ \mathbf{a}_d(u, v) &= \mathbf{a}(u, v) + d \frac{\mathbf{a}(u, v)}{\|\mathbf{a}(u, v)\|}, \text{ with } \|\mathbf{a}\| = \frac{2 \cos v \sin v}{\cos u} \end{aligned} \quad (32)$$

The conchoid surfaces A_d are of degree eight, and are the zero set of the polynomial

$$\begin{aligned} \bar{A}_d(X) &= d^2(x_1^2 + x_2^2)^2x_0^2x_3^2 - (x_1^2 + x_2^2 + x_3^2)(x_3(x_1^2 + x_2^2) - 2x_0x_1x_2)^2, \\ &= d^2(x_1^2 + x_2^2)^2x_0^2x_3^2 - (x_1^2 + x_2^2 + x_3^2)\bar{A}(X)^2. \end{aligned}$$

The inverse foot-point map α^* transforms Plücker's conoid \bar{A} and its conchoid surfaces A_d to \bar{B}^* and a family of rational offset surfaces \bar{B}_d^* . Rational offset parameterizations of B^* and B_d^* can be derived from the parameterization (32). The defining polynomials of these surfaces in dual coordinates read

$$\begin{aligned} \bar{B}^*(U) &= u_0u_3(u_1^2 + u_2^2) + 2u_1u_2(u_1^2 + u_2^2 + u_3^2), \\ \bar{B}_d^*(U) &= d^2(u_3^2)(u_1^2 + u_2^2)^2(u_1^2 + u_2^2 + u_3^2) - \bar{B}^{*2}. \end{aligned}$$

5.3. Pedal surfaces of quadrics

Consider a quadric $\bar{F} \subset \mathbb{P}^3$ as family of tangent planes. Quadratic cylinders and cones are excluded since their pedal 'surfaces' are curves. The dual surface $\bar{F}^* \subset \mathbb{P}^{3*}$ is also a quadric. Using homogeneous plane coordinates $U = (u_0, u_1, u_2, u_3)$, a dual quadric is the zero-set of a quadratic polynomial

$$\bar{F}^*(U) = U^T \cdot A \cdot U, \text{ with } A \in \mathbb{R}^{4 \times 4}, \text{ and } A = A^T. \quad (33)$$

We do not require regularity of A , and will later on in Section 5.3.2 discuss singular dual quadrics. In case that $\det A \neq 0$, and considering $X = (x_0, x_1, x_2, x_3)$ as homogeneous point coordinates, the quadric \bar{F} , whose tangent planes satisfy (33), is the zero-set of the quadratic polynomial

$$\bar{F}(X) = X^T \cdot A^{-1} \cdot X.$$

5.3.1. Pedal surfaces of regular quadrics

Consider a regular dual quadric \overline{F}^* , represented as zero-set of a quadratic polynomial (33). Let $O \notin F$ be the reference point of the foot-point map. There are two main cases to be distinguished. A quadric \overline{F} is called a paraboloid, if the ideal plane $\omega : x_0 = 0$ is tangent to \overline{F} . Otherwise, \overline{F} is either a hyperboloid or an ellipsoid. Let us start with the latter case. We may assume that the dual quadric \overline{F}^* is represented by

$$\overline{F}^* : a_0u_0^2 + a_1u_1^2 + a_2u_2^2 + a_3u_3^2 + u_0(b_1u_1 + b_2u_2 + b_3u_3) = 0.$$

This choice is justified because we may choose ω , and the symmetry planes of the tangential cone with vertex at O as coordinate planes. Applying the foot-point map α and with the abbreviation $\mathbf{x} = (x_1, x_2, x_3)$, its pedal surface $\overline{G} = \alpha(\overline{F}^*)$ is represented by

$$\overline{G} : x_0^2(a_1x_1^2 + a_2x_2^2 + a_3x_3^2) - x_0(\mathbf{x}^2)(b_1x_1 + b_2x_2 + b_3x_3) + a_0(\mathbf{x}^2)^2 = 0. \quad (34)$$

According to Proposition 3, the foot-point map α is the product $\alpha = \sigma \circ \pi$ of the polarity π and the inversion σ . Since $\pi(\overline{F}^*)$ is a quadric, $\overline{G} = \sigma(\pi(\overline{F}^*))$ is the image of a quadric with respect to the inversion σ . Thus \overline{G} is a special instance of a Darboux cyclide, an algebraic surface typically of degree four, whose intersection with ω is the conic $j : x_0 = 0, x_1^2 + x_2^2 + x_3^2 = 0$, with multiplicity two. Note that not all Darboux cyclides are images of quadrics with respect to inversion. Since the highest power of x_0 in (34) is two, the reference point O is a double point of \overline{G} . We mention that parabolic Darboux cyclides are algebraic surfaces of order three, whose intersection with ω contains, besides j , a real line, compare equation (35).

Remark 1. A surface is called Darboux cyclide, if it is the zero-set of a quadratic polynomial $Q(\mathbf{y}) : \mathbf{y}^T \cdot B \cdot \mathbf{y}$ in pentaspherical coordinates $\mathbf{y} = (y_0, y_1, \dots, y_4)$, with $y_0^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2$, which is just $S^3 : \mathbf{y}^T \cdot \text{diag}(-1, 1, 1, 1, 1) \cdot \mathbf{y} = 0$. A cyclide is obtained by applying a stereographic projection $S^3 \rightarrow \mathbb{R}^3$ to the intersection $Q \cap S^3$. This projection is realized by the equations

$$x_0 = y_0 - y_4, x_1 = y_1, x_2 = y_2, x_3 = y_3.$$

Substituting these relations into (34), and taking into account that $(y_0 - y_4)(y_0 + y_4) = y_1^2 + y_2^2 + y_3^2$, gives the quadratic equation

$$a_0(y_0 + y_4)^2 - (y_0 + y_4)(b_1y_1 + b_2y_2 + b_3y_3) + (a_1y_1^2 + a_2y_2^2 + a_3y_3^2) = 0,$$

representing Q . Thus, (34) is a Darboux cyclide.

We turn to the case where \overline{F}^* is a paraboloid. Since ω is a tangent plane of F , we may choose a coordinate system such that F is parameterized by $\mathbf{f}(u, v) = (u, v, au^2 + bv^2 + c)$, with $abc \neq 0$. If a or b are zero, F is a parabolic cylinder, and $\alpha(F)$ is a curve. If $c = 0$, F contains O . Otherwise, \overline{F} is the zero-set of the polynomial $\overline{F}(X) = ax_1^2 + bx_2^2 + cx_0^2 - x_0x_3$. Its dual equation reads

$$\overline{F}^* : -4abu_0u_3 + bu_1^2 + au_2^2 - 4abcu_3^2 = 0.$$

The polarity π from (17) maps \overline{F}^* to a quadric $\pi(\overline{F}^*) : 4abx_0x_3 + bx_1^2 + ax_2^2 - 4abcx_3^2 = 0$. Since ω is tangent to \overline{F} , the origin $O = (1, 0, 0, 0)\mathbb{R}$ is contained in $\pi(\overline{F}^*)$. Consequently, after canceling out the factor x_0 , the pedal surface

$$\overline{G} = \alpha(\overline{F}^*) : 4abx_3(x_1^2 + x_2^2 + x_3^2) + x_0(bx_1^2 + ax_2^2 - 4abcx_3^2) = 0 \quad (35)$$

is a parabolic Darboux cyclide, an algebraic surface of degree three. In case that F is a paraboloid of revolution, thus $a = b$, and additionally O coincides with the focal point of F , thus $c = -\frac{1}{4a}$, this surface is reducible and reads $\overline{G} : (x_1^2 + x_2^2 + x_3^2)(4ax_3 + x_0) = 0$. The first factor $x_1^2 + x_2^2 + x_3^2 = 0$ defines the isotropic cone with vertex at O , the second factor $4ax_3 + x_0 = 0$ is the tangent plane at the vertex $(1, 0, 0, -1/(4a))\mathbb{R}$ of F .

Proposition 8. *The pedal surface $\overline{G} = \alpha(\overline{F}^*)$ of a regular ellipsoid or hyperboloid \overline{F} is a Darboux cyclide of degree four, with the reference point O as double point. The family of offset surfaces \overline{F}_d of the quadric \overline{F} is mapped by α to the conchoid surfaces \overline{G}_d of the cyclide \overline{G} . The pedal surface \overline{G} of a paraboloid \overline{F} is a Darboux cyclide of degree three. Only in case that the reference point O coincides with the focal point of a paraboloid of revolution F , its pedal surface is the tangent plane at F 's vertex.*

In addition to the geometric properties of pedal surfaces of quadrics, we want to discuss quadrics in context with rational offset surfaces. As proved in [9], the offset surfaces F_d of regular quadrics F admit rational parameterizations. The construction is not trivial, and thus we provide an outline. Any regular quadric $\overline{F} \subset \mathbb{P}^3$ is the envelope of a one-parameter family of cones of revolution $C(u)$, with vertices at a focal conic of \overline{F} . It is possible to parameterize these cones of revolution $C(u)$ in a way that the normal vectors of their tangent planes have rational norm. Since offsets of cones of revolution are again cones of revolution, the rationality of the norm of F 's normal vectors holds for its offsets F_d , too.

To construct the pedal surface $\overline{G}_d = \alpha(\overline{F}_d^*)$ of the offsets \overline{F}_d^* of a quadric \overline{F}^* , one computes the family of pedal curves $c = \alpha(C)$ of the family of cones of revolution C . Such a pedal curve c is a rational spherical curve of degree four, and can be constructed in the following way. Let C^\perp be a cone of revolution with vertex O , consisting of lines perpendicular to C 's tangent planes. Let S be the sphere with diameter OV , where V is the vertex of C . Then the pedal curve $c = \alpha(C)$ equals the intersection $S \cap C^\perp$, and thus O is a double point of c . Finally, the pedal surface \overline{G}_d contains a rational one-parameter family of rational spherical quartic curves c , according to the cones of revolution C enveloping \overline{F}_d^* .

Rational offsets of paraboloids and their pedal surfaces. Consider a paraboloid F , and its offset surfaces F_d , we intend to derive explicit rational parameterizations of the offsets and their pedal surfaces G_d , with respect to the reference point $O = (0, 0, 0)$. Let

$$\mathbf{f}(u, v) = (u, v, \frac{1}{2}au^2 + \frac{1}{2}bv^2 + c), \text{ with } abc \neq 0,$$

be a parameterization of F . The corresponding normal vector reads $\mathbf{n}(u, v) = (-au, -bv, 1)$. In order to determine a re-parameterization, so that the resulting normal vector has rational norm, we consider the system of equations

$$\mathbf{n}(u, v) = \lambda(s, t)(\cos s \cos t, \sin s \cos t, \sin t) = \lambda(s, t)\mathbf{m}(s, t). \quad (36)$$

The right hand side of this equation represents the unit vector $\mathbf{m}(s, t)$, scaled by $\lambda(s, t)$. It is more compact to represent $\mathbf{m}(s, t)$ in terms of trigonometric functions, but we either may use any rational parameterization $\mathbf{m}(s, t)$ of S^2 , or use the Weierstrass substitutions $\cos x = (1 - y^2)/(1 + y^2)$, and $\sin x = 2y/(1 + y^2)$, where $y = \tan(x/2)$, to convert the trigonometric representation (36) to a rational one. The system (36) has the obvious solution

$$\lambda(s, t) = 1/\sin t, u = \frac{-\cos s \cos t}{a \sin t}, v = \frac{-\sin s \cos t}{b \sin t}. \quad (37)$$

Performing this reparameterization, the tangent planes E and E_d of F and its offset surface F_d are represented by

$$\begin{aligned} E_d : \mathbf{x} \cdot \mathbf{m}(s, t) &= e(s, t) + d, \\ \text{with } e(s, t) &= -\frac{b \cos^2 s \cos^2 t + a \sin^2 s \cos^2 t - 2abc \sin^2 t}{2ab \sin t}. \end{aligned}$$

The foot-point map α (13) transforms E_d to a parameterization $\mathbf{g}_d(s, t)$ of the pedal surfaces G_d . Since $\|\mathbf{m}\| = 1$, we obtain the polar representation

$$\mathbf{g}(s, t) = (e(s, t) + d)\mathbf{m}(s, t)$$

of the conchoid surfaces G_d of G . Thus the pedal surfaces G_d of the offset surfaces F_d of paraboloids F are rational conchoid surfaces in the sense of Definition 2.

5.3.2. Pedal surfaces of conics as singular dual quadrics

When speaking about pedal surfaces of quadrics and their offsets, it is worth studying dual singular quadrics, too. Consider a dual singular quadric

$$\overline{F}^*(U) = U^T \cdot A \cdot U, \text{ with } A \in \mathbb{R}^{4 \times 4}, A = A^T \text{ and } \text{rk } A = 3, \quad (38)$$

which is formed by the planes passing through the tangent lines of a conic $c = \overline{F}^*$. The null-space of A is the carrier plane γ of c . To study the pedal surface $\alpha(c)$, we use the composition (3) of $\alpha = \sigma \circ \pi$, with the polarity $\pi : \mathbb{P}^{3*} \rightarrow \mathbb{P}^3$ and the inversion $\sigma : \mathbb{P}^3 \rightarrow \mathbb{P}^3$, both with respect to the unit sphere S^2 .

The polar image $Q = \pi(c)$ is a quadratic cone, with vertex $V = \pi(\gamma)$. The generating lines in Q correspond to the pencils of planes in c . The cone Q becomes a cylinder in case that $O \in \gamma$. In case that $O \in c$, a pencil of planes is mapped to points of an ideal line, and Q is a parabolic cylinder. Worth to be mentioned is also the case that $Q = \pi(c)$ is a rotational cone or cylinder. This happens exactly if O is contained in the focal conic d of c . The carrier plane of d is a symmetry plane of γ , and the vertices of d coincide with the focal points of c , and vice versa.

The inversion σ maps a quadratic cone Q to a cyclide $G = \sigma(Q)$, typically of degree four. The family of tangent planes $\tau(u)$ of Q is mapped to a family of spheres $S(u) = \sigma(\tau(u))$, so that G is a canal surface. In case that Q is a cylinder or cone of revolution, it contains also a family of inscribed spheres. Consequently, $G = \sigma(Q)$ is the envelope of two different families of spheres, thus a Dupin cyclide.

Proposition 9. *The pedal surface G of a conic c is a Darboux cyclide, being also a canal surface. The pedal surfaces of the offset surfaces of c , being pipe surfaces with center curve c , are conchoid surfaces G_d of the Darboux cyclide G . In case that the reference point O of the foot-point map is located at the focal conic d of c , the pedal surface G is a Dupin cyclide, being the envelope of a family of spheres in a twofold way.*

Typically the pedal surfaces of conics are canal surfaces of degree four. Degree reductions appear at first if $c = \overline{F}^*$ is a parabola. Since $\omega \in \overline{F}^*$, $O \in \pi(c)$, and in general $\alpha(c)$ is a cubic cyclide. This is illustrated in the following.

Example. We consider the parabola F with parameterization $\mathbf{f}(u) = (u, 0, \frac{a}{2}u^2 + c)$, with $ac \neq 0$, whose tangent planes are $E(u, v) = \mathbb{R}(-\frac{a}{2}u^2 + c, au, v, -1)$. In terms of homogeneous plane coordinates $U = (u_0, \dots, u_3)$, the parabola is the zero set of the polynomial

$$\overline{F}^*(U) = u_1^2 - 2au_0u_3 - 2acu_3^2.$$

The pedal surface $\overline{G} = \alpha(F^*)$ is a parabolic Darboux cyclide, and its defining polynomial reads

$$\overline{G}(X) = x_0(x_1^2 - 2acx_3^2) + 2ax_3(x_1^2 + x_2^2 + x_3^2). \quad (39)$$

We note that applying the polarity π to \overline{F}^* yields the cylinder $\pi(\overline{F}^*)(X) = x_1^2 - 2acx_3^2 + 2ax_0x_3$. The inversion σ maps this surface to the pedal surface (39). In case that $\pi(\overline{F}^*)$ is a cylinder of revolution, it is the envelope of two different one-parameter families of spheres, including the family of tangent planes. Thus, for $2ac = -1$ expressing that the origin is the focal point F , the pedal surface \overline{G} is a parabolic Dupin cyclide.

Since the family of offset surfaces F_d^* is mapped by α to a family of conchoid surfaces G_d , we have a look at rational offset parameterizations of F^* . Therefore we use the reparameterization (37), which maps $E(u, v)$ to

$$E_d(s, t) = \mathbb{R} \left(\frac{\cos^2 s \cos^2 t - 2ac \sin^2 t}{2a \sin t} + d, \cos s \cos t, \sin s \cos t, \sin t \right).$$

In fact we use trigonometric instead of rational functions because of readability. The offsets of the parabola \overline{F}^* are pipe-surfaces \overline{F}_d^* of class four, whose center curve is F . Their dual equation reads

$$\begin{aligned} \overline{F}_d^*(U) &= -4a^2 d^2 u_3^2 (u_1^2 + u_2^2 + u_3^2) + (u_1^2 - 2au_0 u_3 - 2acu_3^2)^2, \\ &= -4a^2 d^2 u_3^2 (u_1^2 + u_2^2 + u_3^2) + \overline{F}^*(U)^2. \end{aligned}$$

The conchoid surfaces \overline{G}_d of the parabolic cyclide \overline{G} are the α -images of \overline{F}_d^* . Consequently, their implicit equation reads

$$\begin{aligned} \overline{G}_d(X) &= -4a^2 d^2 x_0^2 x_3^2 (x_1^2 + x_2^2 + x_3^2) + (x_0(x_1^2 - 2acx_3^2) + 2ax_3(x_1^2 + x_2^2 + x_3^2))^2, \\ &= -4a^2 d^2 x_0^2 x_3^2 (x_1^2 + x_2^2 + x_3^2) + \overline{G}(X)^2. \end{aligned}$$

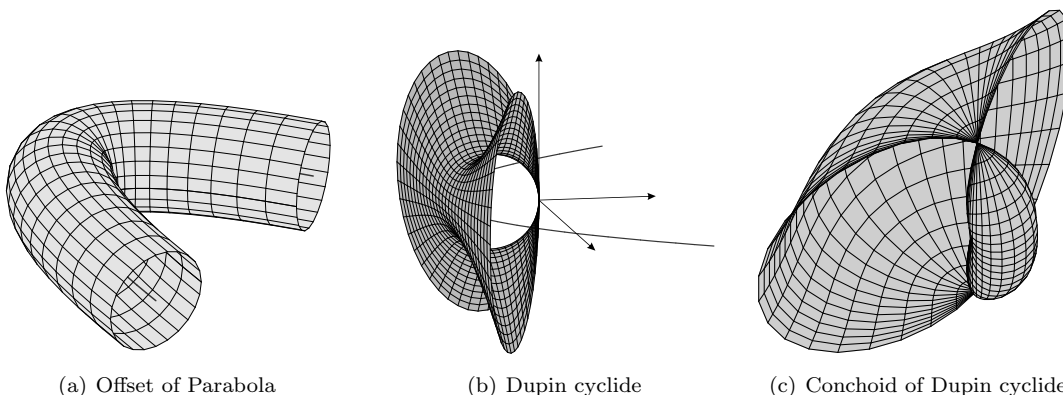


Figure 5: Parabola F , Dupin cyclide G as pedal surface with respect to the parabolas focal point, and conchoid of G

Quadrics of revolution. What we have discussed is far from being a complete classification of the pedal surfaces of quadrics. One important case not being touched so far is the case where $\pi(\overline{F}^*)$ a rotational quadric. This appears if the tangential cone of the reference point O is a cone of revolution. In this case, $\pi(\overline{F}^*)$ is the envelope of a one-parameter family of spheres.

Let $\pi(\overline{F}^*)$ be a rotational quadric, then \overline{G} is the envelope of a one parameter family of spheres, since σ maps spheres to spheres. Thus \overline{G} is a canal surface. As we have already seen, this happens also in case that $\pi(\overline{F}^*)$ is a quadratic cone or cylinder. In the particular case where $\pi(\overline{F}^*)$ is a cone or cylinder of revolution, it is the envelope of two different families of spheres, where tangent planes are counted as spheres. This implies that \overline{G} is a Dupin cyclide.

A further particular case appears when $\pi(\overline{F}^*)$ is a sphere. This happens if the reference point O is the focal point of a rotational quadric \overline{F}^* . Consequently, \overline{G} is a sphere.

5.4. Inverse pedal surfaces of quadrics

While the last section has studied quadrics as dual objects, we consider quadrics G and their conchoid surfaces G_d as set of points. Let $X = (x_0, x_1, x_2, x_3)$ be homogeneous point coordinates

in \mathbb{P}^3 , a quadric \overline{G} is represented as zero-set of a quadratic polynomial

$$\overline{G}(X) = X^T \cdot A \cdot X, \text{ with } A \in \mathbb{R}^{4 \times 4}, \text{ and } A = A^T.$$

If $\det A \neq 0$, the quadric is regular, and in case that $\text{rk } A = 3$, the quadric is a quadratic cone. More degenerated cases are a pair of planes ($\text{rk } A = 2$) and a two-fold plane ($\text{rk } A = 1$). The inverse pedal surface $F^* = \alpha^*(G)$ of a plane G is a paraboloid of revolution, with focal point O . For the remainder of this section we assume that $\text{rk } A = 3$ or 4.

Consider a regular quadric \overline{G} and the reference point O as origin. Assuming that the axes of \overline{G} are parallel to the coordinate axes, we may assume that \overline{G} is the zero-set of the quadratic polynomial

$$\overline{G}(X) = a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + x_0(b_1x_1 + b_2x_2 + b_3x_3).$$

Applying α^* yields \overline{F}^* , which is in fact just the dual object of (34), and its defining polynomial in homogeneous plane coordinates $U = (u_0, u_1, u_2, u_3) = (u_0, \mathbf{u})$ reads

$$\overline{F}^*(U) = u_0^2(a_1u_1^2 + a_2u_2^2 + a_3u_3^2) - u_0\mathbf{u}^2(b_1u_1 + b_2u_2 + b_3u_3) + a_0(\mathbf{u}^2)^2. \quad (40)$$

Dual to the fact that the origin $O = (1, 0, 0, 0)\mathbb{R}$ is a double point of \overline{G} in (34), the ideal plane ω is a double tangent plane of \overline{F}^* .

In [6] it is shown that quadrics $G \subset \mathbb{R}^3$ and their conchoid surfaces G_d allow rational parameterizations \mathbf{g}_d with rational norm $\|\mathbf{g}_d\|$, independently on the chosen reference point. According to Theorem 4, rational parameterizations \mathbf{f}_d of \overline{F}_d^* are obtained by $\mathbf{f}_d = \alpha^*(\mathbf{g}_d)$. Using the composition $\alpha^* = \pi^* \circ \sigma$, the surface \overline{F}^* is the dual object of a cyclide $\sigma(G)$, and the offset surface \overline{F}_d^* is the dual object of a conchoid surface $\sigma(G_d)$ of a cyclide.

Inverse Pedal surfaces of quadratic cones and cylinders. Consider a quadratic cone or cylinder G , as singular quadric. To get insight to the geometric properties of their inverse pedal surfaces we use the decomposition $\alpha^* = \pi^* \circ \sigma$. The surface $\sigma(G)$ is a cyclide, and since G is the envelope of a one-parameter family of tangent planes τ , $\sigma(G)$ is a canal surface, enveloped by the spheres $\sigma(\tau)$. These spheres $\sigma(\tau)$ pass through O . In case $O \in \tau$, $\sigma(\tau) = \tau$ is a plane as well.

Quadratic cones and cylinders are also ruled surfaces, and thus we can use the results from Section 5.2. There we noted that $\alpha^*(G) = F^*$ is the envelope of parabolic cylinders $P = \alpha^*(l)$, being the α^* -images of the generating lines $l \subset G$.

The quadrics $\alpha^*(\tau)$ enveloping F , are typically paraboloids, according to the fact that the spheres $\sigma(\tau)$ contain O , and that the polarity π^* maps a sphere $\sigma(\tau) \ni O$ to a paraboloid. Since $l \subset \tau$, the parabolic cylinders $\alpha^*(l)$ touches the paraboloids $\alpha^*(\tau)$ in points of conics d . Consider the cones of revolution D touching the canal surface $\sigma(G)$ along its characteristic circles $\sigma(l)$. The previously mentioned conics d are just the images of these cones D with respect to π , thus $d = \pi(D)$.

We illustrate this at hand of an example. Consider the cylinder $G : x^2/a^2 + y^2/b^2 - 1 = 0$. In terms of homogeneous coordinates it reads $\overline{G} : x_1^2/a^2 + x_2^2/b^2 - x_0^2 = 0$. The inverse image $\sigma(\overline{G})$ is a cyclide. Applying $\pi : \mathbb{P}^3 \rightarrow \mathbb{P}^{3*}$, one obtains $\alpha^*(\overline{G}) = \overline{F}^*$. These surfaces read

$$\begin{aligned} \sigma(\overline{G}) &: \frac{x_0^2x_1^2}{a^2} + \frac{x_0^2x_2^2}{b^2} = (x_1^2 + x_2^2 + x_3^2)^2, \\ \overline{F}^* &: \frac{u_0^2u_1^2}{a^2} + \frac{u_0^2u_2^2}{b^2} = (u_1^2 + u_2^2 + u_3^2)^2. \end{aligned}$$

The defining polynomial of \overline{F} is rather lengthy, and of degree eight. Thus we provide a parametric representation. Starting with $\mathbf{g}(u, v) = (a \cos u, b \sin u, v)$ for G , the tangent planes of F are

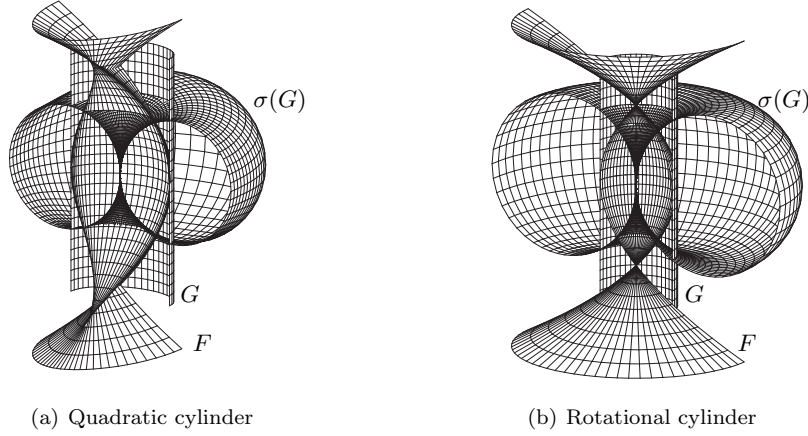


Figure 6: Inverse pedal surfaces of quadratic cylinders

$\alpha^*(\mathbf{g}) = E = \mathbb{R}(-a^2 \cos^2 u - b^2 \sin^2 u - v^2, a \cos u, b \sin u, v)$. An affine parameterization of F , see Figure 6(a), is finally

$$\mathbf{f}(u, v) = \left(-\frac{\cos u}{a} ((a^2 - b^2) \cos^2 u + b^2 - 2a^2 + v^2), -\frac{\sin(u)}{b} ((a^2 - b^2) \cos^2 u - b^2 + v^2), 2v \right).$$

Even the case $a = b$ is interesting, where $\sigma(\overline{G})$ is a torus, whose meridian circles touch the z -axis. The inverse pedal surface is a rotational surface with a parabola $(-b^2 + v^2, 0, 2v)$ as meridian curve, see Figure 6(b).

To represent the whole family of conchoid surfaces G_d of the quadratic cylinder G , one preferably uses a rational polar representation, see [11]. Applying α^* gives a dual representation of the family \overline{F}_d of rational offset surfaces. Let $\mathbf{d} = (d_1, d_2, d_3) = (a \cos u, b \sin u, 0)$ be the foot-point curve with respect to O , a polar representation of G_d reads

$$\mathbf{g}_d(u, t) = \frac{1 + \mathbf{d}^2 t^2 + 2dt}{2t(1 + \mathbf{d}^2 t^2)} (2td_1, 2td_2, 1 - \mathbf{d}^2 t^2).$$

The tangent planes of F_d are $E_d : \mathbf{x} \cdot \mathbf{g}_d = \mathbf{g}_d^2$. Solving the linear system $E_d \cap (E_d)_t \cap (E_d)_u$ yields an affine parameterization $\mathbf{f}_d(u, v)$ of F_d , whose normal vector \mathbf{g}_d has rational length.

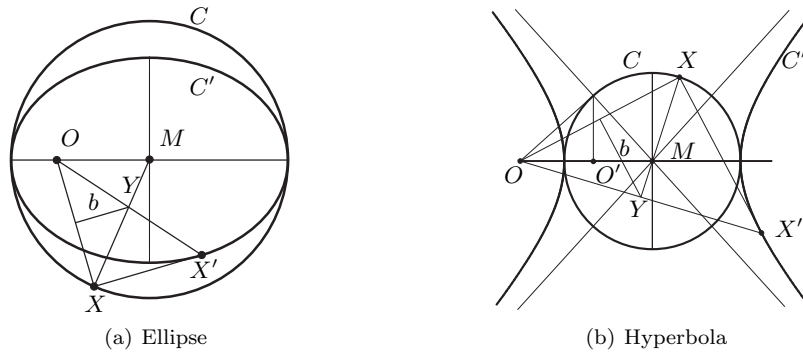


Figure 7: Inverse pedal curves C' of a circle C with respect to point O

Inverse pedal surface of a sphere. As a particular example we discuss the inverse pedal surface F^* of a sphere $G : (x - m^2) + y^2 + z^2 = r^2$, of radius r and centered at $(m, 0, 0)$, with respect to the

reference point $O = (0, 0, 0)$. Since $\sigma(G)$ is a sphere, $F^* = \pi^*(\sigma(G))$ is a quadric. The defining polynomials of \overline{G} and \overline{F}^* in homogeneous coordinates are

$$\begin{aligned}\overline{G}(X) &= x_1^2 + x_2^2 + x_3^2 - 2x_0x_1m + x_0^2(m^2 - r^2), \\ \overline{F}^*(U) &= u_0^2 + 2u_0u_1m + (u_1^2 + u_2^2 + u_3^2)(m^2 - r^2).\end{aligned}$$

Transforming plane to point coordinates one obtains the known result that \overline{F} is a quadric of revolution, with O as focal point,

$$\overline{F}: \frac{r^2(y^2 + z^2)}{a^2} - \frac{x^2}{a} + \frac{2mx}{a} = 1, \text{ with } a = m^2 - r^2.$$

The quadric \overline{F} is an ellipsoid for O being inside of \overline{G} , and a hyperboloid of two sheets for O being outside of \overline{G} . Figure 7 illustrates the 2d-case, the inverse pedal curves of a circle with respect to a reference point O . For $O \in \overline{G}$, the inverse pedal surface \overline{F} degenerates to a single point, according to Thales theorem. Considering the dual surface in that particular case, \overline{F}^* is a bundle of planes, passing through that single point.

According to Theorem 4, a particular parameterization $\mathbf{g}(u, v)$ of G with rational norm $\|\mathbf{g}(u, v)\|$, see [12] is mapped by α^* to a parameterization $\mathbf{f}(u, v)$ of F , whose normal vector has rational length. In this way, the conchoid surfaces of spheres correspond to the offsets of ellipsoids and hyperboloids of revolution.

Corollary 10. *The conchoid surfaces G_d of a sphere G are in birational correspondence to the offset surfaces F_d of ellipsoids or hyperboloids of revolution F , depending on whether the reference point O is inside or outside of G .*

6. Conclusion

The present article studies relations between families of rational offset surfaces and rational conchoid surfaces. The foot-point map α transforms a family of offset surfaces to a family of conchoid surfaces, where the reference points of the foot-point map and the conchoid construction coincide. Since α is a birational map, birational invariants are transformed from one family to the other. The same properties hold for the inverse foot-point map. The relations between offset surfaces and conchoid surfaces are demonstrated at hand of pedal surfaces and inverse pedal surfaces of ruled surfaces and quadrics.

There is a close relation to bisector surfaces. Considering a surface $G \subset \mathbb{R}^3$, and a fixed reference point $O = (0, 0, 0)$. The bisector surface $B(G, O)$ of G and O is the envelope of symmetry planes S of O and a moving point $\mathbf{g} \in G$. Scaling the inverse pedal surface $F^* = \alpha^*(G)$ by the factor $1/2$ gives $B(G, O)$. More details on this construction may be found in [10].

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