# Kokotsakis meshes and flexible quad meshes

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# Outline

1. Flexible Kokotsakis meshes
2. Kokotsakis' flexible tesselation
3. Rigidity of a quadrangular cylinder tiling

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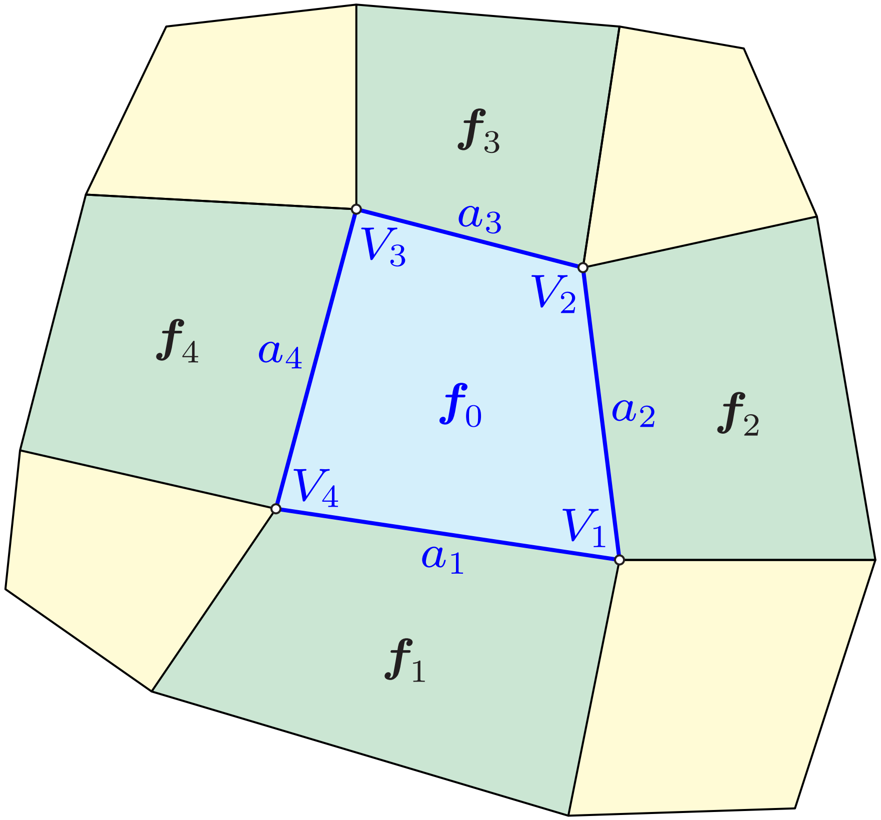
**Acknowledgement:** Georg Nawratil, TU Wien

# 1. Flexible Kokotsakis meshes



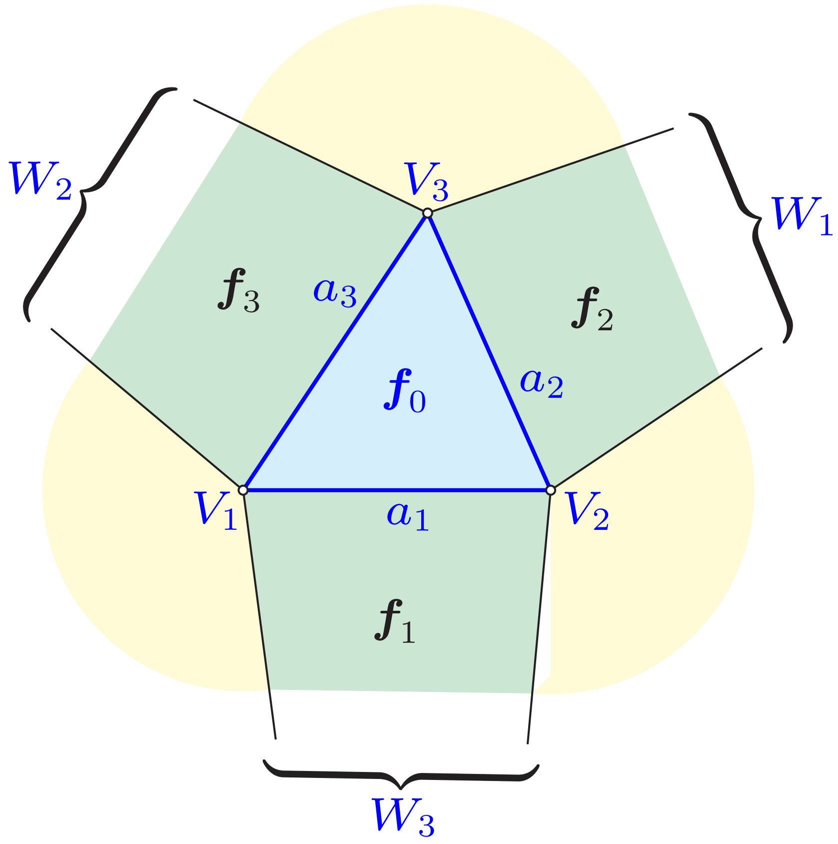

Special case $n = 4$

A **Kokotsakis mesh** is a polyhedral structure consisting of an $n$-sided central polygon $\boldsymbol{f}_0$ surrounded by a belt of polygons.

Each side $a_i$, $i = 1, \ldots, n$, of $\boldsymbol{f}_0$ is shared by a polygon $\boldsymbol{f}_i$. At each vertex $V_i$ of $\boldsymbol{f}_0$ four faces are meeting.

Each face is a rigid body; only the dihedral angles can vary ('rigid origami'). Can it be continuously flexible ?

# 1. Flexible Kokotsakis meshes



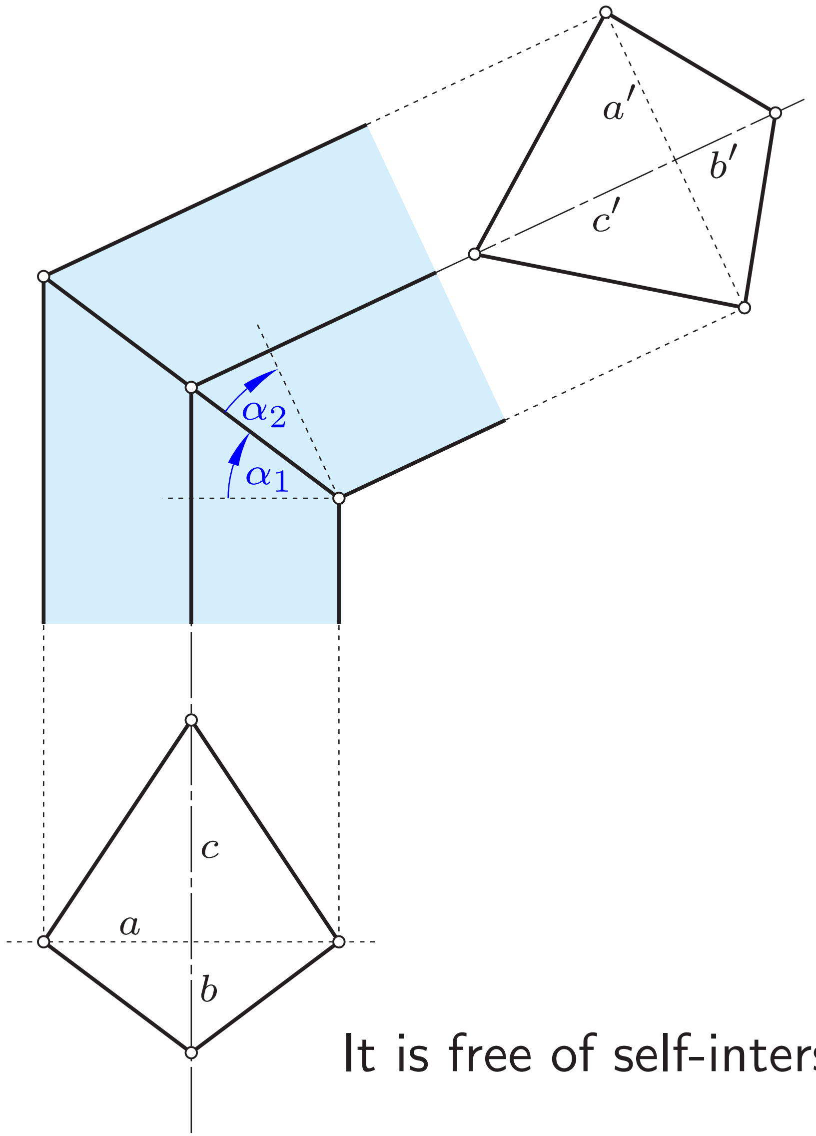

Special case: $n = 3$

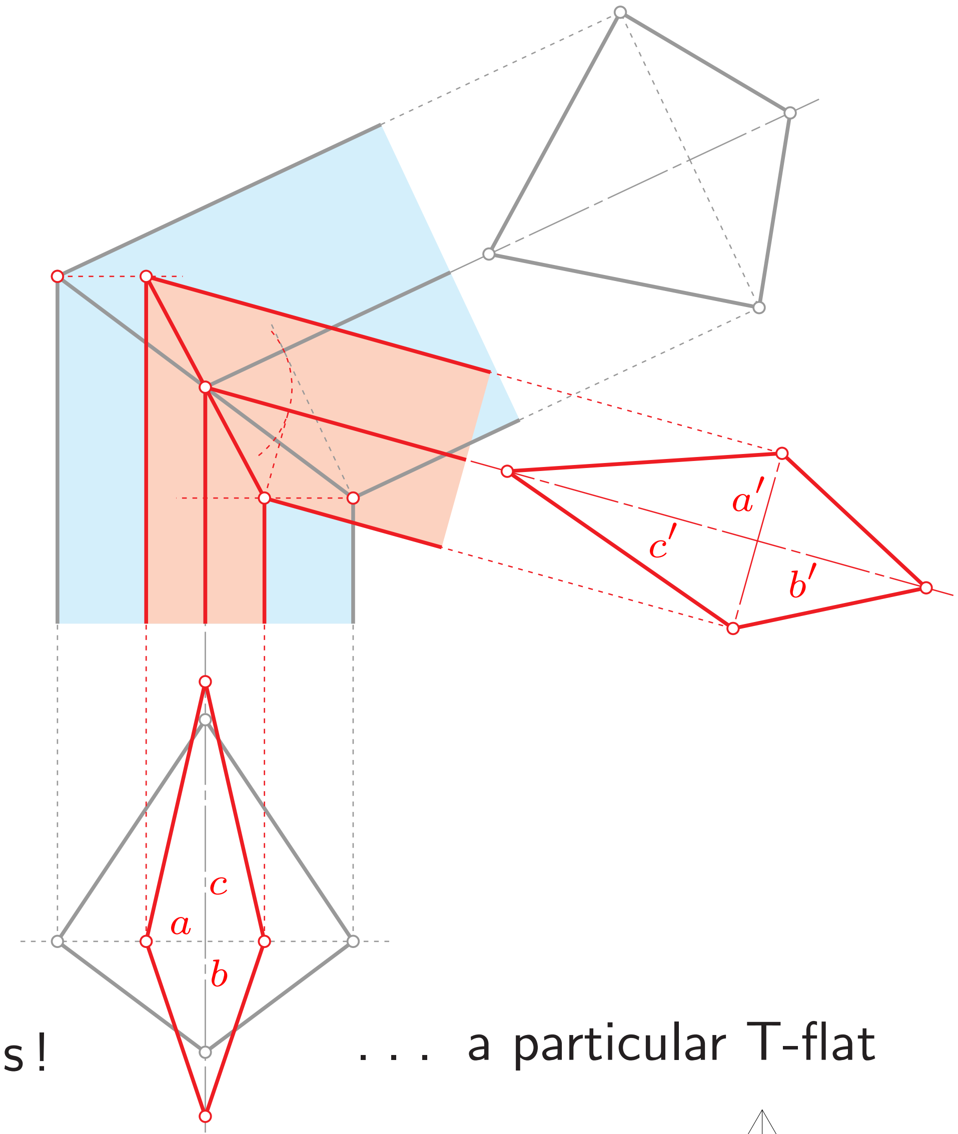
A Kokotsakis mesh for $n = 4$ is also called Neunflach [German] (nine-flat) (KOKOTSAKIS 1931, SAUER 1932)

For $n = 3$ the Kokotsakis mesh is equivalent to an octahedron with $V_1V_2V_3$ and $W_1W_2W_3$ as opposite triangular faces.

This offers an alternative approach to R. BRICARD's *flexible octahedra*.

G. Nawratil: There is exactly one flexible octahedron with vertices at infinity.



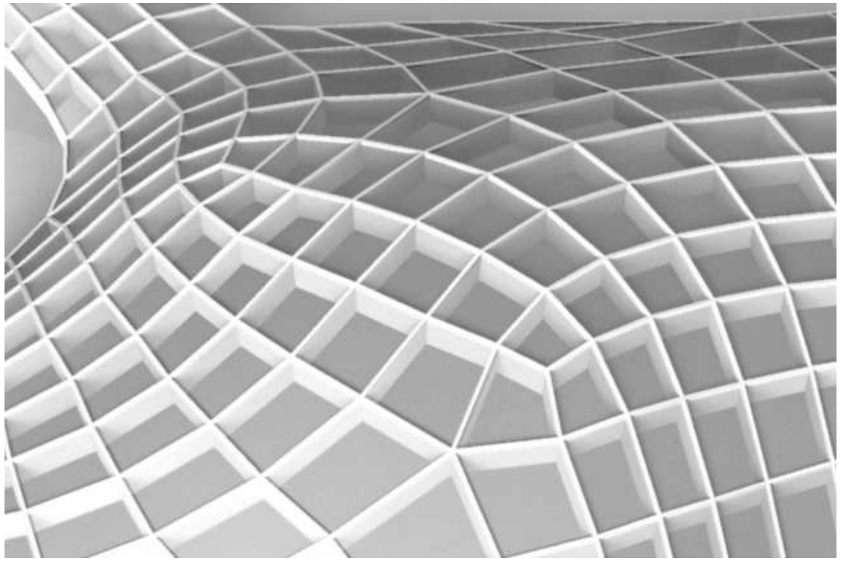

It is free of self-intersections !

. . . a particular T-flat

# 1. Flexible Kokotsakis meshes

In *discrete differential geometry* there is a interest on polyhedral structures composed of quadrilaterals (quadrilateral surfaces).

If all quadrilaterals are planar, they form a *discrete conjugate net* = **quad mesh**.



H. Pottmann, Y. Liu, J. Wallner, A. Bobenko, W. Wang: *Geometry of Multi-layer Freeform Structures for Architecture.* ACM Trans. Graphics **26** (3) (2007), SIGGRAPH 2007

# 1. Flexible Kokotsakis meshes

In ‘Freeform Architecture’ most of the surfaces are designed as *polyhedral surfaces*

— like the Capital Gate in Dubai (height = 160 m, inclination $18^\circ$)
Steel construction: Wagner Biro, Austria

# 1. Flexible Kokotsakis meshes

**Theorem:** [Bobenko, Hoffmann, Schief 2008]

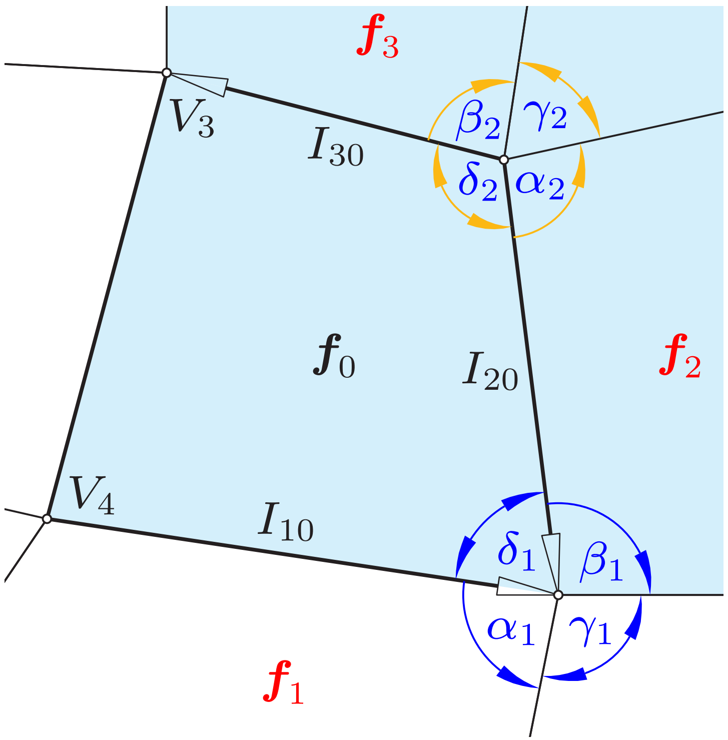
*A discrete conjugate net in general position is continuously flexible* $\iff$ *all its* $3 \times 3$ *complexes are continuously flexible.*

A historical model of a flexible Kokotsakis meshes:

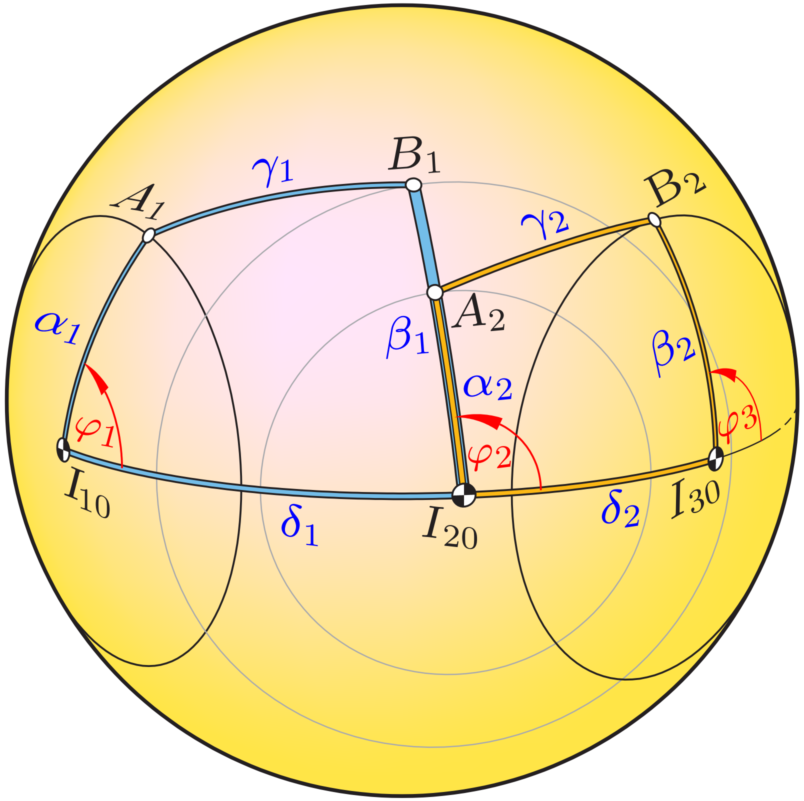
The classification of all flexible Kokotsakis meshes $(n = 4)$ has **recently** be finished:

Ivan Izmestiev: *Classification of flexible Kokotsakis polyhedra with quadrangular base,* preprint, 74 p., May **2013**

# 1. Flexible Kokotsakis meshes




Transmission from $\boldsymbol{f}_1$ to $\boldsymbol{f}_3$ via $V_1$ and $V_2$.

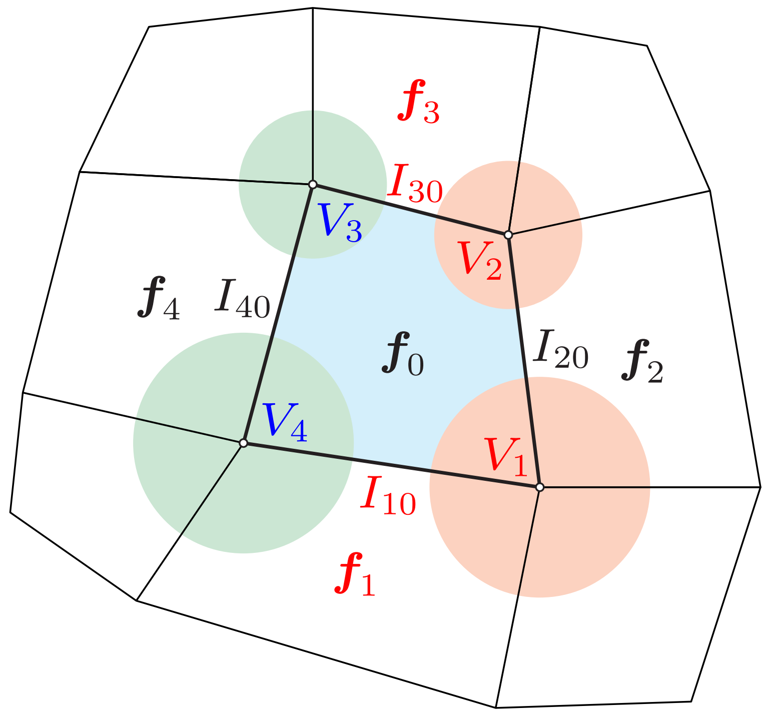



The composition of two spherical four-bars.

# 1. Flexible Kokotsakis meshes

Continuous flexibility of a Kokotsakis mesh for $n = 4$ means:

The transmission from $\boldsymbol{f}_1$ to $\boldsymbol{f}_3$ can be decomposed in two different ways, via $V_1$ and $V_2$ – or via $V_4$ and $V_3$.



# 1. Flexible Kokotsakis meshes

Conditions for one flexible case:

$\alpha_1 + \beta_2 = \delta_1 + \delta_2$

$\mathrm{s}\alpha_1\mathrm{s}\gamma_1 : \mathrm{s}\beta_2\mathrm{s}\gamma_2 = \mathrm{s}\beta_1\mathrm{s}\delta_1 : \mathrm{s}\alpha_2\mathrm{s}\delta_2 =$
$(\mathrm{c}\beta_1\mathrm{c}\delta_1 - \mathrm{c}\alpha_1\mathrm{c}\gamma_1) : (\mathrm{c}\beta_2\mathrm{c}\gamma_2 - \mathrm{c}\alpha_2\mathrm{c}\delta_2)$

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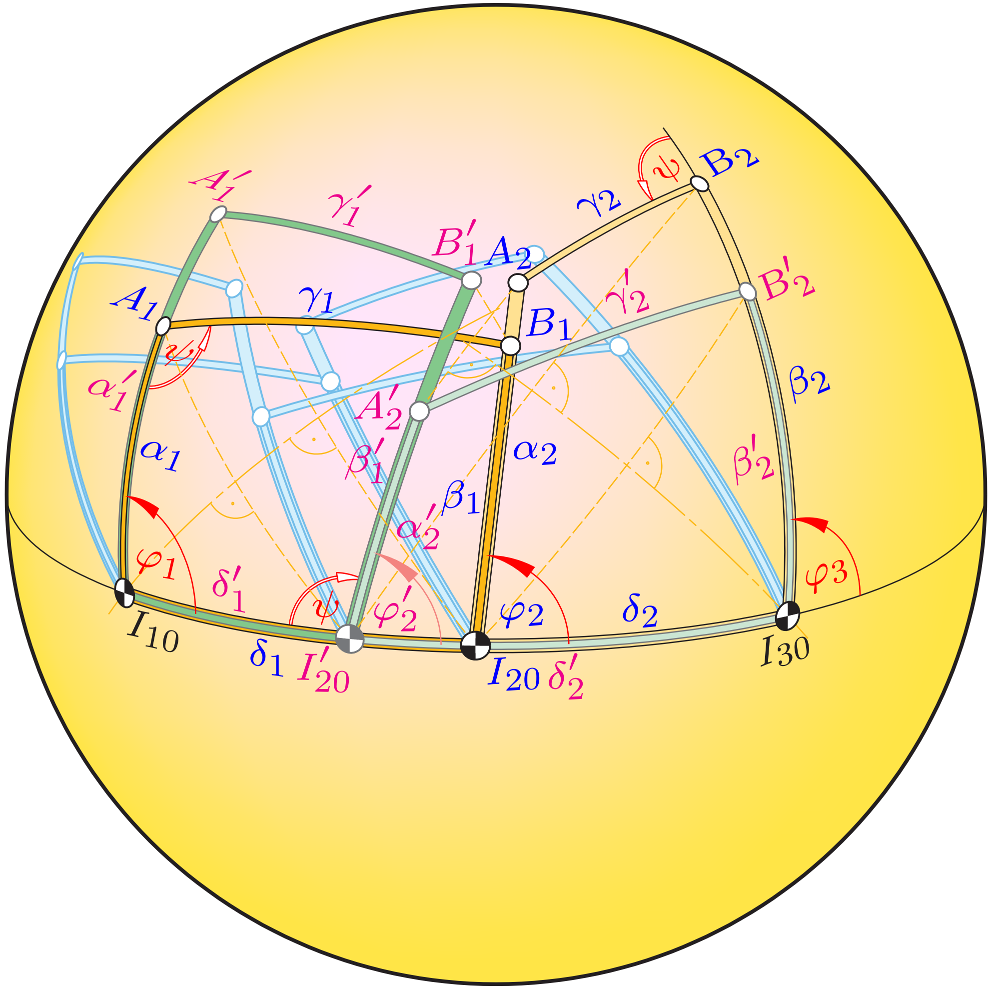
Right: The spherical image of this case with two decompositions.

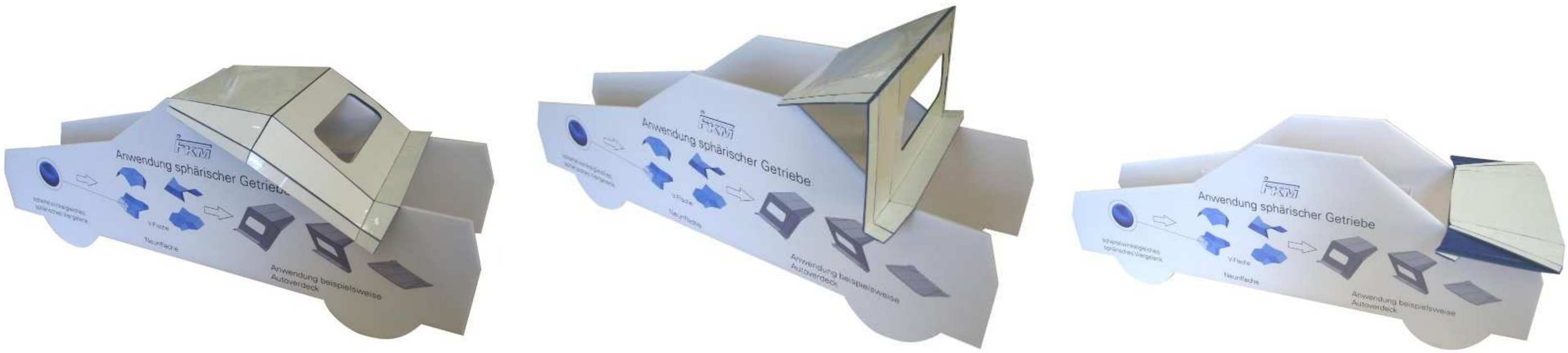
A second pose is shown in light-blue.



# 1. Flexible Kokotsakis meshes

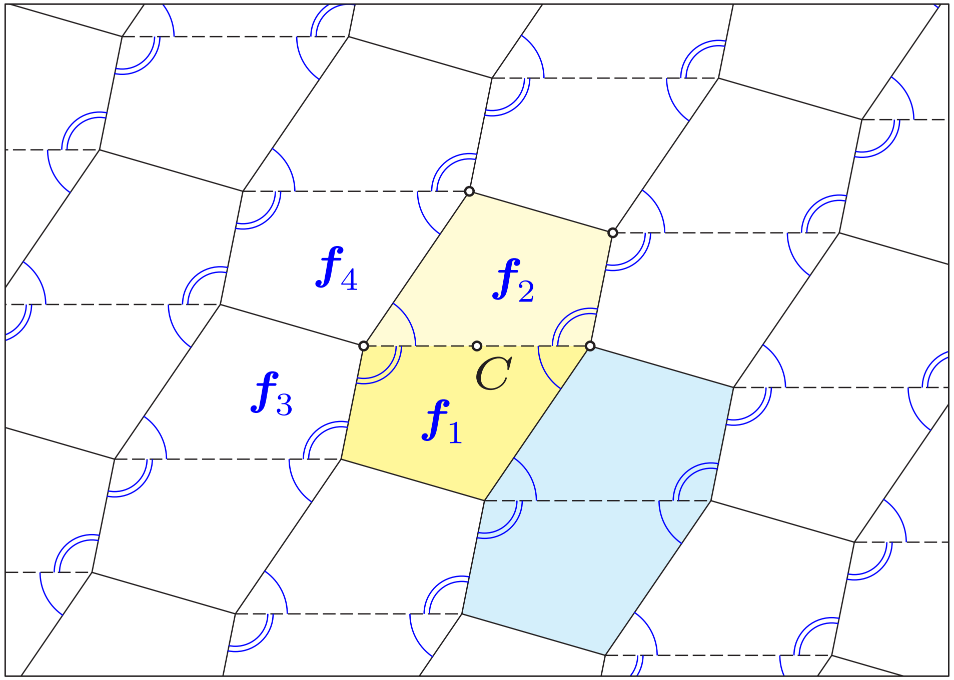
Only in very particular cases we know something about the geometry of the flexions obtained during the self-motions of continously flexible Kokotsakis meshes and quad meshes,

e.g., application of a discrete Voss surface in




Nadja Posselt: *Synthese von zwangläufig beweglichen 9-gliedrigen Vierecksflachen,* diploma thesis, TU Dresden 2010

# 2. Kokotsakis' flexible tesselation




A. Kokotsakis, 1932
Athens

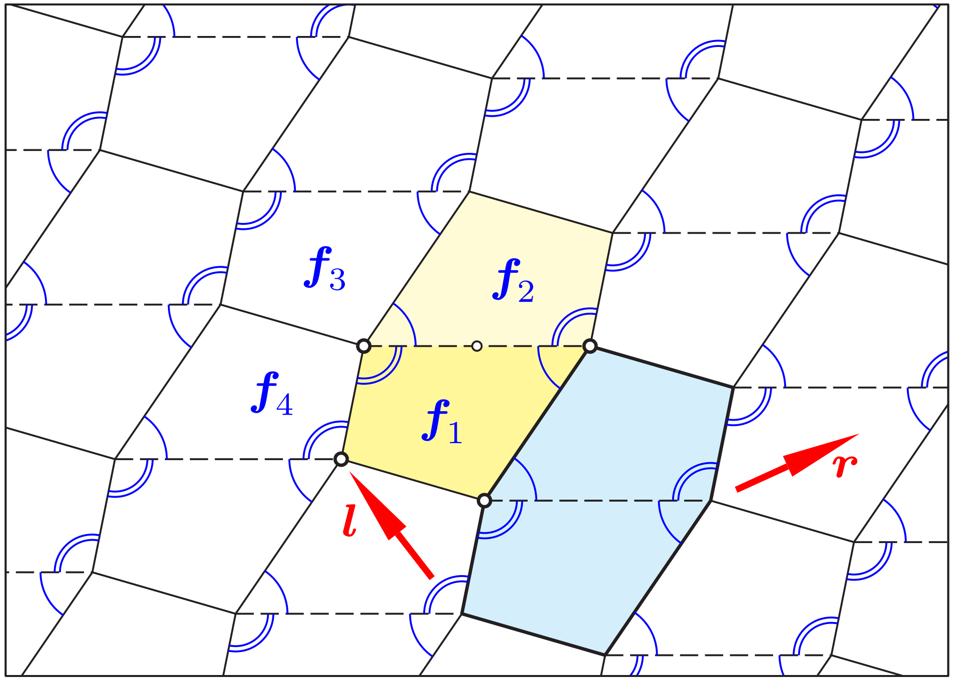
Any plane quadrangle is a tile for a regular tessellation of the plane (wallpaper group $\mathbf{p2}$, generically).

It is obtained by applying

- iterated $180°$-rotations about the midpoints of the sides of an initial quadrangle or
- by applying iterated translations on a centrally symmetric hexagon.

For a convex $f_1$ this polyhedral structure is continuously flexible.

# 2. Kokotsakis' flexible tesselation




A. Kokotsakis, 1932
Athens

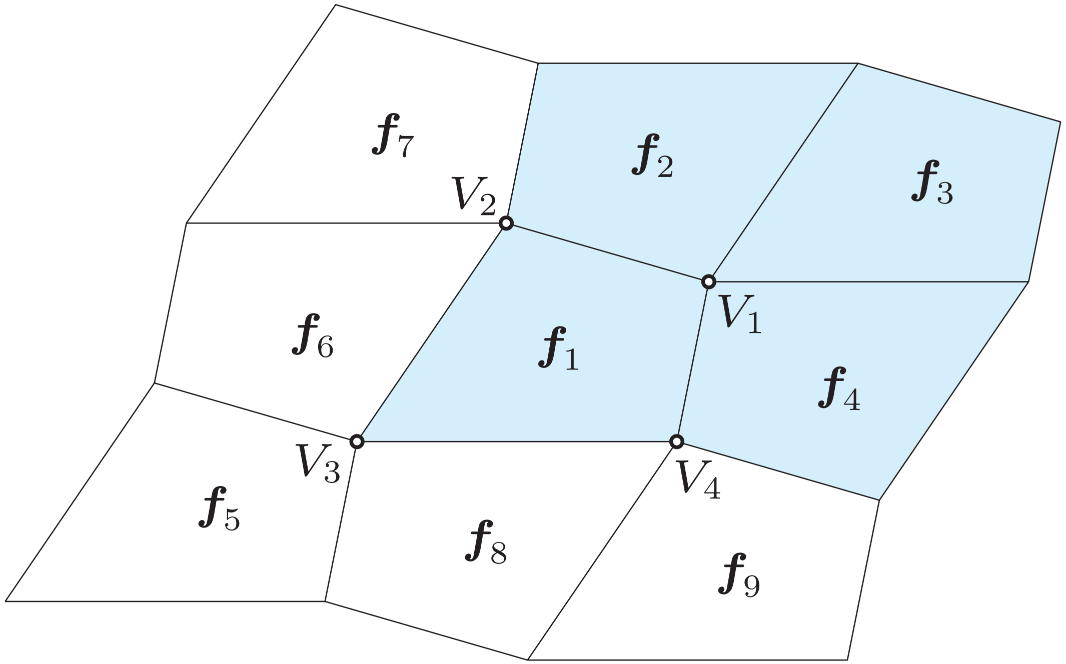
Any plane quadrangle is a tile for a regular tessellation of the plane.

It is obtained by applying

- iterated $180°$-rotations about the midpoints $C$ of the sides of an initial quadrangle or
- by applying iterated translations $l, r$ on a centrally symmetric hexagon.

For a convex $f_1$ this polyhedral structure is continuously flexible.

# 2. Kokotsakis' flexible tesselation




We pick out a $3 \times 3$ complex from this tessellation. Why it is flexible ?

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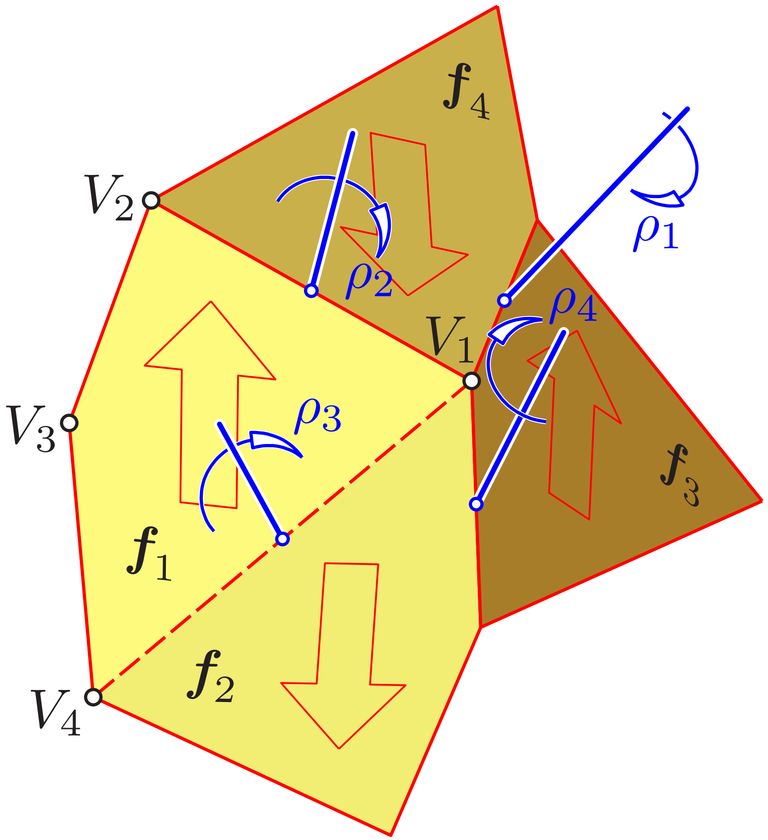
We first focus on the four-sided pyramide with vertex $V_1$.

Due to the required convexity no interior angle is $> \pi$ . Therefore this pyramid is continuously flexible.

# 2. Kokotsakis' flexible tesselation




In each pose for any two neighbouring faces there is a 180°-rotation $\rho_i$ which interchanges these two faces.

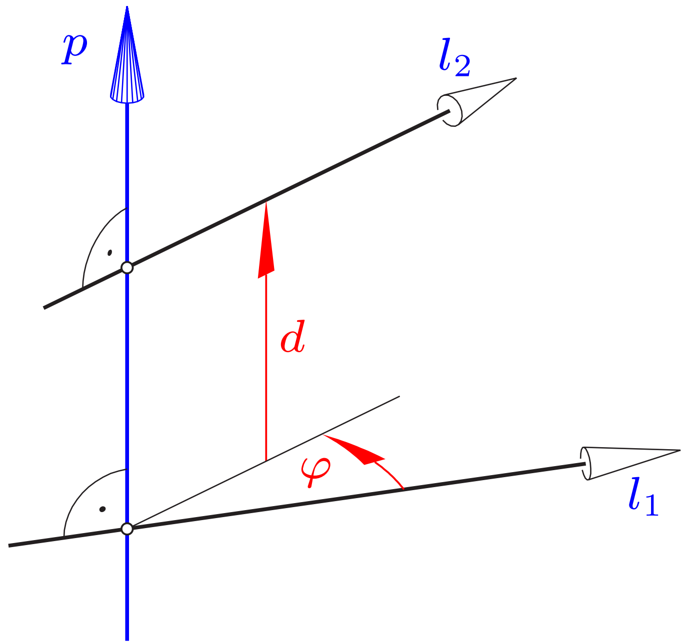
The axis is located in a bisector plane and passes through the midpoint of the common edge.

# 2. Kokotsakis' flexible tesselation

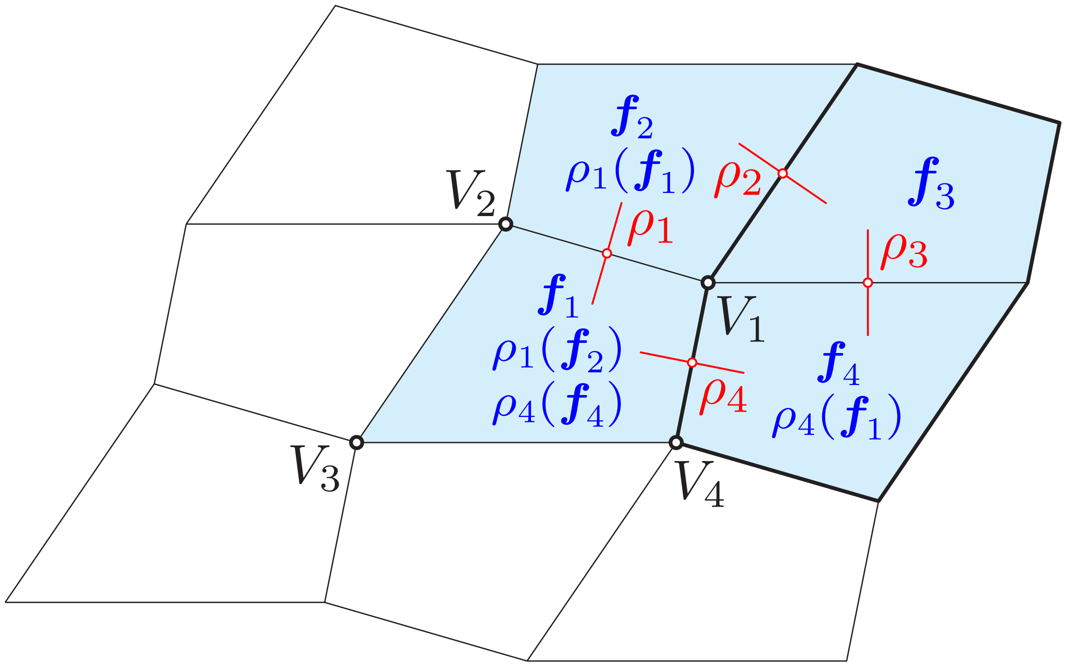
The product of any two $180^\circ$-rotations $\rho_1, \rho_2$ about respective axes $l_1$ and $l_2$ is a helical motion $\rho_2 \circ \rho_1$ about the common perpendicular $p$:

angle of rotation: $2\varphi$

length of translation: $2d$



# 2. Kokotsakis' flexible tesselation

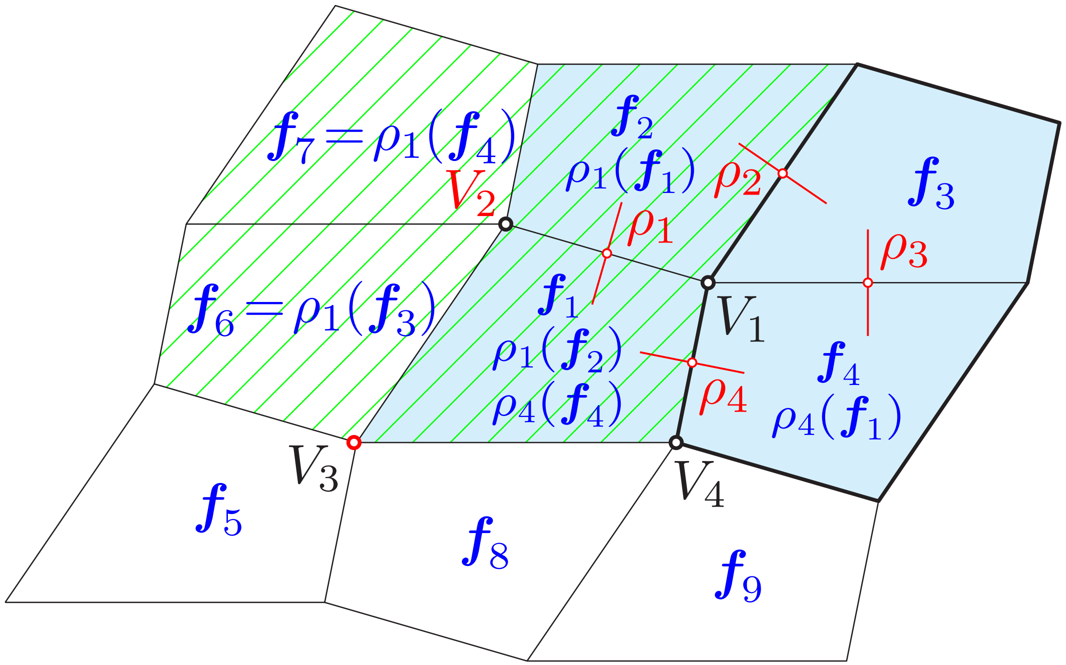



After applying all four $180°$-rotations consecutively to the quadrangle $f_1$, this is mapped via $f_2$, $f_3$, $f_4$ onto itself, hence

$$\rho_3 \circ \rho_4 = \rho_2 \circ \rho_1 \,.$$

$\Longrightarrow$ the four axes have a common perpendicular $p$, and all vertices $V_1, \ldots, V_4$ are at the same distance from $p$.

# 2. Kokotsakis' flexible tesselation

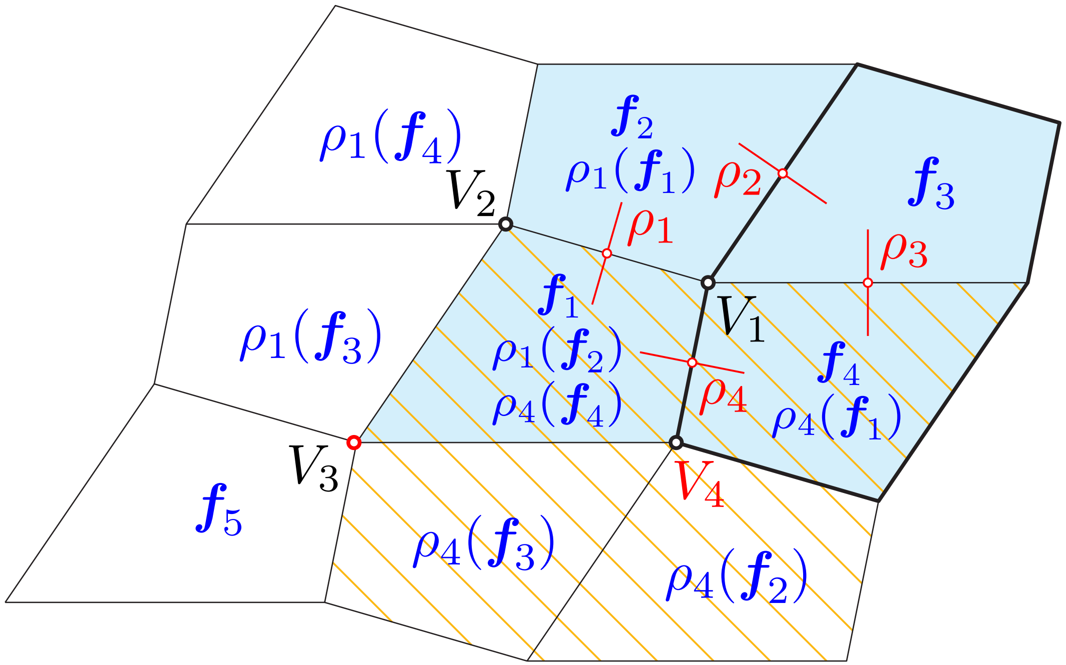



$\rho_1$ maps the pyramide with vertex $V_1$ onto the pyramide with vertex $V_2$.

(There would also be a second possibility to continue the flection of the $2 \times 2$ mesh to $f_6$ and $f_7$.)

There are two possibilities to continue the flection onto the fourth pyramide with vertex $V_3$ .

# 2. Kokotsakis' flexible tesselation

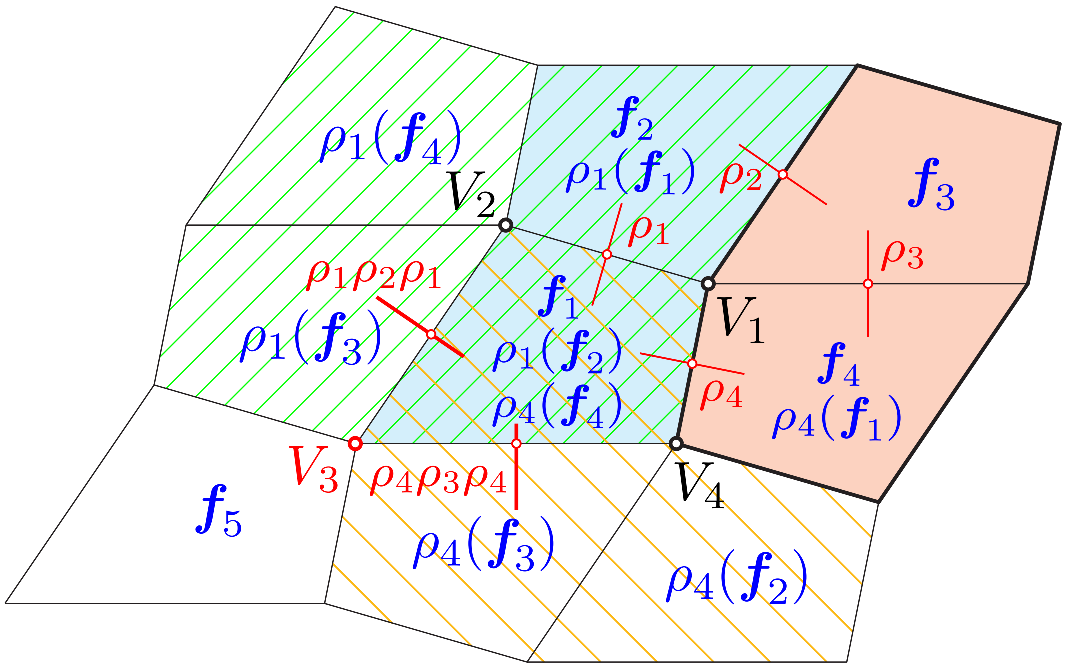



$\rho_1$ maps the pyramide with vertex $V_1$ onto the pyramide with vertex $V_2$.

$\rho_4$ maps the pyramide with vertex $V_1$ onto the pyramide with vertex $V_4$.

There are two possibilities to continue the flection onto the fourth pyramide with vertex $V_3$ .

# 2. Kokotsakis' flexible tesselation



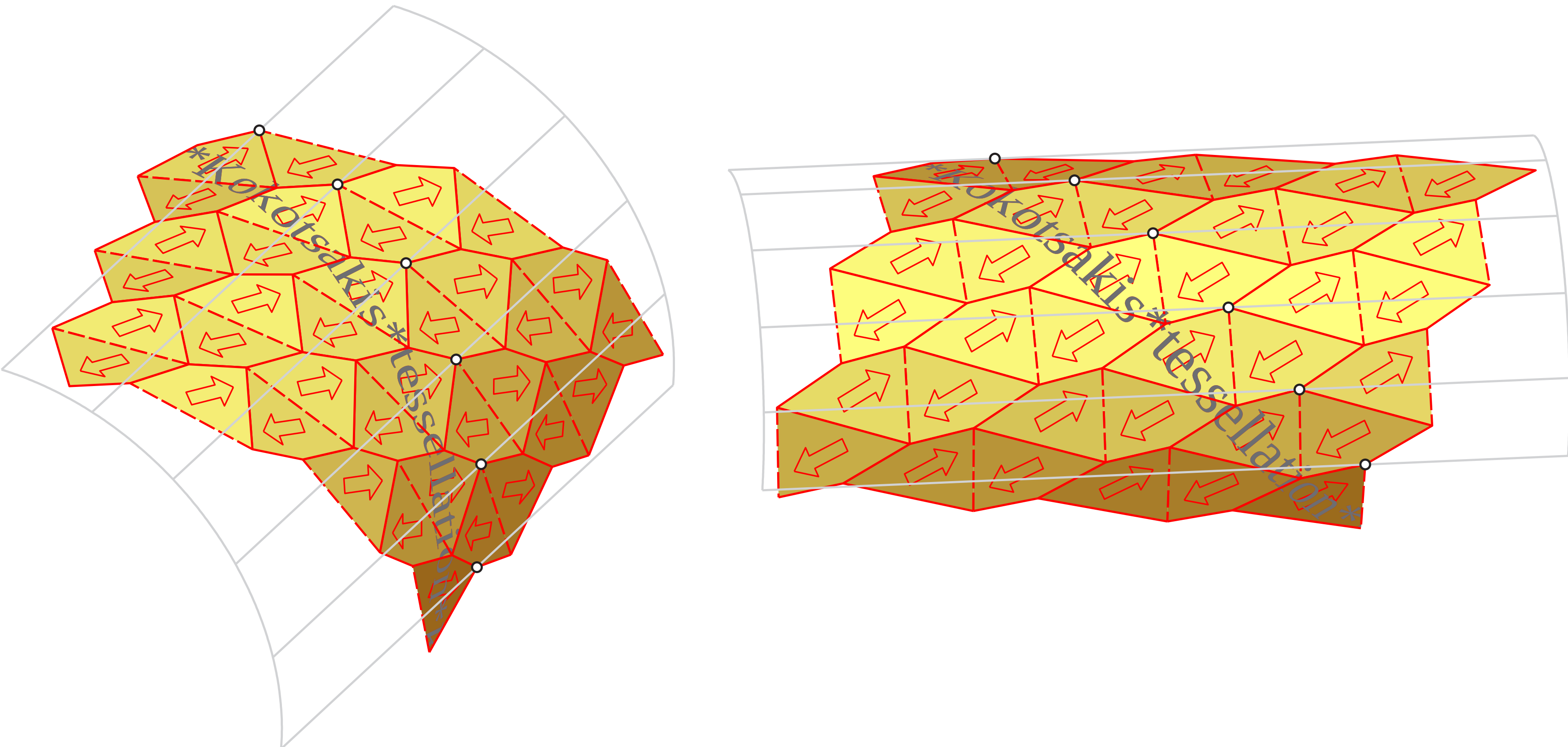

$\rho_1 \circ \rho_2 \circ \rho_1$ maps the pyramide with vertex $V_2$ onto the pyramide with vertex $V_3$.

$\rho_4 \circ \rho_3 \circ \rho_4$ maps the pyramide with vertex $V_4$ onto the pyramide with vertex $V_3$.

$f_5$ is the image of $f_2 = \rho_1(f_1)$ under $\rho_1 \circ \rho_2 \circ \rho_1$, and image of $f_4 = \rho_4(f_1)$ under $\rho_4 \circ \rho_3 \circ \rho_4$, as $f_5 = \rho_1 \circ \rho_2(f_1) = \rho_4 \circ \rho_3(f_1)$.

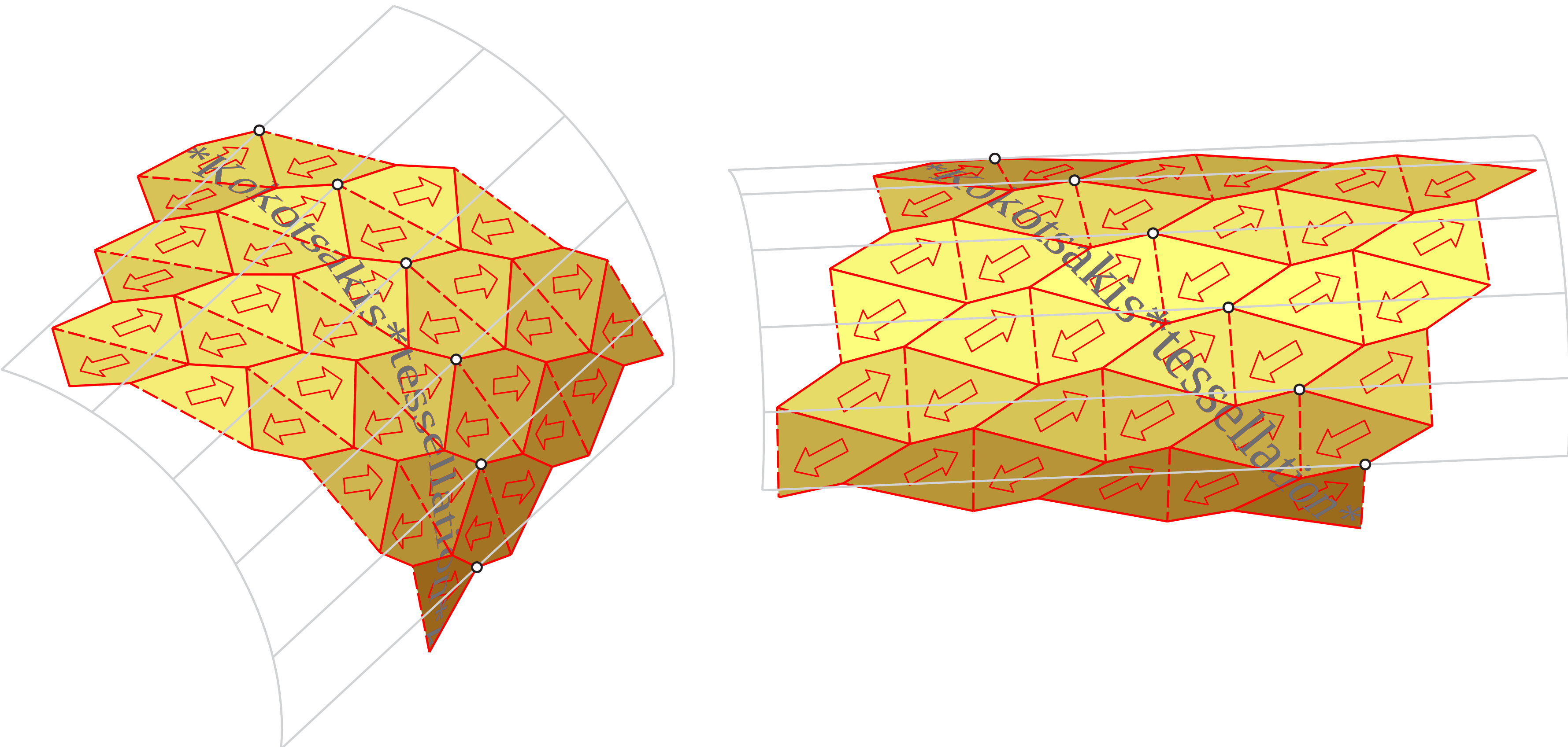
The complete pose arises from the "hexagon" $f_3 \cup f_4$ under iterations of the helical motions $r = \rho_2 \circ \rho_1$ and $l = \rho_4 \circ \rho_1$. This can be continued to a $m \times n$ mesh.

# 2. Kokotsakis' flexible tesselation



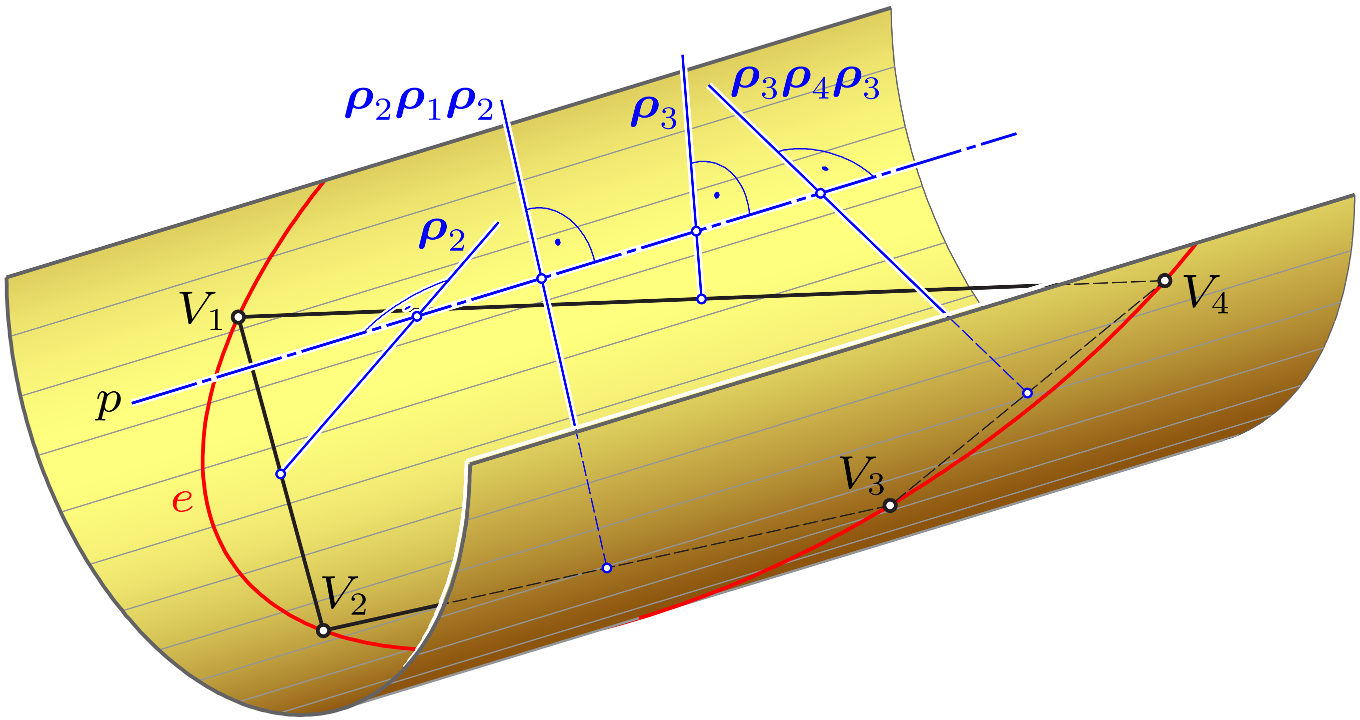
At each flexion obtainable by a continuous self-motion of an $m \times n$ tesselation mesh all vertices are located on a right circular cylinder (discrete conjugate quadrangular net on this cylinder)

# 2. Kokotsakis' flexible tesselation



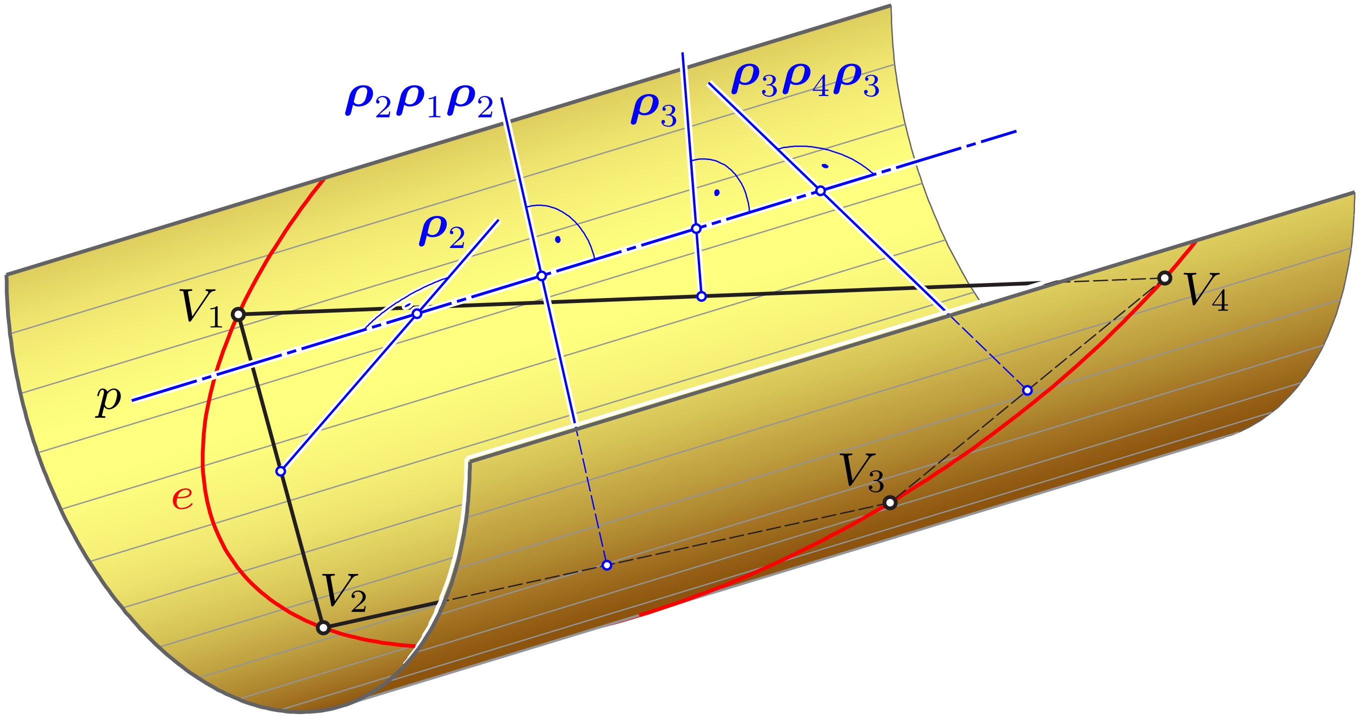

Each flexion of an $\infty \times \infty$ tesselation mesh is periodic. Generically, its group of isomorphisms is generated by coxial helical motions $\boldsymbol{l}, \boldsymbol{r}$ and by $\rho_4$. It is isomorphic to that of the flat case $(\mathbf{p2})$.

# 2. Kokotsakis' flexible tesselation



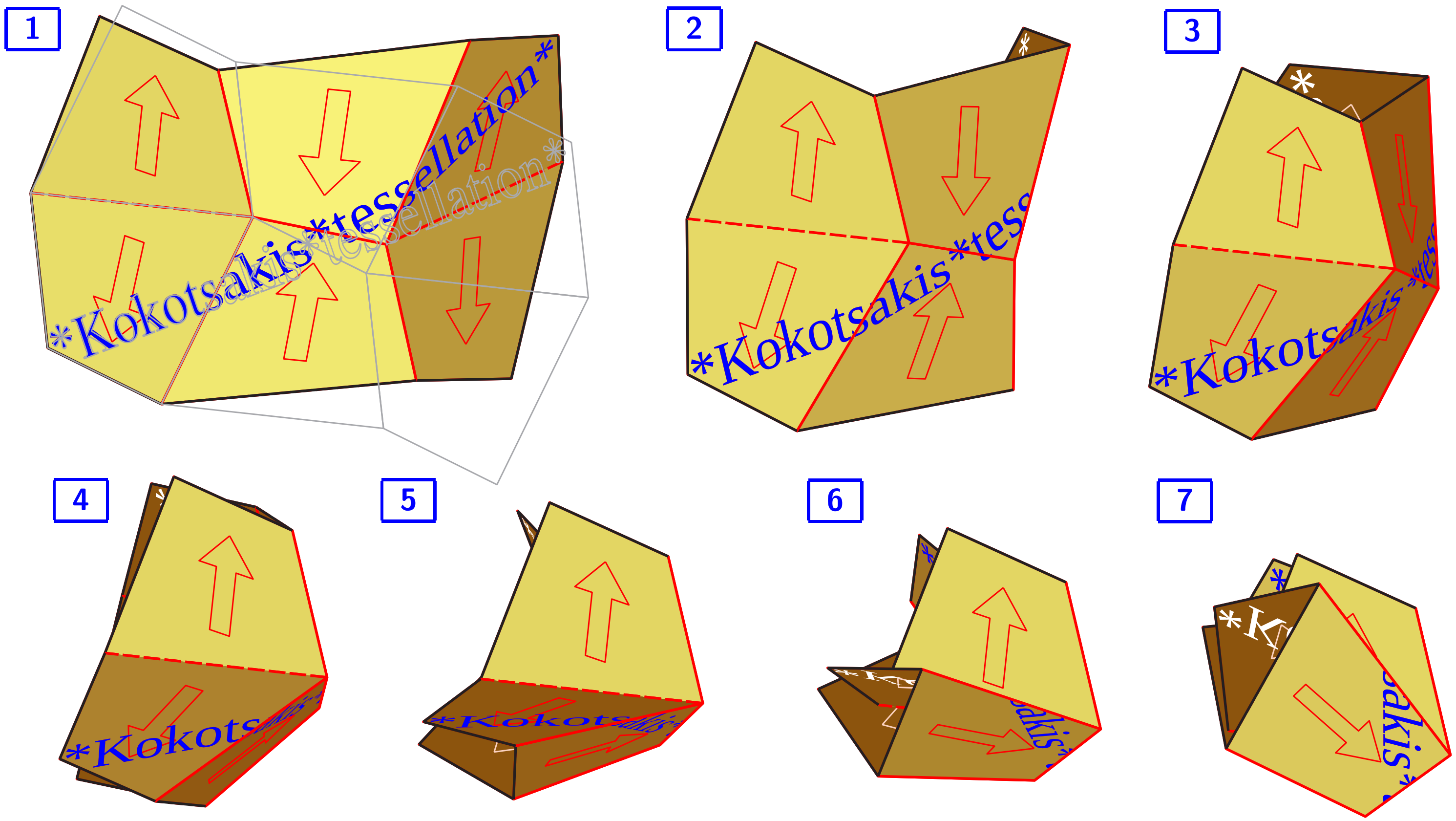

Another method to obtain the flexions: The plane spanned by $\boldsymbol{f}_{11}$ intersects the circumcylinder along an ellipse $e$.

# 2. Kokotsakis' flexible tesselation



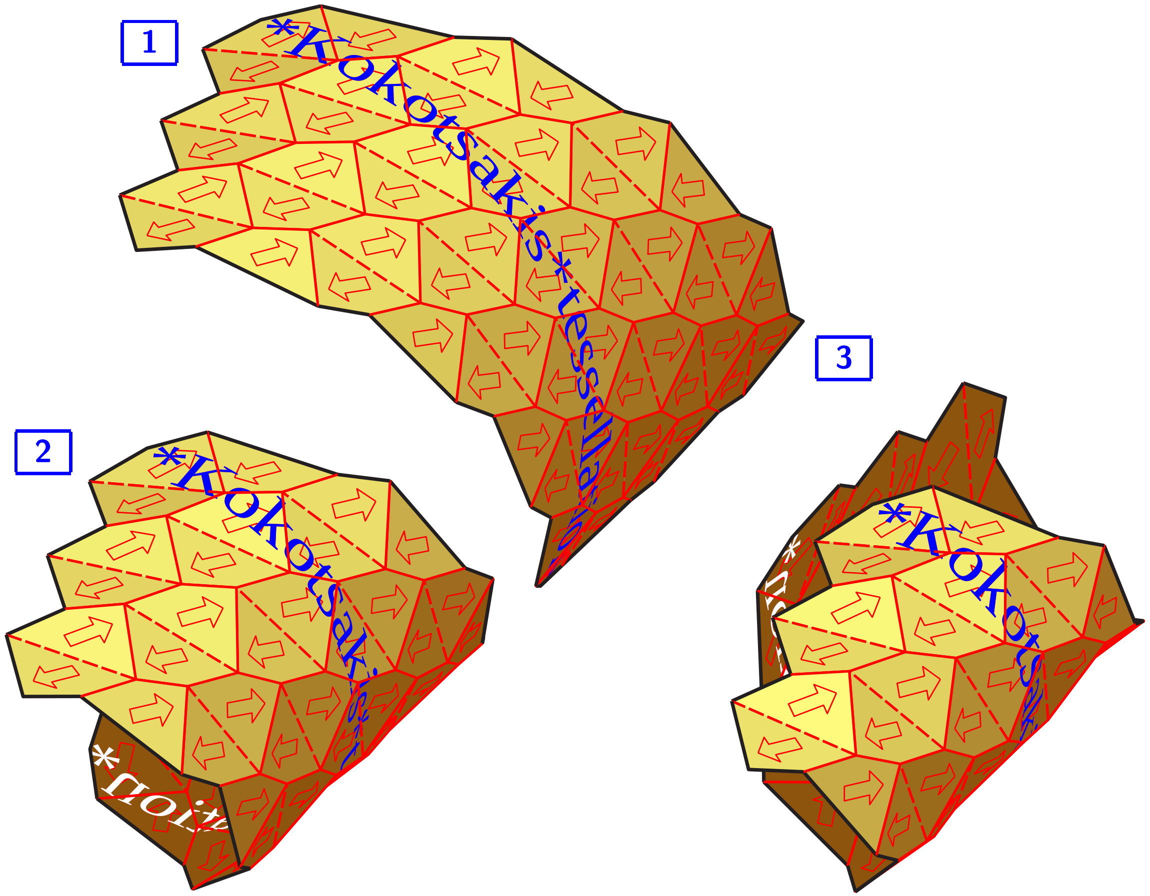

Conversely, there is a one-parameter set of ellipses passing through a given convex quadrangle. For each ellipse there are two cylinders of revolution passing through. Each such cylinder defines a flexion of a tessellation mesh (TM).

# 2. Kokotsakis' flexible tesselation




For quadrangles with circumcircle the mesh (= Voss surface) admits a second flat pose with coinciding circumcircles coincide (not free of self-intersections !)

# 3. Flexion which tiles the cylinder




Left: Flexions of a $9 \times 6$ TM

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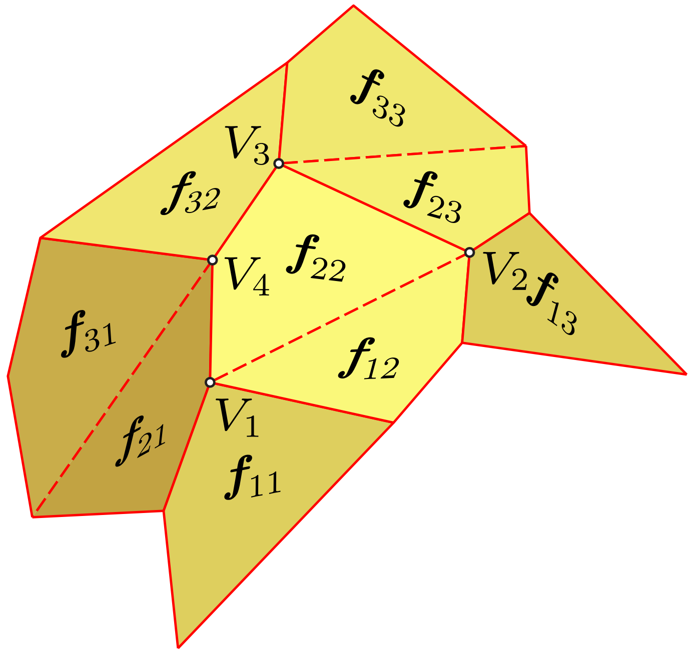
Under which conditions the flexion is *horizontally closing*, i.e., the right border zig-zag fits exactly to the left border — apart from a vertical shift ?

(the trapezoidal case with aligned borders is excluded)

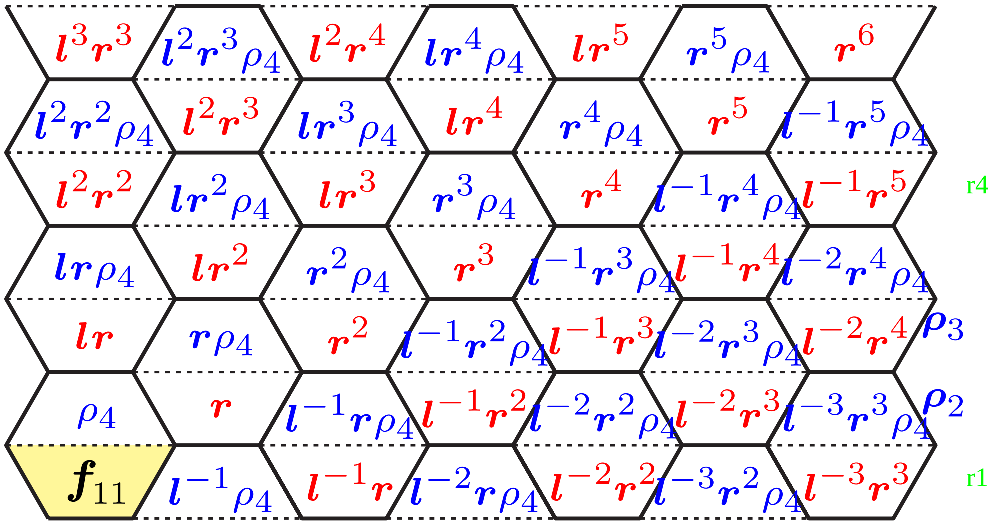
# 3. Flexion which tiles the cylinder

A $m \times n$ **tessellation mesh** is a grid of $m \times n$ quadrangles denoted by $\boldsymbol{f}_{ij}$, $1 \le i \le m$, $1 \le j \le n$

$$\begin{array}{ccccc} \boldsymbol{f}_{m1} & \boldsymbol{f}_{m2} & \boldsymbol{f}_{m3} & \cdots & \boldsymbol{f}_{mn} \\ \vdots & \vdots & \vdots & & \vdots \\ \boldsymbol{f}_{21} & \boldsymbol{f}_{22} & \boldsymbol{f}_{23} & \cdots & \boldsymbol{f}_{2n} \\ \boldsymbol{f}_{11} & \boldsymbol{f}_{12} & \boldsymbol{f}_{13} & \cdots & \boldsymbol{f}_{1n} \end{array}$$



# 3. Flexion which tiles the cylinder




This scheme of a $7 \times 7$ TM shows which product of $\boldsymbol{l}$, $\boldsymbol{r}$ and $\rho_4$ must be applied to $\boldsymbol{f}_{11}$ to obtain $\boldsymbol{f}_{ij}$.

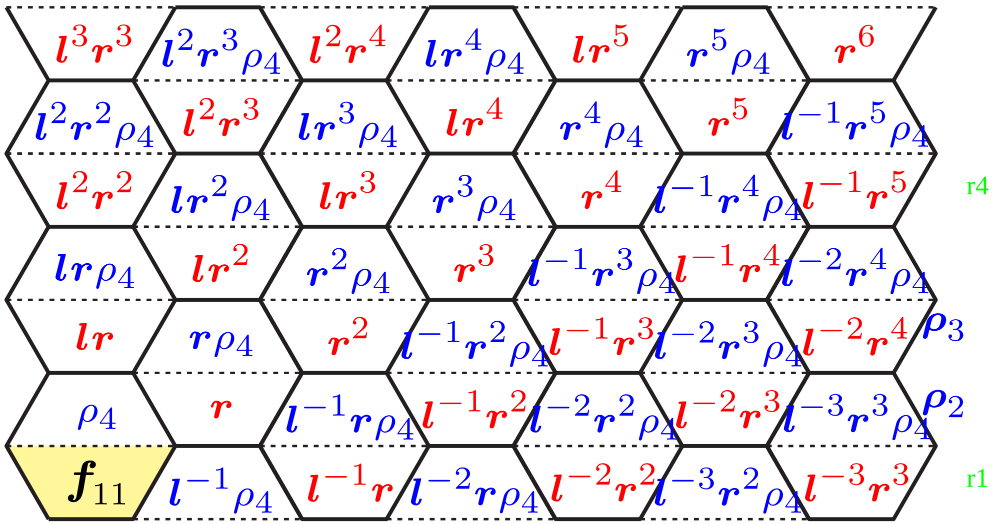
Each flexion of an $\infty \times \infty$ TM is periodic; the group of isomorphism acts transitively on the quadrangles.

**Lemma:** $\boldsymbol{f}_{11}$ can be transformed into $\boldsymbol{f}_{ij}$ according to

$$\boldsymbol{f}_{ij} = \begin{cases} \boldsymbol{l}^{\frac{i-j}{2}} \boldsymbol{r}^{\frac{i+j}{2}-1}(\boldsymbol{f}_{11}) \\ \text{for } i+j \equiv 0 \pmod 2, \\ \boldsymbol{l}^{\frac{i-j-1}{2}} \boldsymbol{r}^{\frac{i+j-3}{2}} \rho_4(\boldsymbol{f}_{11}) \\ \text{for } i+j \equiv 1 \pmod 2. \end{cases}$$

($\boldsymbol{r} = \rho_2 \circ \rho_1$ and $\boldsymbol{l} = \rho_4 \circ \rho_1$)

# 3. Flexion which tiles the cylinder




For *odd* $m$ $f_{2\,n+1} = r(f_{1n})$ is identical with $f_{k+2\,1}$ of the most-left row, $k \equiv 1 \pmod 2$, iff

$$l^{\frac{1-n}{2}} r^{\frac{n+1}{2}} = l^{\frac{k+1}{2}} r^{\frac{k+1}{2}}\, d(2\pi)$$

Since the involved helical motions commute pairwise, we obtain

$$l^{-\frac{n+k}{2}} \circ r^{\frac{n-k}{2}} = d(2\pi).$$

**Theorem:**
A flexion of an $m \times n$ TM closes horizontally with vertical shift $k$

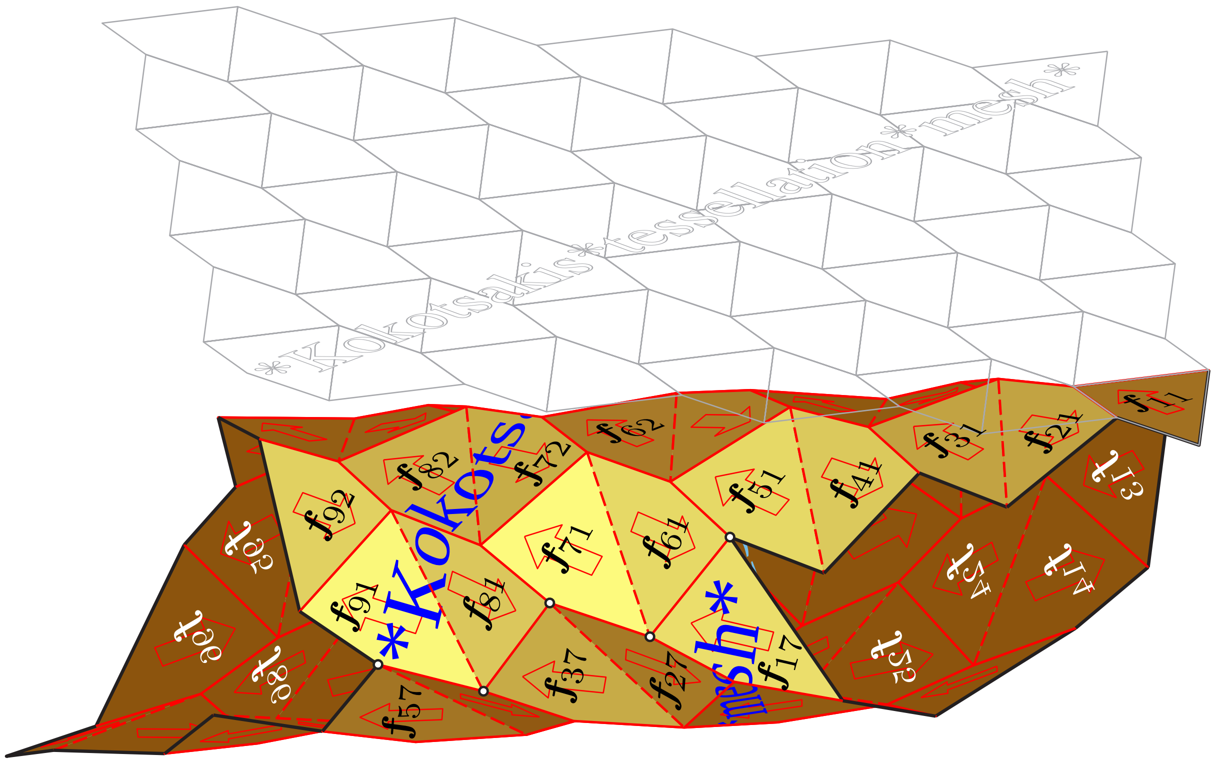
$\Longleftrightarrow$

$$\exists\ a, b \in \mathbb{Z}:\ l^a r^b = d(2\pi).$$

$n = -a + b$, shift $k = -a - b$.

# 3. Flexion which tiles the cylinder

Example: Horizontally closing $7 \times 9$ TM satisfying $\boldsymbol{l}^{-6}\boldsymbol{r} = d(2\pi)$:




How to obtain a closing version ?

- Either numerically by minimizing a certain distance.
- Or by starting with two coaxial helical motions $\boldsymbol{r}, \boldsymbol{l}$ obeying $\boldsymbol{l}^a\boldsymbol{r}^b = d(2\pi)$ and $\rho_4$.

# 3. Flexion which tiles the cylinder

After specifying $\rho_1, \rho_2, \rho_4$ and any first vertex $V_1$ a quadrangle $V_1 \ldots V_4$ is defined. However, this will not be planar.

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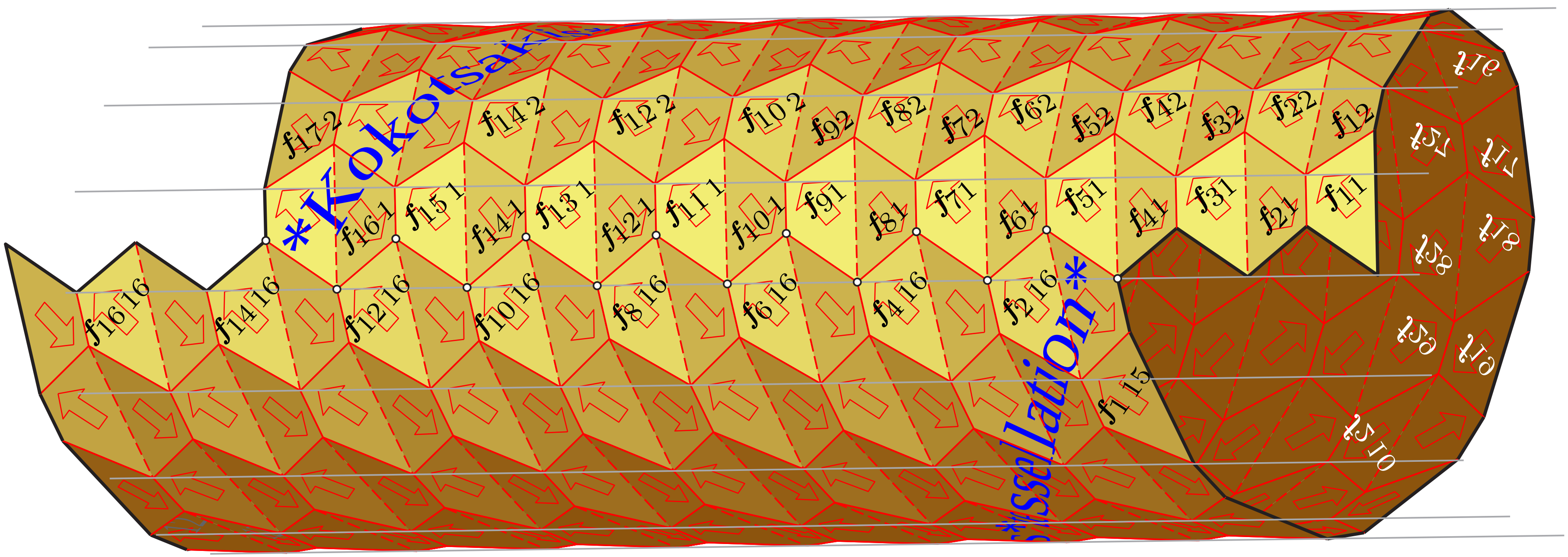
**Lemma:**
For planarity of the quadrangle $V_1 \ldots V_4$ it is necessary and sufficient to specify $V_1$ on a ruled surface of degree 3.

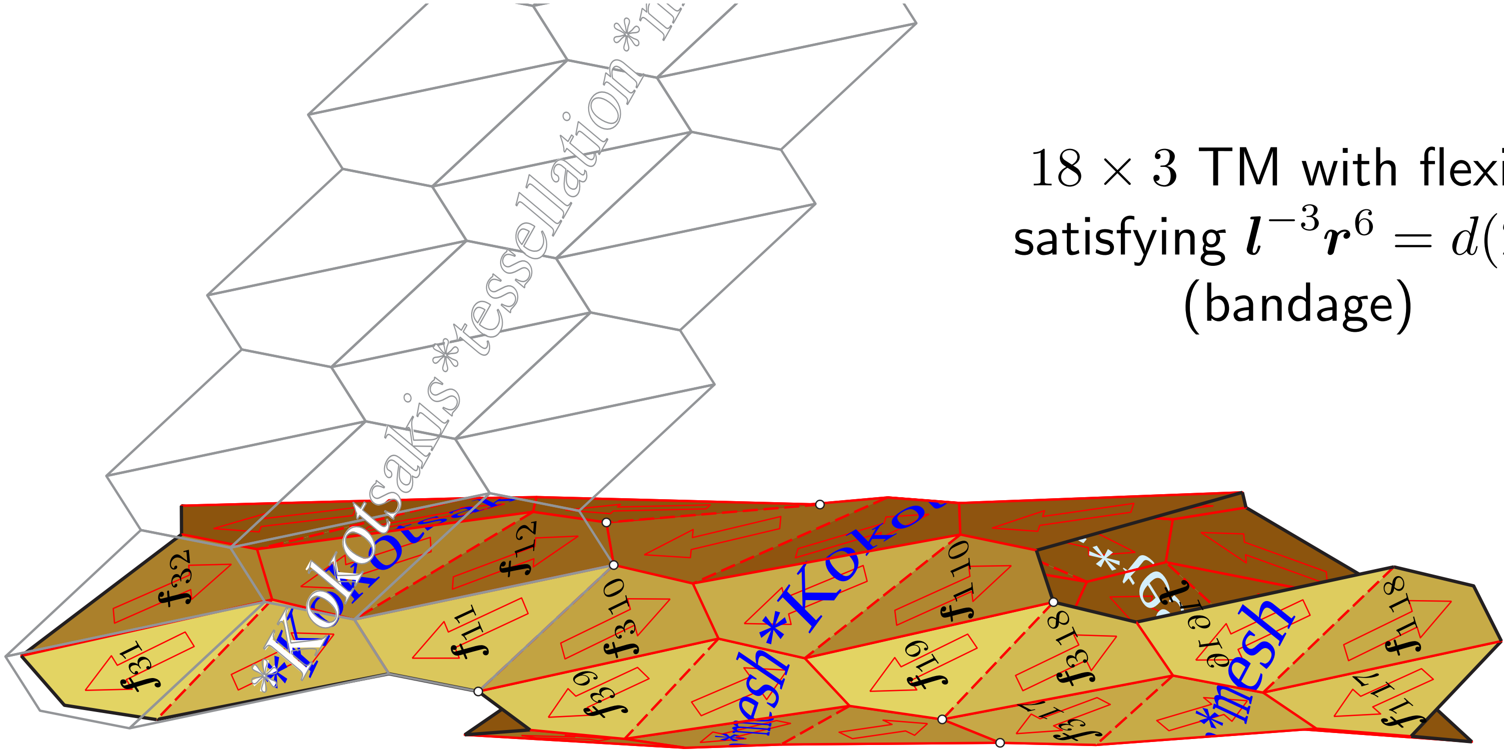
# 3. Flexion which tiles the cylinder

A basic trapezoid yields as nontrivial closing flexion of a $16 \times 17$ TM a modified *Schwarz lantern* satisfying $l^{-10}r^{6} = d_{2\pi}$ and $k = 4$.



# 3. Flexion which tiles the cylinder

**Theorem:** For given axes of the $180°$-rotations $\rho_1, \rho_2, \rho_4$ there is a two-parameter set of planar convex quadrangles for tiling a cylinder.



$18 \times 3$ TM with flexion satisfying $\boldsymbol{l}^{-3}\boldsymbol{r}^6 = d(2\pi)$ (bandage)

# Rigidity of a quadrangular cylinder tiling

**Theorem:**
When at a horizontally closing flexion of an $m \times n$ TM the right and the left border line are glued together — at least at one vertex — then in the case of a non-cyclic base quadrangle the resulting quad mesh is infinitesimally rigid.

*Remark:* For a cyclic quadrangle $V_1 \dots V_4$ the closing pose is flat. But this is trivially infinitesimally flexible.

*Proof:* We keep $\boldsymbol{f}_{11}$ fixed and assume that $\boldsymbol{f}_{1-k\,m+1}$ coincides with $\boldsymbol{f}_{11}$. We will confirm that there is no non-trivial infinitesimal motion of the mesh such that any vertex of $\boldsymbol{f}_{1-k\,1+m}$ obtains a zero-velocity.

We parametrize the self-motion of the TM by any parameter $u$ .

# Rigidity of a quadrangular cylinder tiling

For all $u \in I$ there is a helical motion about an axis $p(u)$ through angle $\varphi(u)$ and with translational length $l(u)$ with $\boldsymbol{f}_{11} \mapsto \boldsymbol{f}_{1-k\,m+1}$.

In a coordinate frame attached to $\boldsymbol{f}_{11}$ let $\boldsymbol{p}(u)$ be the unit vector of the axis $p(u)$ and $\boldsymbol{m}(u)$ the position vector of the intersection between $p(u)$ and $[\boldsymbol{f}_{11}]$.

The director cone of the surface of axes is quadratic; we may assume $\dot{\boldsymbol{p}} \neq \boldsymbol{0}$ due to a regular parametrization of this cone.

The points $\boldsymbol{m}(u)$ are located on the nine-point-conic of the quadrangle $\boldsymbol{f}_{11}$; hence $\boldsymbol{m}(u)$ can never coincide with any vertex.

We set up the helical motion mapping $\boldsymbol{x} \in \boldsymbol{f}_{11}$ onto $\boldsymbol{x}' \in \boldsymbol{f}_{1-k\,m+1}$ by

$$\boldsymbol{x}' - \boldsymbol{m} = \cos\varphi(\boldsymbol{x} - \boldsymbol{m}) + \big[(1 - \cos\varphi)\big(\boldsymbol{p} \cdot (\boldsymbol{x} - \boldsymbol{m})\big) + l\,\big]\,\boldsymbol{p} + \sin\varphi\,[\boldsymbol{p} \times (\boldsymbol{x} - \boldsymbol{m})]\,.$$

# Rigidity of a quadrangular cylinder tiling

Differentiation by $u$ yields for the infinitesimal motion of $\boldsymbol{f}_{1-k\,1+m}$

$$\begin{aligned}\dot{\boldsymbol{x}}' - \dot{\boldsymbol{m}} = {} & -\dot\varphi \sin\varphi(\boldsymbol{x}-\boldsymbol{m}) - \cos\varphi\,\dot{\boldsymbol{m}} + \Big[\dot\varphi\sin\varphi\big(\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{m})\big) \\ & + (1-\cos\varphi)\big(\dot{\boldsymbol{p}}\cdot(\boldsymbol{x}-\boldsymbol{m})\big) - (1-\cos\varphi)(\boldsymbol{p}\cdot\dot{\boldsymbol{m}}) + \dot{l}\,\Big]\boldsymbol{p} \\ & + \Big[(1-\cos\varphi)\big(\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{m})\big)+l\,\Big]\,\dot{\boldsymbol{p}} + \dot\varphi\cos\varphi\,[\boldsymbol{p}\times(\boldsymbol{x}-\boldsymbol{m})] \\ & + \sin\varphi\,\Big[\big(\dot{\boldsymbol{p}}\times(\boldsymbol{x}-\boldsymbol{m})\big)-\big(\boldsymbol{p}\times\dot{\boldsymbol{m}}\big)\Big]\,.\end{aligned}$$

In the horizontally closing pose $u = u_0$ with $\varphi(u_0) = 2\pi$ and $l(u_0) = 0$ remains:

$$\dot{\boldsymbol{x}}' = \boldsymbol{v}_{\boldsymbol{x}'} = \dot{l}\,\boldsymbol{p} + \dot\varphi\,[\boldsymbol{p}\times(\boldsymbol{x}-\boldsymbol{m})]\,.$$

$\Longrightarrow$ also the instantaneous motion of $\boldsymbol{f}_{1-k\,1+m}$ is a helical motion about $p(u_0)$ with angular velocity $\dot\varphi(u_0)$ and translational velocity $\dot{l}(u_0)$.

# Rigidity of a quadrangular cylinder tiling

In the rotational case $\dot{l} = 0$, $\dot{\varphi} \neq 0$ point $\boldsymbol{m}$ would be fixed, but $\boldsymbol{m}$ never coincides with one vertex of $\boldsymbol{f}_{1-k\,1+m}$.

Only under $\dot{\varphi} = \dot{l} = 0$ there exists a fixed vertex; the complete face $\boldsymbol{f}_{1-k\,1+m}$ has a stillstand. However, $\dot{l} = 0$ for $u = u_0$ will lead to a contradiction.

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Let $(\tau, t)$ and $(\sigma, s)$ denote the angles of rotation and lengths of translation of the helical motions $\boldsymbol{l}(u)$ and $\boldsymbol{r}(u)$, respectively. The horizontally closing implies

$$a, b \in \mathbb{Z} \setminus \{(0,0)\} \quad \text{with} \quad \varphi(u) = a\tau + b\sigma, \quad l(u) = at + bs.$$

# Rigidity of a quadrangular cylinder tiling

From $\boldsymbol{l}\colon\ \boldsymbol{v}_1 \mapsto \boldsymbol{v}_3$ and $\boldsymbol{r}\colon\ \boldsymbol{v}_2 \mapsto \boldsymbol{v}_4$ with $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_4$ as vertices of $\boldsymbol{f}_{11}$ we obtain

$$s(u) = (\boldsymbol{v}_3 - \boldsymbol{v}_1) \cdot \boldsymbol{p}(u), \ \ t(u) = (\boldsymbol{v}_4 - \boldsymbol{v}_2) \cdot \boldsymbol{p}(u).$$

$\Longrightarrow\ \ \dot{l} = a\dot{t} + b\dot{s}$. Hence, $l = \dot{l} = 0$ for $u = u_0$ results in

$$\boldsymbol{n} \cdot \boldsymbol{p}(u_0) = \boldsymbol{n} \cdot \dot{\boldsymbol{p}}(u_0) = 0 \ \text{ for } \ \boldsymbol{n} := a(\boldsymbol{v}_4 - \boldsymbol{v}_2) + b(\boldsymbol{v}_3 - \boldsymbol{v}_1) \neq \boldsymbol{0}\,.$$

The linearly independent vectors $\boldsymbol{p}(u_0)$ and $\dot{\boldsymbol{p}}(u_0)$ span a tangent plane of the quadratic director cone, and this plane is orthogonal to the vector $\boldsymbol{n}$ which — as a linear combination of the two diagonal vectors of $\boldsymbol{f}_{11}$ — lies in a symmetry plane of the cone. Hence for a non-degenerate cone, i.e., for a non-cyclic base quadrangle, $\boldsymbol{p}(u_0)$ must be a direction vector of one of the generators in the symmetry plane parallel $[\boldsymbol{f}_{11}]$. But these generators are no more axes of cylinders of rotation through the base quadrangle. □

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