Strophoids, a family of cubic curves with remarkable properties

Hellmuth Stachel

stachel@dmg.tuwien.ac.at  —  http://www.geometrie.tuwien.ac.at/stachel
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1. Definition of Strophoids

**Definition:** An irreducible cubic is called **circular** if it passes through the absolute circle-points. A circular cubic is called **strophoid** if it has a double point (= node) with orthogonal tangents. A strophoid without an axis of symmetry is called **oblique**, otherwise **right**.

\[ S: (x^2 + y^2)(ax + by) - xy = 0 \]

with \( a, b \in \mathbb{R}, (a, b) \neq (0, 0) \). In fact, \( S \) intersects the line at infinity at \((0 : 1 : \pm i)\) and \((0 : b : -a)\).
1. Definition of Strophoids

The line through $N$ with inclination angle $\varphi$ intersects $S$ in the point

$$X = \left( \frac{s \varphi \ c^{2} \varphi}{a \ c \varphi + b \ s \varphi}, \ \frac{s^{2} \varphi \ c \varphi}{a \ c \varphi + b \ s \varphi} \right).$$

This yields a parametrization of $S$. $\varphi = \pm 45^\circ$ gives the points $G, G'$. The tangents at the absolute circle-points intersect in the focus $F$. 
1. Definition of Strophoids

The polar equation of \( S \) is

\[
S: \quad r = \frac{1}{a \sin \varphi + b \cos \varphi}.
\]

The inversion in the circle \( K \) transforms \( S \) into the curve \( H \) with the polar equation

\[
H: \quad r = \frac{a}{\sin \varphi} + \frac{b}{\cos \varphi}.
\]

This is an equilateral hyperbola which satisfies

\[
H: \quad (x - b)(y - a) = ab.
\]
1. Definition of Strophoids

The curve $\mathcal{H}$ has two axes of symmetry $mF \leftrightarrow N$ which implies that the inverse curve $S$ is self-invers with respect to two circles through $N$ with centers $G, G'$. The curve $S$ is the envelope of circles centered on confocal parabolas $P_1$ and $P_2$. 
1. Definition of Strophoids

The product of the polarity and the inversion in $\mathcal{K}$ is the pedal transformation $t \mapsto T$ w.r.t. $N$.

Polar to $\mathcal{H}$ is the parabola $\mathcal{P}$.

**Theorem:** The strophoid $S$ is the pedal curve of the parabola $\mathcal{P}$ with respect to $N$.

The parabola’s directrix $m$ is parallel to the asymptote of $S$. $F$ is the midpoint between $N$ and the parabola’s focus $F_p$. 
1. Definition of Strophoids

Tangents \( t \) of the parabola \( \mathcal{P} \) intersect \( S \) beside the pedal point \( T \) in two real or conjugate complex points \( Q \) and \( Q' \).

**Definition:** \( Q \) and \( Q' \) are called associated points of \( S \).

\( Q \) and \( Q' \) are associated iff the lines \( QN \) and \( Q'N \) are harmonic w.r.t. the tangents at \( \overline{A} \).

For given \( t \) the points \( Q \) and \( Q' \) lie on a circle centered on \( g \).
2. Associated Points

Projective properties of cubics with a node:

There is a 1-1 correspondence between $S$ and lines through $N$, except $N$ corresponds to $t_1$ and $t_2$.

The involution $\alpha$ which fixes $t_1, t_2$ determines pairs $X, X'$ of associated points.

Involutions which exchange $t_1$ and $t_2$ determine involutions $\beta$ on $S$ with $N \mapsto N$ and several properties, e.g., there exists an ‘associated’ involution $\beta' = \alpha \circ \beta = \beta \circ \alpha$. 
2. Associated Points

• $\beta$ has a center $Z$ such that $X, X\beta, Z$ are collinear.

• The centers $Z$ of $\beta$ and $Z'$ of $\beta'$ are associated.

• The lines $Z'X, Z'X\beta$ correspond in an involution which fixes $Z'N$ and the line through the fixed points $Y, Y'$ of $\beta$.

• For associated points, the diagonal points $XY \cap X'Y'$ and $XY' \cap X'Y$ are again on $S$. The tangents at corresponding points intersect on $S$. 
2. Associated Points

On the equicevian cubic $S$, the following pairs of points are associated:

- $Q, Q'$,
- the absolute circle-points,
- The focal point $F$ and the point $F'$ at infinity,
- $G, G'$ on the line $g \perp NF$. 
2. Associated Points

**Theorem:**

- For each pair \((Q, Q')\) of associated points, the lines \(NQ, NQ'\) are symmetric w.r.t. the bisectors \(t_1, t_2\) of \(\triangle BNC\).
- The **midpoint** of associated points \(Q, Q'\) lies on the **median** \(m = NF'\).
- The **tangents** of \(S\) at associated points meet each other at the point \(T' \in S\) associated to the pedal point \(T\) on \(t = QQ'\).
- For each point \(P \in S\), the lines \(PQ\) and \(PQ'\) are symmetric w.r.t. \(PN\).
2. Associated Points

We recall:

**Theorem:** Given three aligned points $A, A'$ and $N$, the locus of points $X$ such that the line $XN$ bisects the angle between $XA$ and $XA'$, is the Apollonian circle.

The second angle bisector passes through the point $\tilde{N}$ harmonic to $N$ w.r.t. $A, A'$. 
2. Associated Points

Theorem: Given the non-collinear points $A, A'$ and $N$, the locus of points $X$ such that the line $XN$ bisects the angle between $XA$ and $XA'$, is a strophoid with node $N$ and associated points $A, A'$.

The respectively second angle bisectors are tangent to the parabola $\mathcal{P}$. 
Theorem: The strophoid $S$ is the locus of focal points $(Q, Q')$ of conics $\mathcal{N}$ which contact line $AN$ at $A$ and line $A'N$ at $A'$.

The axes of these conics are tangent to the parabola $\mathcal{P}$.
Let the tangents to $C$ at $T$ and $T'$ intersect at $Q$. Then
\[ \alpha = \angle TF_2Q = \angle QF_2\overline{T}. \]

On the other hand, the tangent $t$ at $T$ bisects the angle between $TF_1$ and $TF_2$. 
3. Strophoids as a Geometric Locus

The points of contact of tangents drawn from a fixed point $N$ to confocal conics as well as the foot points of normals through $N$ lie on a strophoid.
3. Strophoids as a Geometric Locus

The curve $\mathcal{V}$ of intersection between the sphere (radius $2r$) and the vertical right cylinder (radius $r$) is called Viviani’s window.

Central projections with center $C \in \mathcal{V}$ and a horizontal image planes map $\mathcal{V}$ onto a strophoid.
3. Strophoids as a Geometric Locus

\[ \psi \]

\[ \psi' \]

\[ \psi'' \]

\[ \psi''' \]

\[ N' \]

\[ N'' \]

\[ N''' \]

\[ X' \]

\[ X'' \]

\[ X''' \]

\[ \lambda \]

\[ \lambda' \]

\[ \lambda'' \]

\[ 45^\circ \]

\[ xy \]

\[ z \]

\[ y \]

\[ x \]

\[ \psi \] lies also on a **cone of revolution** and on a **torus**.

Points of \( \psi \) have **equal geographic longitude and latitude**.
3. Strophoids as a Geometric Locus

Two particular examples of flexible octahedra where two faces are omitted. Both have an axial symmetry (types 1 and 2)

Below: Nets of the two octahedra.
According to R. Bricard there are 3 types of **flexible octahedra** (four-sided double-pyramids). Those of type 3 admit two flat poses. In each such pose, the pairs \((A, A')\), \((B, B')\), and \((C, C')\) of **opposite vertices** are associated points of a strophoid \(S\).
3. Strophoids as a Geometric Locus

In plane kinematics, points with trajectories of stationary curvature is a strophoid $S$ as well as the locus $C$ of corresponding centers of curvature.
3. Strophoids as a Geometric Locus

The *elementary geometry of triangles* seems to be an endless story.

- Clark Kimberling’s *Encyclopedia of Triangle Centers* shows a list of 7,622 remarkable points (available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html)

- Bernard Gibert’s *Cubics in the Triangle Plane* shows a list of 721 related cubics (available at http://bernard.gibert.pagesperso-orange.fr/index.html)
3. Strophoids as a Geometric Locus

For any point \( P \neq A, B, C \) the segments \( AA_P, BB_P, \) and \( CC_P \), are called **cevians** of the point \( P \).

Giovanni Ceva, 1647-1734, Milan/Italy.
3. Strophoids as a Geometric Locus

The point $E$ is called **equicevian**, if its three cevians have the same lengths, i.e., $AA_E = BB_E = CC_E$.

An equicevian point is called **improper** if it lies on one side line of the triangle (like $P$), otherwise **proper** (like $E$).

There exist $\leq 6$ improper equicevian points. We focus in the sequel on proper ones.
3. Strophoids as a Geometric Locus

A point $P$ is called $A$-equicevian iff $\overline{BB_P} = \overline{CC_P}$.

**Theorem:** All $A$-equicevian points lie on the line $BC$ or on the $A$-equicevian cubic $S_A : H_A(X, Y) = 0$, where

$$H_A(X, Y) = (vX - uY)(X^2 + Y^2) + uv(X^2 - Y^2) - (u^2 - v^2 + 1)XY - (vX + uY) - uv.$$

Analogue $B$-equicevian points ($\overline{AB_P} = \overline{CC_P}$) and $C$-equicevian points ($\overline{AB_P} = \overline{BC_P}$).
3. Strophoids as a Geometric Locus

All equicevian cubics are strophoids. $E_1, E_2 \in S_A \cap S_B \cap S_C$ are proper equicevian points.
Theorem: For each triangle $ABC$, the remaining equicevian points are identical with the two real and two complex conjugate focal points of the Steiner circumellipse $S$.

The Steiner circumellipse $S$ of $ABC$ is the (unique) ellipse centered at the centroid $G$ and passing through its vertices.
**Theorem:** When $a$ and $b$, with $a \geq b$, denote the semiaxes of the Steiner circumellipse $S$ of $ABC$, the cevians of the real foci have the length $3a/2$. The length of the cevians through the imaginary foci is $3b/2$.

Thank you for your attention!

• S. Abu-Saymeh, M. Hajja: *Equicevian points on the medians of a triangle.* preprint.


