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The HEUREKA-Polyhedron

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Introduction

Buckminster Fuller's Jitterbug consists of eight equilateral triangles. Each vertex represents a spherical joint between two triangles (see [1], [2]). At the HEUREKA-polyhedron[1] these twelve spherical joints are replaced by joints with degree 2 of freedom. Buckminster Fuller had already studied the one-parametric motion M_O of the Jitterbug that converts a regular octahedron via a regular icosahedron into a cuboctahedron (Fig. 2). This motion M_O is now constrained for the HEUREKA-polyhedron. The proof for this fact reveals that this polyhedron allows also a second one-parametric motion M_T (Fig. 8). At the Jitterbug this motion recently has been recognized by H. F. Verheyen [3]. It converts a twofold covered regular tetrahedron via a truncated tetrahedron into a regular octahedron.

[1] A 15 m high model of this polyhedron was exhibited at the national research exposition of Switzerland 1991 in Zurich.

Definition of the one-parametric motion M_O

Let a regular octahedron $\mathbf{O}$ be given with edges of length 1. In each face of $\mathbf{O}$ we inscribe an equilateral triangle in such a way that on each edge there is a common vertex for the triangles inscribed in the two neighbouring faces (Fig. 1). All these twelve new vertices divide their edges in the same ratio. If the lengths of the two segments are denoted by t and $(1-t)$, then the side-length of the inscribed triangles reads

$$l(t) = \sqrt{3t^2 - 3t + 1}.$$

The angle τ of rotation of each inscribed triangle against the original face matches the equations

$$\sin\tau = \frac{t\sqrt{3}}{2\sqrt{3t^2-3t+1}}, \quad \cos\tau = \frac{2-3t}{2\sqrt{3t^2-3t+1}}, \quad \tan\tau = \frac{t\sqrt{3}}{2-3t}.$$

Additionally we transform this figure by a dilation with dilatation factor $1/l(t)$. Then the sides of all eight inscribed triangles have unit length.

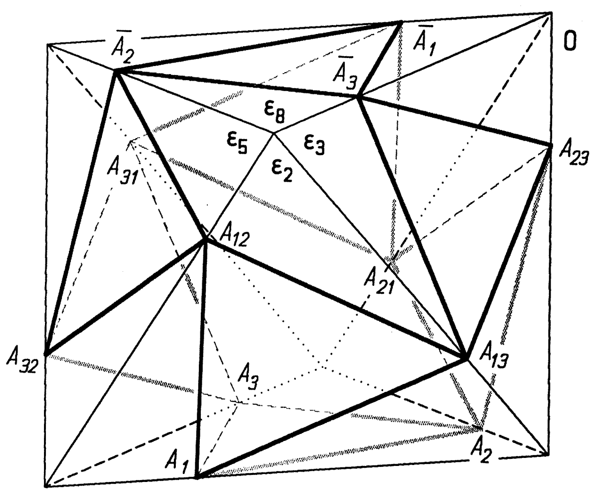

Figure 1. Octahedron $\mathbf{O}$ *and HEUREKA-polyhedron* $\mathbf{H}$

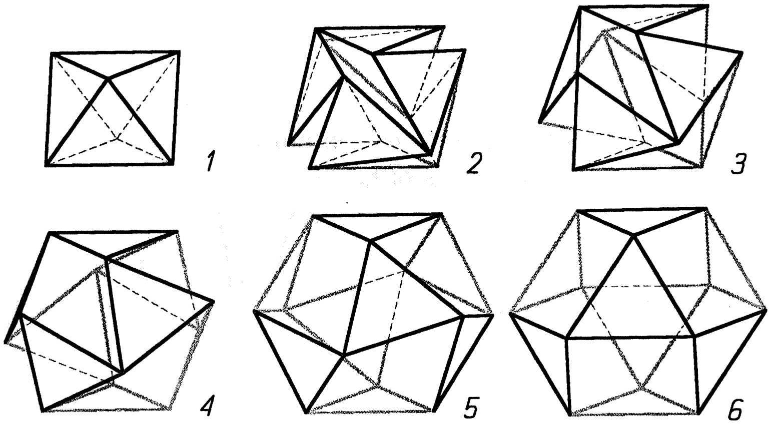

Figure 2. The one-parametric motion M_O

While the parameter t is varying from 0 to 1 the kinematic chain **H** formed by these triangles of side length 1 performs a one-parametric motion M_O (Fig. 2). The dihedral angle φ between two neighbouring faces remains unchanged. We calculate

$$\tan\frac{\varphi}{2} = \sqrt{2}, \;\; \sin\frac{\varphi}{2} = \sqrt{\frac{2}{3}}, \;\; \frac{\varphi}{2} = 54,73..^\circ, \;\; \cos\varphi = -\frac{1}{3}, \;\; \varphi = 109,47..^\circ.$$

In the initial position and in the end position the triangles cover the faces of **O**. We obtain for $t = 0,5$ the triangular faces of a cuboctahedron. For $t = (\sqrt{5}-1)/2 = 0,6180..$ (golden section) or for $t = (3-\sqrt{5})/2 = 0,3819..$ the convex closure of the triangles is a regular icosahedron; the dilatated octahedron **O** is one of the five regular octahedra that can be circumscribed about this icosahedron.

In half of the faces of **O** the increasing t defines a clockwise rotation, if it is seen from outside. In this case the included faces of **H** will be called *positive*; their planes define a regular tetrahedron circumscribed to **H** and **O**. The other faces of **H** are called *negative*.

Let S_O denote the symmetry-group of the octahedron **O**. In order to preserve the set **H** of inscribed triangles, a direct motion $\delta \in S_O$ has to transform the positive faces again into positive faces. In this case δ maps each circumscribed tetrahedron onto itself. The indirect motions of S_O that preserve **H** have to commute the two tetrahedra.

Hence for $t \neq 0,5$ the symmetry-group S_H of **H** is isomorphic to the group $S_O/\{id, \sigma\}$, provided $\sigma \in S_O$ is the reflection in a plane perpendicular to one edge of **O**. Group S_H contains eight 120°-rotations about the face-axes of **O**, three 180°-rotations about vertex-axes and the identity together with the products of each of these isometries with the reflection in the center of **O**. The symmetry operations contained in S_H induce just the even permutations of the four face-axes of **O**.

The HEUREKA-polyhedron

The defined one-parametric motion M_O of **H** preserves the dihedral angle between each two neighbouring triangles. Therefore this structure will still be flexing, if the lines perpendicular to the corresponding planes at the common vertex are linked together by a connecting sector outside of **O** with center angle $(\pi - \varphi)$. The perpendicular lines are the axes of rotation of this rigid sector against the adjacent triangles thus forming the special joints used at the HEUREKA-polyhedron (Fig. 3). This completed structure shall again be denoted by **H**.

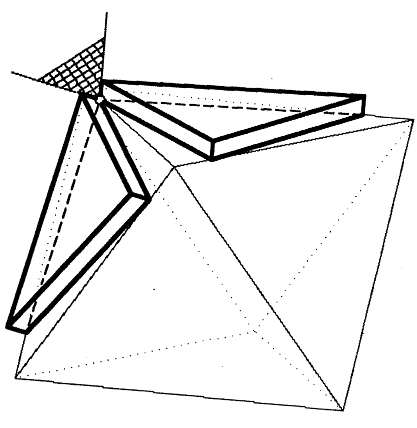

Figure 3. Joints of the HEUREKA-polyhedron

Let us fix one triangle $A_1A_2A_3$ of **H** in a horizontal position (Fig. 1). Then the opposite triangle $\overline{A}_1\overline{A}_2\overline{A}_3$ performs a pure translation in vertical direction. Any remaining vertex of **H** will be denoted by A_{ij} provided this

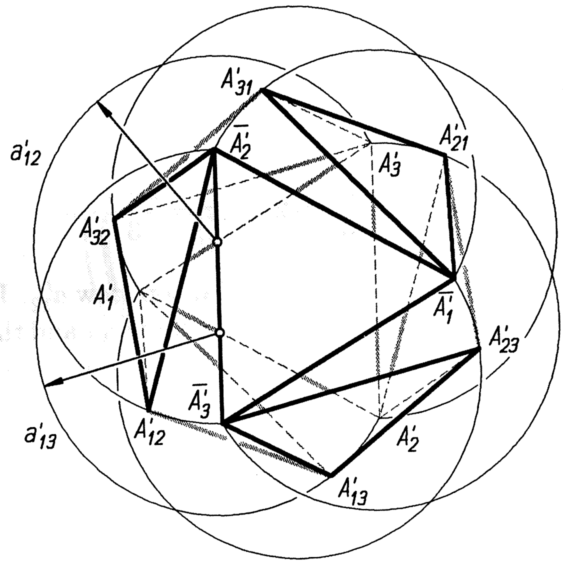

Figure 4. Top view of point paths under M_O

point is endpoint of edges through A_i and $\overline{A}_j$ respectively. In the following we discuss the path of A_{ij}:

The triangle $A_1A_{12}A_{13}$ is symmetric to the fixed triangle $A_1A_2A_3$ with respect to the plane χ_1 bisecting the angle between the corresponding planes. The plane χ_1 through A_1 has a constant inclination. Therefore χ_1 is tangent to a cone Ψ_1' of revolution with the vertex A_1 and with the half-vertex-angle $(\pi - \varphi)/2$. Any plane tangent to Ψ_1' is a possible mirror plane unless it contains interior points of the base triangle $A_1A_2A_3$. We can produce the motion of the triangle $A_1A_{12}A_{13}$ around point A_1 by rolling a cone Ψ_1 on the fixed cone Ψ_1', where Ψ_1 is congruent to Ψ_1'. Under this rolling of cones the point A_{13} traces a spheric trochoid a_{13} on the unit sphere Σ_1 with center A_1. a_{13} can also be obtained by reflecting A_2 in admissible tangent planes of Ψ_1'. The lines joining A_2 and A_{13} have the constant inclination $(\pi-\varphi)/2$ and they intersect Σ_1 in A_2 and A_{13}. Between each two vertical planes through A_2 the intersecting circle with Σ_1 and the line A_2A_{13} make similar figures. Hence we get the top view a_{13}' of a_{13} by transforming the equator of Σ_1 under a dilatation with center A_2 and with the dilatation-factor

$$\cos^2 \frac{(\pi - \varphi)}{2} = \sin^2 \frac{\varphi}{2} = \frac{2}{3}.$$

a'_{13} is a 240°-degree circular arc with radius $2/3$ (Fig. 4). a_{13} is part of a quartic of the first kind with one node.

In a coordinate system with vertical z-axis, with $A_1 = (0,0,0)$ and $A_2 = (0,1,0)$ the path a_{13} reads

$$a_{13}: \quad x = \frac{2}{3}\sin 2\tau, \quad y = \frac{1}{3} + \frac{2}{3}\cos 2\tau, \quad z = \frac{2\sqrt{2}}{3}\sin\tau, \quad 0 \le \tau \le \frac{2\pi}{3}.$$

Analogously the path a_{12} of A_{12} has a circular top view a'_{12}. In each position the two lines A_2A_{13} and A_3A_{12} are perpendicular to χ_1 and therefore parallel to each other. Due to the theorem of the circumference-angle we conclude that the image points A'_{13} and A'_{12} in the top view are running along their circular paths a'_{13} and a'_{12} with the same velocity at each moment. This could also be confirmed using the parameter equations of a_{12} and a_{13}. The symmetry operations of **H** reveal that the velocities of all image points A'_{ij} are equal.

The one-parametric motion M_O of H is constrained

The degree of freedom of the HEUREKA-polyhedron **H** is $F \geq 1$. We show $F = 1$ for all positions that can continuously be obtained from positions of M_O. This is done by proving that in this case the faces are located in the planes of a regular octahedron **O**. Hence each vertex of **H** is lying on an edge of **O** and this characterizes M_O. But there are still other motions without any bifurcation to M_O.

Let any position of the completed structure **H** with 8 triangles and 12 sectors be given. We begin our discussion by defining the spherical component of the motion: Each face is represented by a perpendicular unit vector pointing to the outside where the joining sectors are situated. The endpoints of these vectors define a graph with 8 knots and 12 bars on the unit sphere Σ (Fig. 5). The knots are seen as turning points between the bars. All bars have the same spherical length $(\pi - \varphi)$ and they enclose six spherical quadrangles $Q_1, \ldots, Q_6$.[2]

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[2] The interior of these spherical quadrangles is unique by the definition that it contains the shorter arcs of the spherical diagonals.

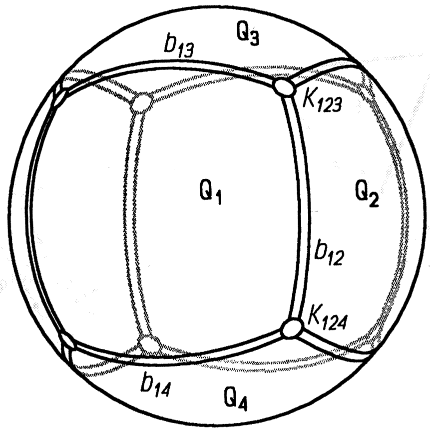

Figure 5. Spherical component of M_O.

There are 3 cases:

1. *These quadrangles cover the whole unit sphere.*
2. *None of these quadrangles degenerates, but they don't cover Σ.*
3. *At least two bars coincide.*

Case 1. If the quadrangles cover the whole unit sphere, then *the spherical framework is rigid. The eight triangles of* **H** *are always parallel to the faces of a regular octahedron.*

Proof. Spherical quadrangles of equal side-length have two axes of symmetry. It can easily be proved that the area A of such a quadrangle is maximal if and only if the diagonals have the same length. In all positions of our one-parametric motion M_O the 8 knots are the vertices of a cube. Then all six quadrangles are of maximal area and they just cover Σ. Therefore the area of each quadrangle is $A \leq \frac{2\pi}{3}$, and there is only one way to cover the whole sphere by six quadrangles of this type. ■

We will prove that the planes of the 8 faces form either a regular octahedron or a regular tetrahedron: Let us denote these planes by $\varepsilon_1, \ldots, \varepsilon_8$. We suppose that the positive faces are characterized by even indices and that the pairs $(\varepsilon_1, \varepsilon_8)$, $(\varepsilon_2, \varepsilon_7)$, $(\varepsilon_3, \varepsilon_6)$, $(\varepsilon_4, \varepsilon_5)$ determine opposite faces. In each plane ε_i the lines of intersection with the adjacent planes form a regular triangle that is circumscribed to the face of **H**. Let the vertices of the

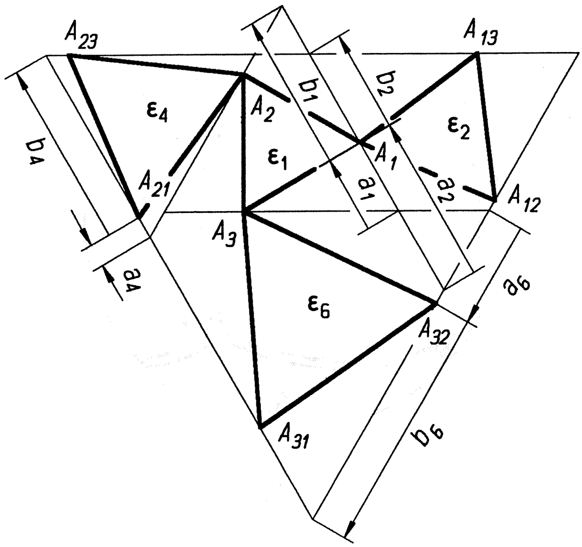

Figure 6. Faces in ε_1, ε_2, ε_4, ε_6

face in ε_i divide the sides of the trace triangle in segments of length a_i and b_i.

We suppose that the faces adjacent to ε_1 in the positive order are ε_2, ε_4, ε_6. From the parallelity to the planes of an octahedron we deduce that also the planes ε_3, ε_7, ε_5 meet ε_1 in a regular triangle. Fig. 6 shows ε_1 together with the planes ε_2, ε_4 and ε_6 after rotation about their trace into ε_1 (cf. Fig. 1). By comparing the segments on the sides of the trace triangle in ε_1 with those in ε_2, ε_4 and ε_6 we deduce the equations

$$b_2 - b_1 = a_4 - a_1, \qquad b_4 - b_1 = a_6 - a_1, \qquad b_6 - b_1 = a_2 - a_1.$$

Using the terms

$$A_{ij} := a_i - a_j, \qquad B_{ij} := b_i - b_j$$

we obtain

$$\begin{array}{llllll}
B_{21} = A_{41}, & B_{41} = A_{61}, & B_{61} = A_{21}, & B_{43} = A_{23}, & B_{23} = A_{83}, & B_{83} = A_{43} \\
B_{65} = A_{85}, & B_{85} = A_{25}, & B_{25} = A_{65}, & B_{47} = A_{87}, & B_{87} = A_{67}, & B_{67} = A_{47}, \\
B_{12} = A_{32}, & B_{32} = A_{52}, & B_{52} = A_{12}, & B_{14} = A_{74}, & B_{74} = A_{34}, & B_{34} = A_{14}, \\
B_{16} = A_{56}, & B_{56} = A_{76}, & B_{76} = A_{16}, & B_{58} = A_{38}, & B_{38} = A_{78}, & B_{78} = A_{58}.
\end{array}$$

We resolve these equations for A_{i1} and $B_{i1}, i \in \{3,5,7\}$:

$$\begin{aligned}
A_{41} &= A_{23} = A_{21} - A_{31} \Rightarrow A_{31} = A_{21} - A_{41},\\
A_{21} &= A_{65} = A_{61} - A_{51} \Rightarrow A_{51} = A_{61} - A_{21},\\
A_{61} &= A_{47} = A_{41} - A_{71} \Rightarrow A_{71} = A_{41} - A_{61},\\
A_{41} &= B_{43} = B_{41} - B_{31} \Rightarrow B_{31} = A_{61} - A_{41},\\
A_{21} &= B_{25} = B_{21} - B_{51} \Rightarrow B_{51} = A_{41} - A_{21},\\
A_{61} &= B_{67} = B_{61} - B_{71} \Rightarrow B_{71} = A_{21} - A_{61}.
\end{aligned}$$

By paying regard to these solutions we derive

$$\begin{aligned}
B_{32} = A_{52} \quad &\Rightarrow \quad A_{61} - A_{41} - A_{41} = A_{61} - A_{21} - A_{21} \quad \Rightarrow \quad A_{41} = A_{21},\\
B_{56} = A_{76} \quad &\Rightarrow \quad A_{41} - A_{21} - A_{21} = A_{41} - A_{61} - A_{61} \quad \Rightarrow \quad A_{21} = A_{61},
\end{aligned}$$

hence

$$\begin{aligned}
A := A_{21} = A_{41} = A_{61}, &\qquad A_{31} = A_{51} = A_{71} = 0,\\
B_{21} = B_{41} = B_{61} = A, &\qquad B_{31} = B_{51} = B_{71} = 0.
\end{aligned}$$

Additionally we find

$$B_{58} = A_{52} \quad \Rightarrow \quad B_{51} - B_{81} = A_{51} - A_{21} \quad \Rightarrow \quad B_{81} = A.$$

We summarize:

$$\begin{aligned}
a_1 = a_3 = a_5 = a_7, &\qquad b_1 = b_3 = b_5 = b_7,\\
a_2 = a_4 = a_6 = a_8, &\qquad b_2 = b_4 = b_6 = b_8,
\end{aligned}$$

$$a_2 - a_1 = b_2 - b_1 \quad \textit{or} \quad d := a_1 - b_1 = a_2 - b_2.$$

Finally we take into account that the lengths a_1, b_1 and a_2, b_2 of the segments on the trace triangles have to define inscribed triangles with unit length.[3] This implies

$$\begin{aligned}
a_1^2 - a_1 b_1 + b_1^2 = a_2^2 - a_2 b_2 + b_2^2 = 1;\\
a_1^2 - a_1 d = a_2^2 - a_2 d \quad \Longrightarrow \quad (a_1 - a_2)(a_1 + a_2 - d) = 0.
\end{aligned}$$

The first solution reads $a_1 = a_2$, $b_1 = b_2$. This means that the planes of the eight faces form a octahedron as stated.

[3] Note that this condition is not fulfilled in Fig. 6.

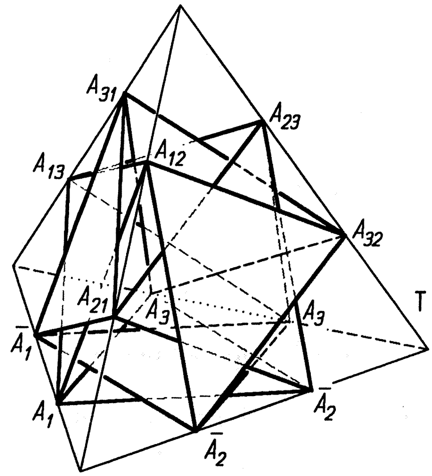

Figure 7. Tetrahedron **T** *and HEUREKA-polyhedron* **H**

The second solution

$$a_2 = -(a_1 - d) = -b_1, \quad b_2 = a_2 - d = -a_1$$

gives raise to a new motion M_T of the HEUREKA-polyhedron; at the Jitterbug this motion recently has been noticed by H. F. Verheyen [3]. In this case the pairs of opposite planes $(\varepsilon_1, \varepsilon_8)$, $(\varepsilon_2, \varepsilon_7)$, $(\varepsilon_3, \varepsilon_6)$, $(\varepsilon_4, \varepsilon_5)$ are coinciding and they form the planes of a regular tetrahedron **T** (Fig. 7) with dihedral angle $\pi - \varphi = 70,52..^\circ$. Therefore the special joints of the HEUREKA-polyhedron are consistent with this situation provided the oriented lines normal to ε_1, ε_3, ε_5, ε_7 are pointing to the outside of **T** and they are opposite to those of ε_2, ε_4, ε_6, ε_8 respectively.

Similar to the original explanation of the HEUREKA-motion M_O we can define the one-parametric motion M_T by inscribing a symmetric pair of regular triangles in each face of **T**. M_T converts a twofold covered tetrahedron with fourfold edges (ratio $r = 0$ on the edges of **T**) into a octahedron with a twofold covering of half of the faces and with twofold edges $(r = 1 : 1)$. If the vertices of **H** make three equal spaces on the edges of **T** $(r = 1 : 2)$ (position 4 in Fig. 8), then the vertices of **H** form a truncated tetrahedron; the edges of **H** are diagonals of the four hexagons.

Theorem. *The octahedral motion M_O and the tetrahedral motion M_T are constrained motions for the HEUREKA-polyhedron* **H**.

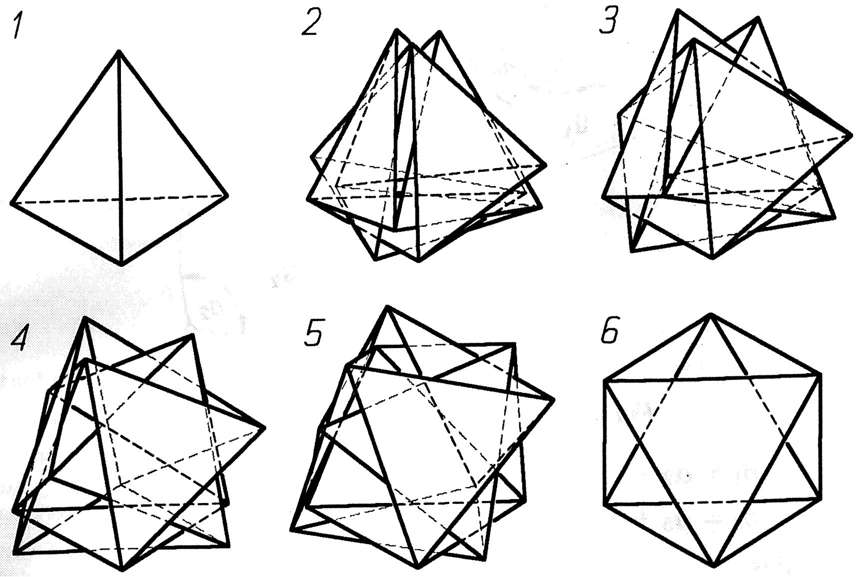

Figure 8. The one-parametric motion M_T

Case 2. None of these quadrangles degenerates but they don't cover Σ: *This case turns out to be impossible on a sphere.*

Proof. Let us assume that there is such a graph. There must be a bar b_{12} whose adjacent quadrangles Q_1, Q_2 have common interior points (Fig. 9). At the endpoints K_{123} and K_{124} of b_{12} the two remaining bars b_{13}, b_{23} and b_{14}, b_{24} can be completed to quadrangles Q_3, Q_4 due to their symmetry. Then also the quadrangles Q_6, Q_5 opposite to Q_1, Q_2 resp. are defined.

We may assume that the boundary of the union U of all six quadrangles consists of the bars b_{12}, b_{13}, b_{35}, b_{56}, b_{46}, b_{24}. If necessary we first have to change the indices of Q_3 and Q_4. The vertices K_{145} and K_{236} are interior points of U.

Let α_i, β_i denote the measures of the pairwise congruent angles in Q_i. Due to NEPER's rule the following equation holds:

$$\cot\frac{\alpha_i}{2}\cot\frac{\beta_i}{2} = \cos(\pi - \varphi) = \frac{1}{3}.$$

We assume that the β-angles meet at the interior points K_{145} and K_{236}. Then we obtain

$$\beta_1 + \beta_4 + \beta_5 = \beta_2 + \beta_3 + \beta_6 = 2\pi.$$

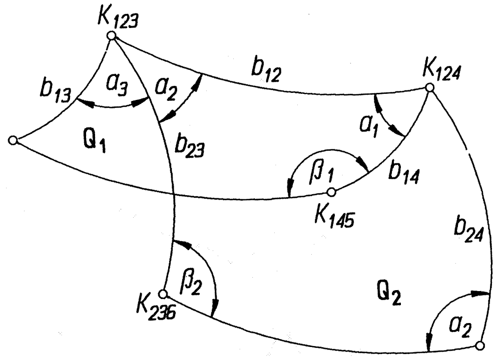

Figure 9. Spherical graph in case 2.

$$\beta_1 = \alpha_2 + \alpha_3, \qquad \beta_3 = \alpha_1 + \alpha_5, \qquad \beta_5 = \alpha_3 + \alpha_6,$$
$$\beta_6 = \alpha_5 + \alpha_4, \qquad \beta_4 = \alpha_6 + \alpha_2, \qquad \beta_2 = \alpha_4 + \alpha_1.$$

This implies

$$\alpha_2 + \alpha_3 + \alpha_6 = \alpha_1 + \alpha_4 + \alpha_5 = \pi,$$

hence

$$\sum_{i=1}^{6} \alpha_i = 2\pi.$$

On the other hand on a sphere we have

$$\alpha_1 + \beta_1 = \alpha_1 + \alpha_2 + \alpha_3 > \pi, \qquad \alpha_6 + \beta_6 = \alpha_4 + \alpha_5 + \alpha_6 > \pi,$$

and this leads to the contradiction $\sum_{i=1}^{6} \alpha_i > 2\pi$. ■

Case 3. At least two bars coincide:

Let us assume that the quadrangle Q_1 collapses into a dyad caused by the coincidence of the knots K_{145} and K_{123}. By completing $K_{145} = K_{123}, K_{124}$ and K_{246} to a quadrangle we get $Q_2 = Q_4$ and thus $K_{236} = K_{456}$; the quadrangle Q_6 collapses too. We further obtain $Q_3 = Q_5$. The graph can additionally degenerate, if one of the quadrangles Q_2 or Q_3 or both collapse. In any case the spherical graph allows a two-parametric motion, without any bifurcation to the graph of case 1. However until now no corresponding motion of the HEUREKA-polyhedron could be found. There are some reasons to conjecture that M_O and M_T are the only motions that can be achieved by **H**.

References

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[2] A. C. Edmondson, *A Fuller Explanation. The Synergetic Geometry of Buckminster Fuller.* Birkhäuser, Boston, Basel, Stuttgart 1987, p. 159–169.

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