The Computational Fundamentals of Spatial Cycloidal Gearing

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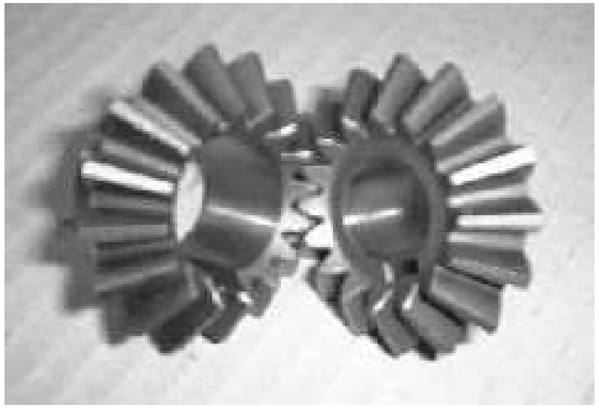
1. Introduction

WANTED: A gear set in order to transmit the rotary motion from wheel Σ_2 about axis a_{21} to the second wheel Σ_3 with axis a_{31} such that the *ratio of angular velocities ω_{31}/ω_{21} is constant.*

spur gears

a_{21}, a_{31} parallel

bevel gears

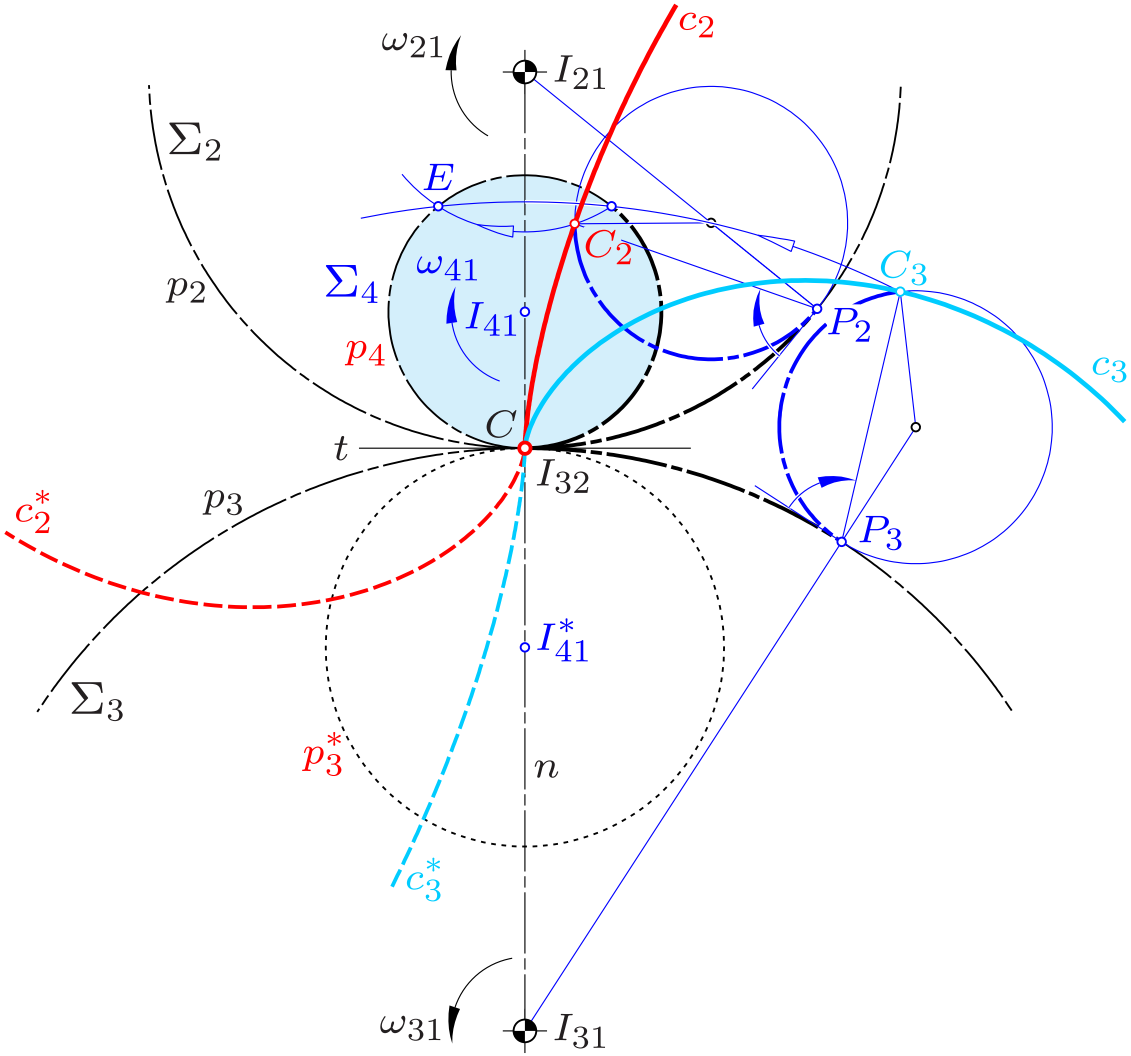


a_{21}, a_{31} crossing

skew gears
(e.g., worm gears)

a_{21}, a_{31} skew

2. Reuleaux's principle of gearing in the plane

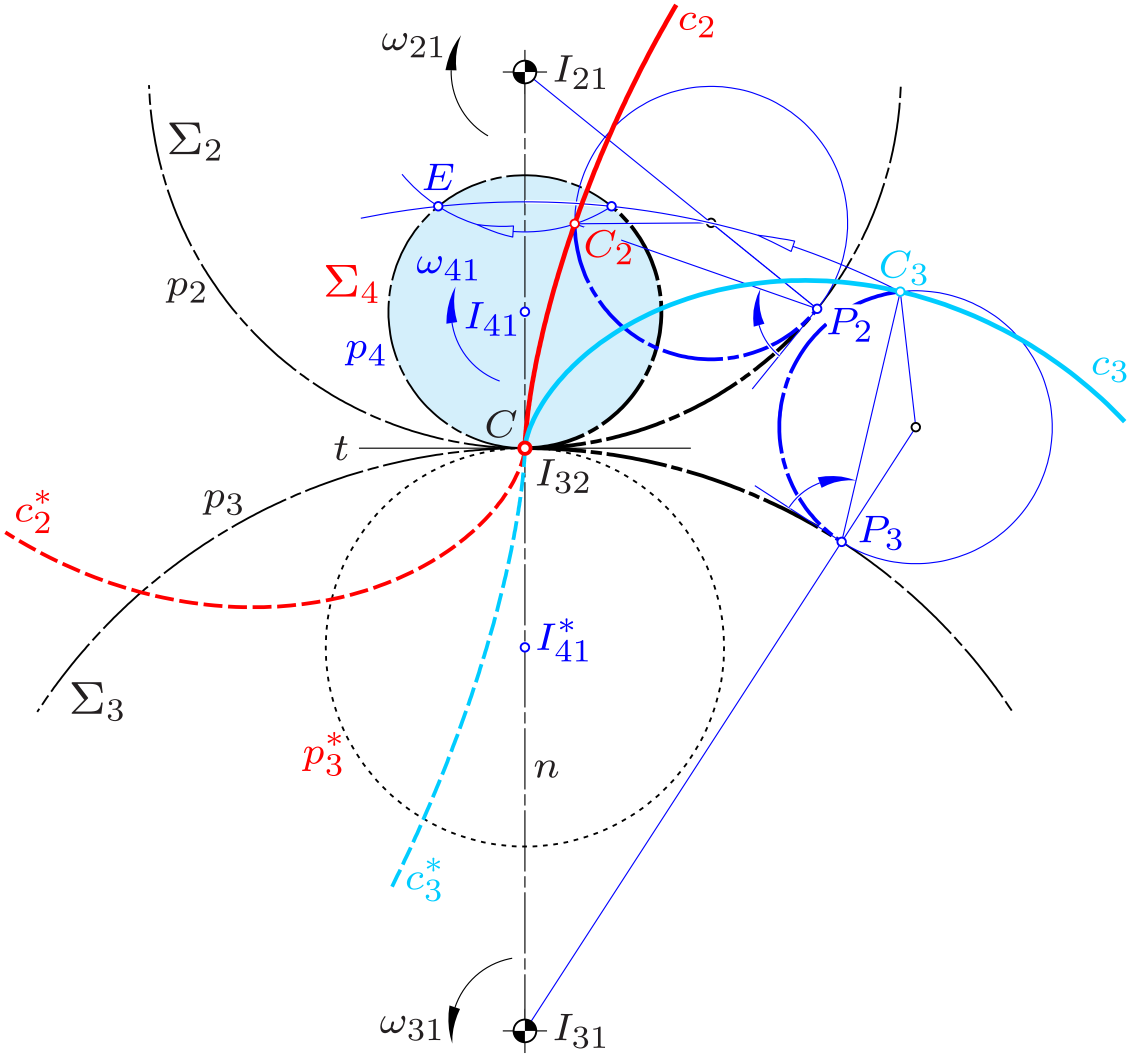


Principle of gearing by
? . . . Ch.E.L. Camus (1733),
F. Reuleaux (1875)

Let a smooth auxiliary curve p_4 roll on the polodes p_2, p_3. Then any point C attached to p_4 traces conjugate tooth profiles c_2, c_3.

At *cycloidal gears* (see left) p_4 is a circle and $C \in p_4$.

2. Reuleaux's principle of gearing in the plane



Adaption for 3D:

The auxiliary curve p_4 defines a system Σ_4 which can move such that for the relative poles holds: $I_{42} = I_{43} = I_{23}$.

Comments on Reuleaux's principle in the plane

Almost all conjugate tooth profiles can be generated by Reuleaux's principle. E.g., for involute gearing the auxiliary curve is $p_4 \subset \Sigma_4$ is a logarithmic spiral.

Generalization:

Instead of a point $C \in \Sigma_4$ a smooth curve $c_4 \subset \Sigma_4$ can be given. Then the *envelopes* c_2, c_3 of c_4 under Σ_4/Σ_2 and Σ_4/Σ_3, resp., give *conjugate tooth flanks*.

Spherical kinematics:

Reuleaux's principle is also valid in the spherical case, i.e., for bevel gears.

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3. Reuleaux's principle of gearing in 3-space

GIVEN: Rotations Σ_2/Σ_1, Σ_3/Σ_1 about fixed skew axes a_{21}, a_{31} with angular velocites ω_{21}, ω_{31}.

Question: Is there a frame Σ_4 moving such that the instantaneous screw motions of Σ_4/Σ_2, Σ_4/Σ_3 and Σ_3/Σ_2 are equal, i.e., equal axis and screw parameter, but different velocities.

We find the answer by means of dual vectors.

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Dual vectors in spatial kinematics

bijection: *Spear* $g \;\mapsto\;$ *dual unit vector* $\widehat{\boldsymbol{g}} = \boldsymbol{g} + \varepsilon \boldsymbol{g}_0$ under $\varepsilon^2 = 0$,
$\widehat{\boldsymbol{g}} \cdot \widehat{\boldsymbol{g}} = \boldsymbol{g} \cdot \boldsymbol{g} + 2\boldsymbol{g} \cdot \boldsymbol{g}_0 = 1 + \varepsilon\, 0 = 1.$

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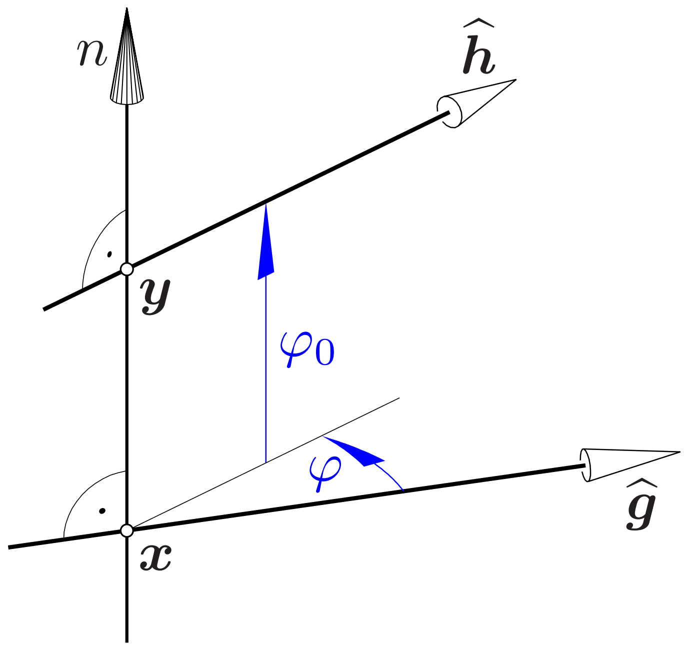
$\widehat{\varphi} = \varphi + \varepsilon \varphi_0$ is called *dual angle*.

$\cos \widehat{\varphi} = \widehat{\boldsymbol{g}} \cdot \widehat{\boldsymbol{h}} = \boldsymbol{g} \cdot \boldsymbol{h} + \varepsilon(\boldsymbol{g}_0 \cdot \boldsymbol{h} + \boldsymbol{g} \cdot \boldsymbol{h}_0)$,

$$\sin \widehat{\varphi}\, \widehat{\boldsymbol{n}} = \widehat{\boldsymbol{g}} \times \widehat{\boldsymbol{h}} = \\ = \boldsymbol{g} \times \boldsymbol{h} + \varepsilon[(\boldsymbol{g}_0 \times \boldsymbol{h}) + (\boldsymbol{g} \times \boldsymbol{h}_0)].$$



Dual vectors in spatial kinematics

Bijection: *Instantaneous motion* of $\Sigma_i/\Sigma_j \;\mapsto\;$ *twist* $\widehat{\boldsymbol{q}}_{ij} = \widehat{\omega}_{ij}\,\widehat{\boldsymbol{p}}_{ij}$
with $\widehat{\boldsymbol{p}}_{ij}$ as spear of the instantaneous axis and $\widehat{\omega}_{ij} = \omega_{ij} + \varepsilon\omega_{ij0}$ with
ω_{ij} and ω_{ij0} as instantaneous angular- and translatory velocity, respectively.

Aronhold-Kennedy Theorem:

Given: $\Sigma_1, \Sigma_2, \Sigma_3$ and instantaneous twists $\widehat{\boldsymbol{q}}_{21}, \widehat{\boldsymbol{q}}_{31}$ of Σ_2/Σ_1, Σ_3/Σ_1, resp.,
$\Longrightarrow\;\; \widehat{\boldsymbol{q}}_{32} = \widehat{\boldsymbol{q}}_{31} - \widehat{\boldsymbol{q}}_{21}$ is the twist of the relative motion Σ_3/Σ_2.

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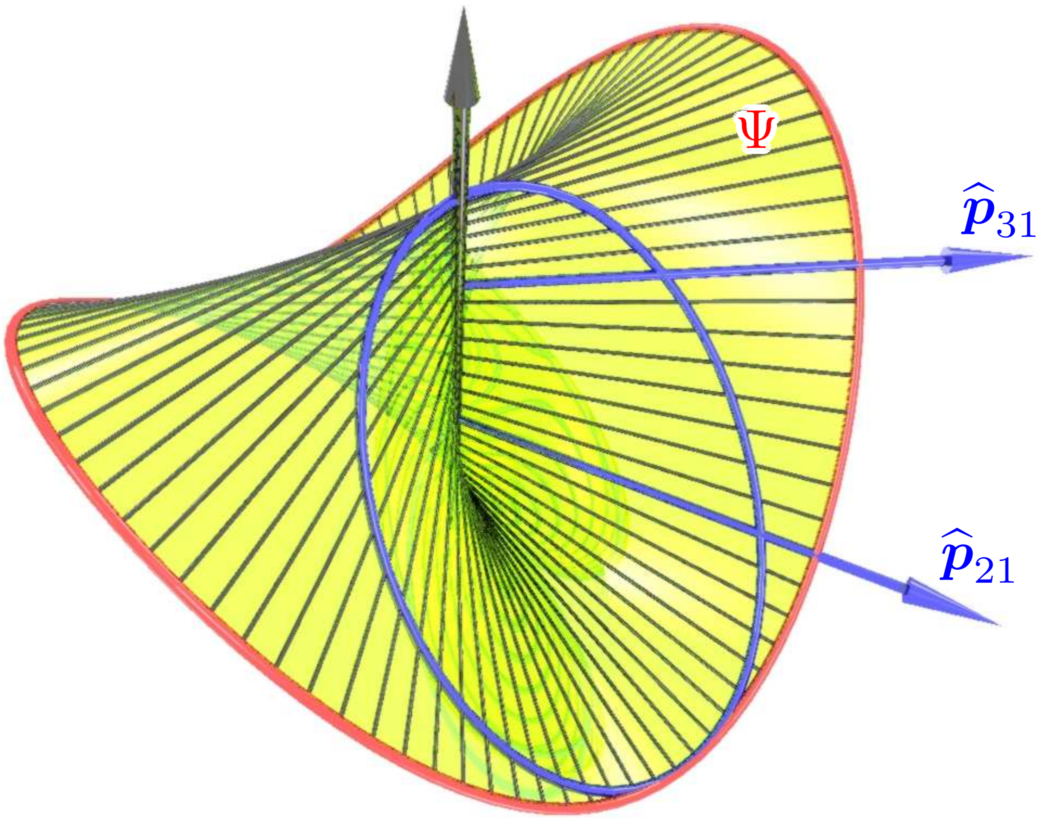
Plücker's conoid

Let Σ_2 and Σ_3 rotate about fixed skew axes $\widehat{\boldsymbol{p}}_{21}$, $\widehat{\boldsymbol{p}}_{31}$ with variable angular velocities ω_{21}, ω_{31}, respectively.



Then in any case the twist

$$\widehat{\omega}_{32}\,\widehat{\boldsymbol{p}}_{32} = \omega_{31}\,\widehat{\boldsymbol{p}}_{31} - \omega_{21}\,\widehat{\boldsymbol{p}}_{21}$$

is a real linear combination of $\widehat{\boldsymbol{p}}_{21}$ and $\widehat{\boldsymbol{p}}_{31}$ $\Longrightarrow$

$\widehat{\boldsymbol{p}}_{32}$ is located on Plücker's conoid Ψ (= cylindroid).

The position of $\widehat{\boldsymbol{p}}_{32}$ on Ψ defines the pitch $h_{32} = \omega_{320}/\omega_{32}$ uniquely.

3. Reuleaux's principle of gearing in 3-space

GIVEN: Rotations Σ_2/Σ_1, Σ_3/Σ_1 about fixed skew axes $\widehat{\boldsymbol{p}}_{21}$, $\widehat{\boldsymbol{p}}_{31}$ with angular velocites ω_{21}, ω_{31}.

Question: Is there a frame Σ_4 moving such that the instantaneous screw motions of Σ_4/Σ_2, Σ_4/Σ_3 and Σ_3/Σ_2 are equal, i.e., equal axis and screw parameter, but different velocities.

$$\widehat{\boldsymbol{q}}_{42} = \lambda_2 \widehat{\boldsymbol{q}}_{32}, \quad \widehat{\boldsymbol{q}}_{43} = \lambda_3 \widehat{\boldsymbol{q}}_{32}, \quad \lambda_1, \lambda_2 \in \mathbb{R}, \implies \widehat{\boldsymbol{q}}_{41} - \widehat{\boldsymbol{q}}_{i1} = \lambda_i(\widehat{\boldsymbol{q}}_{31} - \widehat{\boldsymbol{q}}_{21}),$$

$$\widehat{\boldsymbol{q}}_{41} = \lambda_1 \widehat{\boldsymbol{q}}_{31} + (1 - \lambda_1)\widehat{\boldsymbol{q}}_{21} = (1 + \lambda_2)\widehat{\boldsymbol{q}}_{31} - \lambda_2 \widehat{\boldsymbol{q}}_{21}$$

Necessary and sufficient: The instantaneous axes $\widehat{\boldsymbol{p}}_{41}$ of Σ_4/Σ_1 are located on Plücker's conoid Ψ and the pitch h_{41} is corresponding to this position.

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A tribute to Martin Disteli

M. Disteli: *Über die Verzahnung der Hyperboloidräder mit geradlinigem Eingriff.* Z. Math. Phys. **59**, 244–298 (1911)

Martin Disteli, (1862–1923)
Dresden, Karlsruhe

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8. Über einige Sätze der kinematischen Geometrie, welche der Verzahnungslehre zylindrischer und konischer Räder zugrunde liegen. Ebenda 56, 1908.
9. Über die Verzahnung der Hyperboloidräder mit geradlinigem Eingriff. Ebenda 59, 1911.
10. Über das Analogon der Savaryschen Formel und Konstruktion in der kinematischen Geometrie des Raumes. Ebenda 62, 1913.

Arbeiten die Axoide erledigt, so gilt es weiter die Verzahnung der zugehörigen Räderpaare durchzuführen. Dies geschieht in den folgenden Abhandlungen im Anschlusse an die Ballsche Schraubentheorie. Wir können in Rücksicht auf den uns zugemessenen Raum schon deshalb nicht auf Einzelheiten eingehen, weil die Darstellung hier vielfach insofern nicht ganz glücklich ist, als bei der Zusammenfassung der schönen und neuen Resultate die darin vorkommenden Gebilde nur durch Buchstaben bezeichnet werden, deren Bedeutung oft auf einer verwickelten Erklärung beruht. Es wäre wünschenswert, wenn diese Teile einer verständlicheren Bearbeitung unterzogen würden. Wir bemerken zum Schlusse nur noch, daß in der letzten Schrift die sogenannte Savarysche Formel der ebenen Bewegungslehre, welche den augenblicklichen Krümmungskreis der Bahnkurve irgendeines Punktes des bewegten Systems zu bestimmen erlaubt, falls die augenblicklichen Krümmungskreise der beiden Polbahnen bekannt sind, auf die hier in Betracht kommenden räumlichen Bewegungen ausgedehnt werden. An die Stelle des Krümmungskreises tritt dann die augenblickliche Striktionsschraubenfläche der Axoide. Gerade in der Behandlung der zuletzt skizzierten Probleme zeigte sich Distelis besondere Stärke.

Inhalt.

4. Consequences for skew gearing

In analogy to cycloid gearing in the plane, we specify the axis $\widehat{\boldsymbol{p}}_{41}$ of Σ_4/Σ_1 on Plücker's conoid and keep it fixed in the machine frame Σ_1.

Then we move simultaneously Σ_2 with the twist $\widehat{\boldsymbol{q}}_{21}$, Σ_3 with twist $\widehat{\boldsymbol{q}}_{31}$ and Σ_4 with twist $\widehat{\boldsymbol{q}}_{41}$ (axis $\widehat{\boldsymbol{p}}_{41}$ and pitch h_{41}) such that the relative axes $\widehat{\boldsymbol{p}}_{42}$, $\widehat{\boldsymbol{p}}_{43}$ and $\widehat{\boldsymbol{p}}_{32}$ coincide.

Main Theorem: For each line $\widehat{g}$ attached to Σ_4 the ruled surfaces Φ_2, Φ_3 traced by $\widehat{g}$ under Σ_4/Σ_2 and Σ_4/Σ_3, resp., are conjugate tooth flanks.

The contact takes place at of points of $\widehat{g}$.

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4. Consequences for skew gearing

Proof: The distribution of tangent planes of Φ_2 and Φ_3 along the line $\widehat{\boldsymbol{g}}$ is defined by the derivatives $\dot{\widehat{\boldsymbol{g}}}$, i.e., by

$$\Sigma_4/\Sigma_2 : \ \dot{\widehat{\boldsymbol{g}}} = \widehat{\boldsymbol{g}} \times \widehat{\boldsymbol{q}}_{42}, \qquad \Sigma_4/\Sigma_3 : \ \dot{\widehat{\boldsymbol{g}}} = \widehat{\boldsymbol{g}} \times \widehat{\boldsymbol{q}}_{43}.$$

Since $\widehat{\boldsymbol{q}}_{42}$ and $\widehat{\boldsymbol{q}}_{43}$ are real multiples of $\widehat{\boldsymbol{q}}_{32}$, the same holds for $\dot{\widehat{\boldsymbol{g}}}$. They have the same distribution of tangent planes. □

The axodes of Σ_3/Σ_2 are one-sheet hyperboloids of revolution Π_2, Π_3.
Under Σ_4/Σ_2 and Σ_4/Σ_3 a ruled helical surface Π_4 (auxiliary surface) is sliding and rolling (German: schroten) along the hyperboloids.

In analogy to planar are spherical cycloidal gearing we start with the generator $\widehat{\boldsymbol{g}} = \widehat{\boldsymbol{p}}_{32} =$ ISA.

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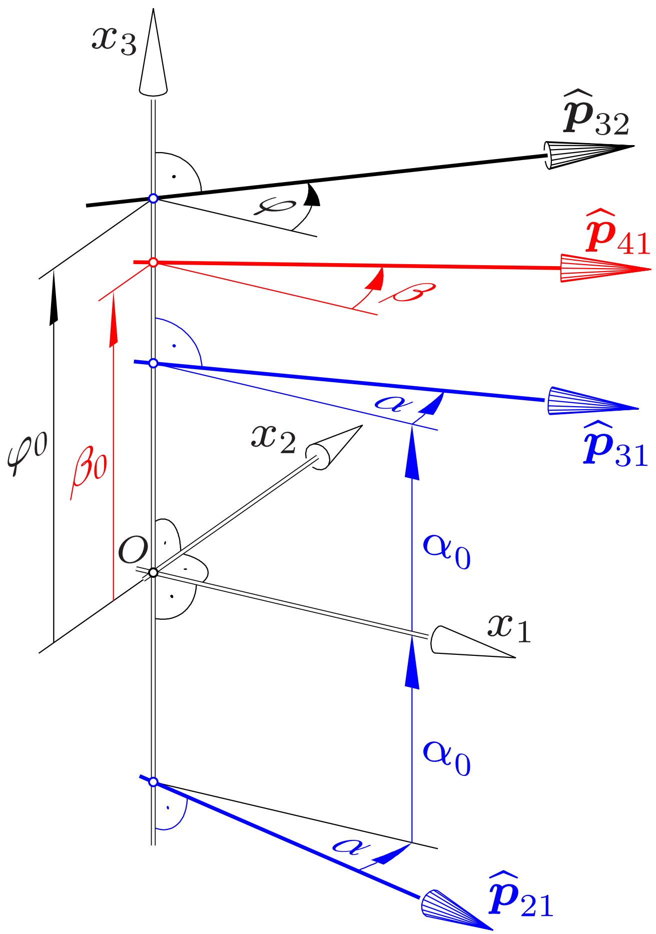

There are 4 axes involved — all located on Plücker's conoid

- Axes $\widehat{\boldsymbol{p}}_{21}$, $\widehat{\boldsymbol{p}}_{31}$ of the wheels,
- the ISA $\widehat{\boldsymbol{p}}_{32}$, and
- the axis $\widehat{\boldsymbol{p}}_{41}$ of the auxiliary system Σ_4.

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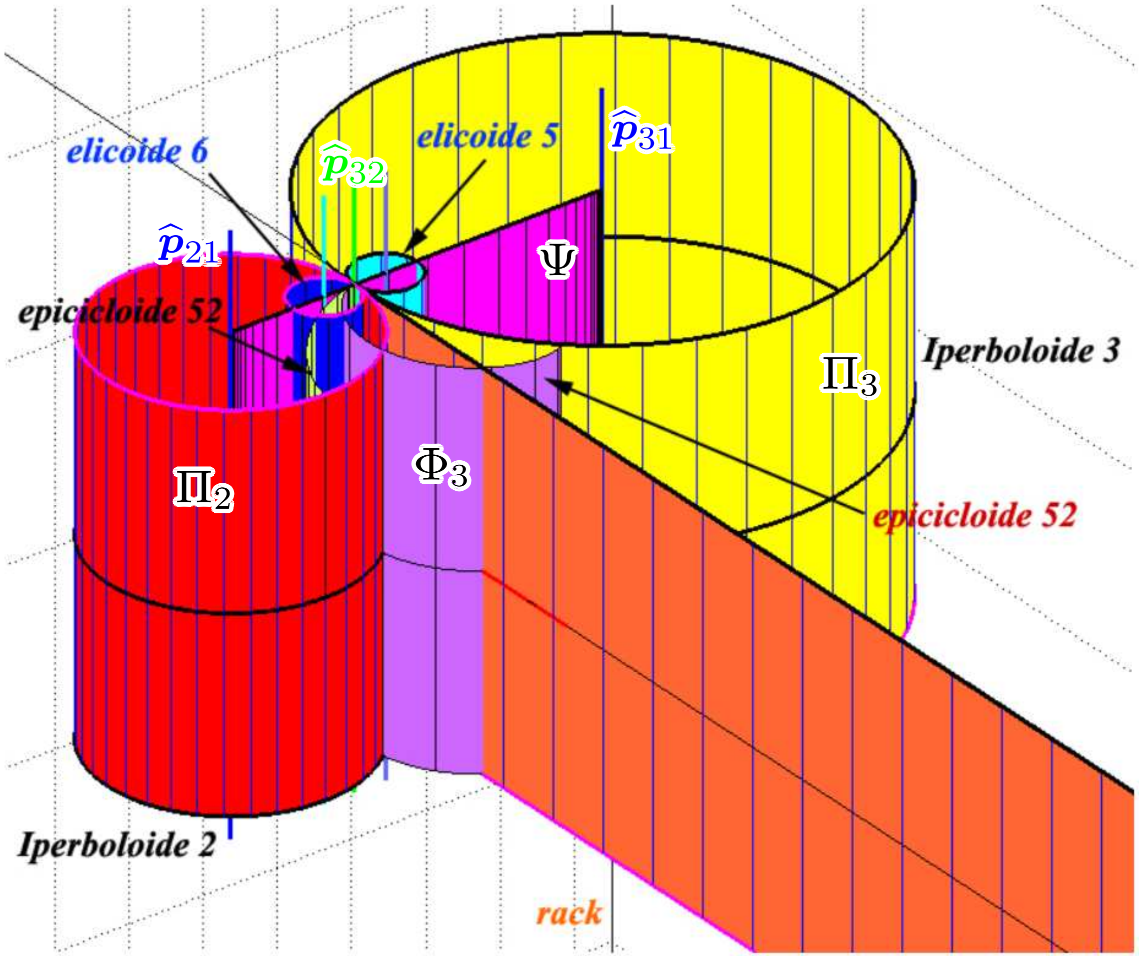
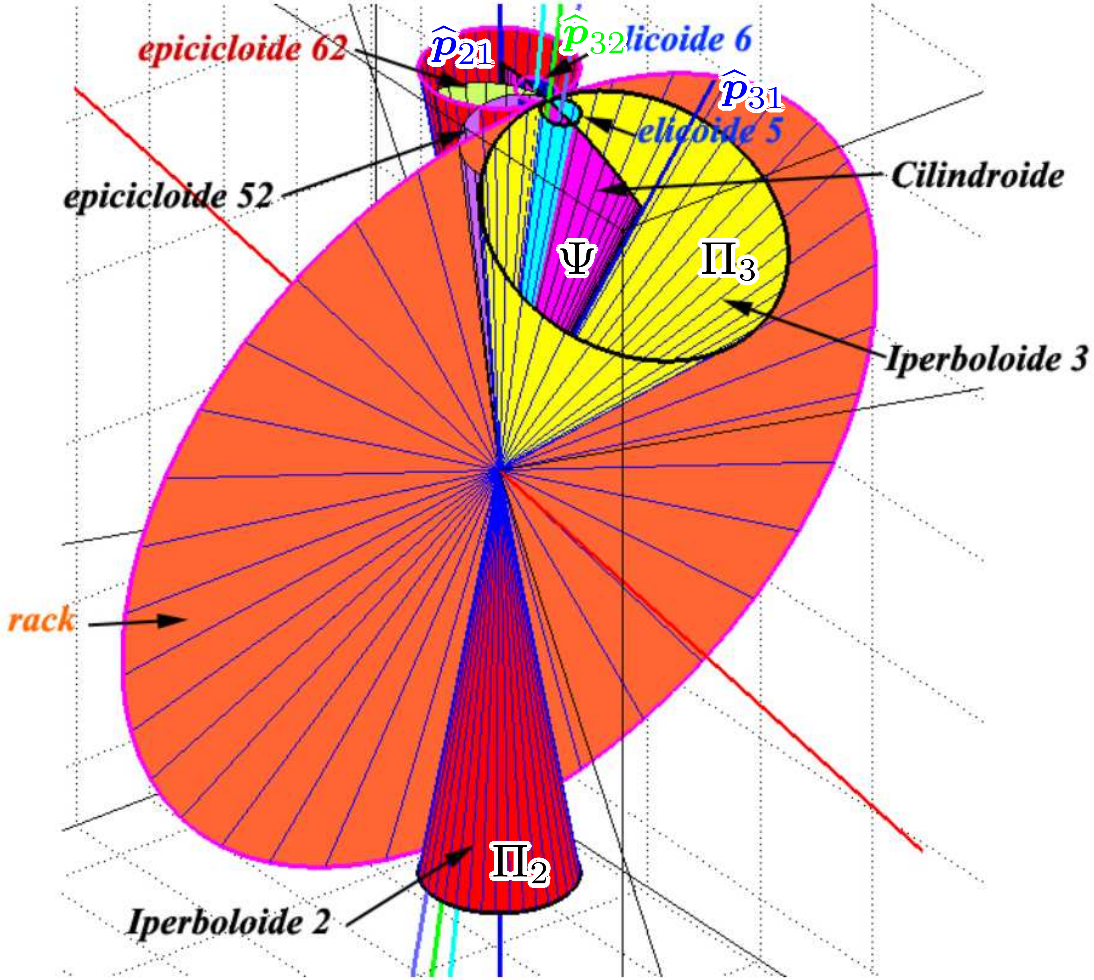


planar case

4. Consequences for skew gearing



spherical case

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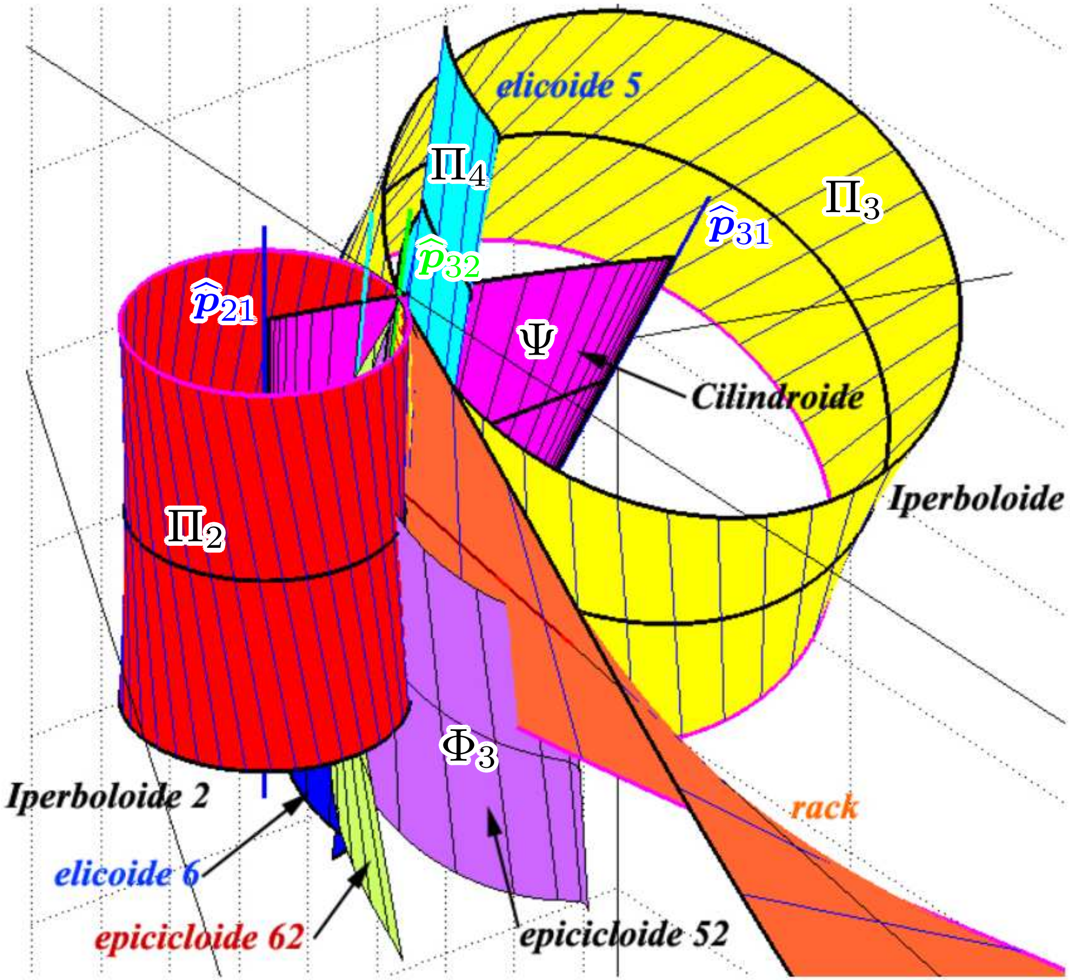
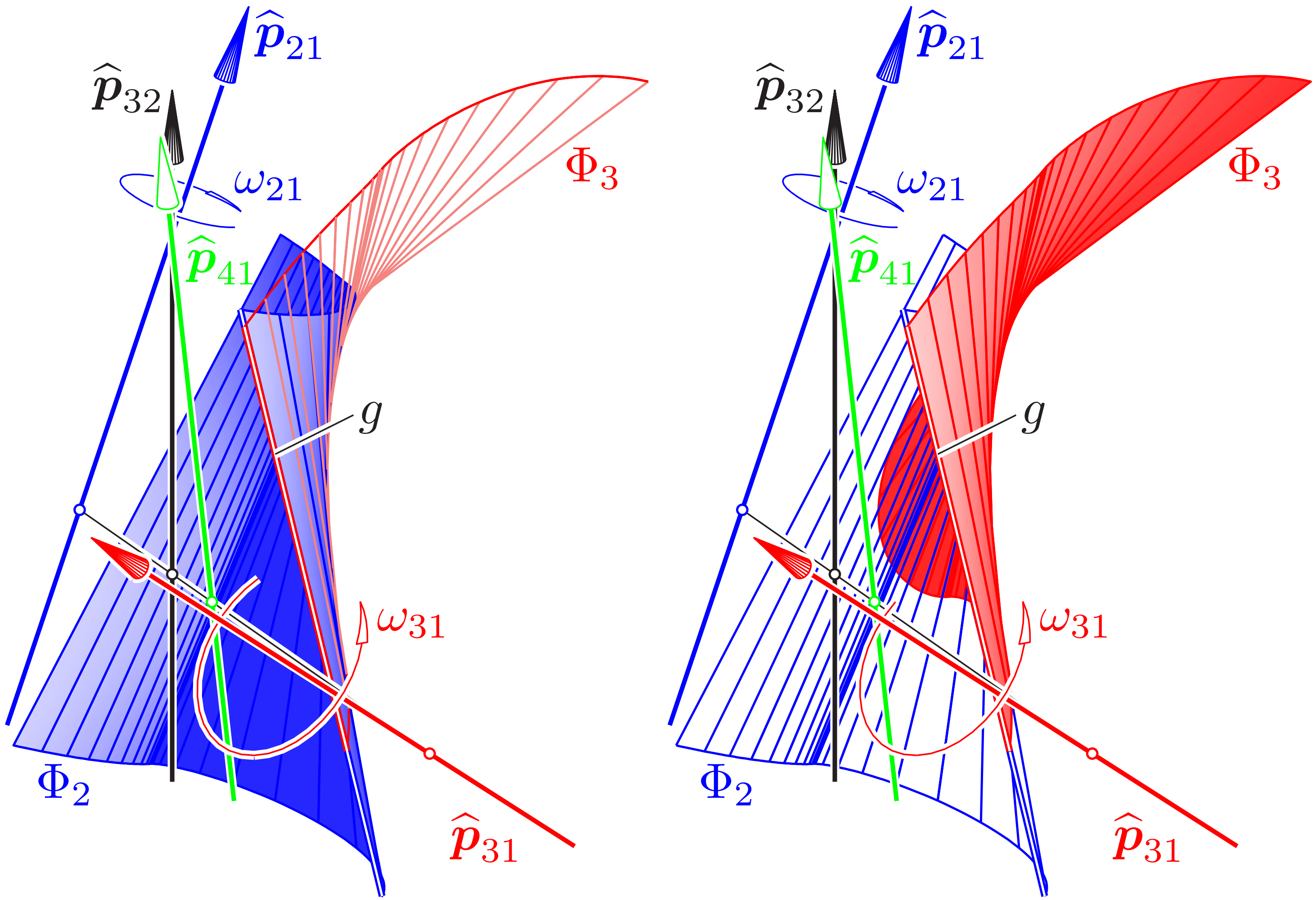


skew case

4. Consequences for skew gearing



conjugate
tooth flanks

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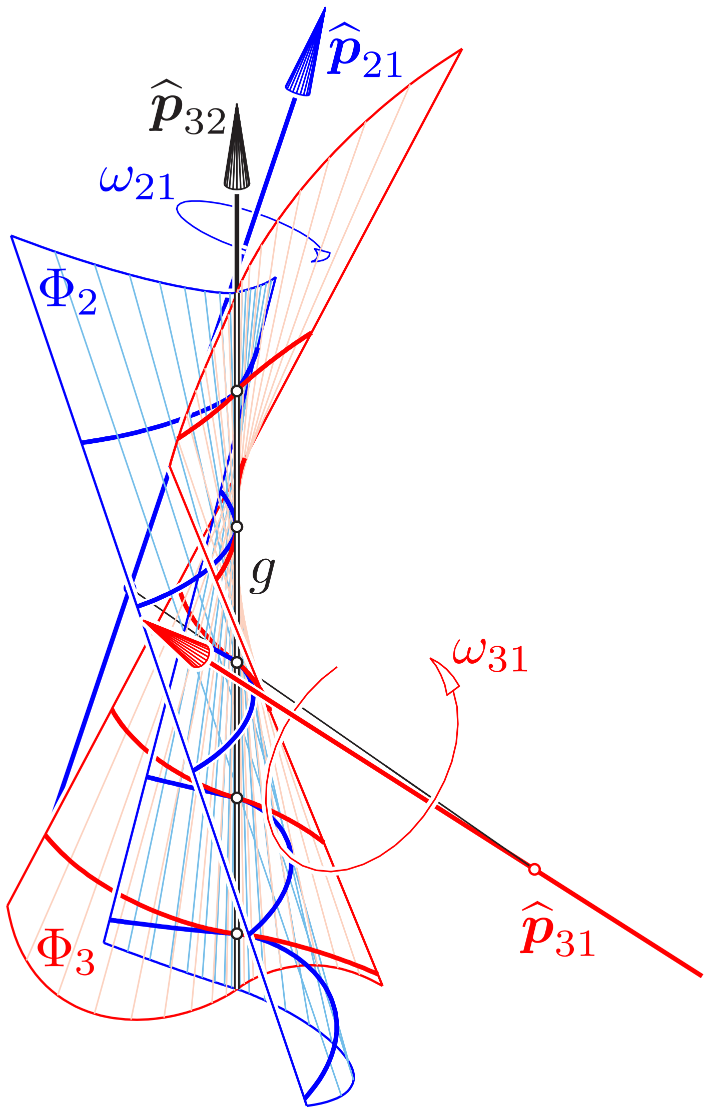

These are conjugate cycloidal tooth flanks Φ_2, Φ_3.

The displayed red and blue curves are the intersections with planes orthogonal to the line g of contact.

In this particular case g is the ISA and therefore a singular line of the ruled surface — like a cusp in the plane.

Each tooth flank is composed from two parts glued together along g. In can be proved that such pairs of surfaces have a common tangent plane at each point of g.

4. Consequences for skew gearing

Theorem: For each surface Φ_4 attached to Σ_4 the envelopes Φ_2, Φ_3 of under Σ_4/Σ_2 and Σ_4/Σ_3, resp., are conjugate tooth flanks.

Proof: For any pose of Φ_4, a point C is a point of contact between Φ_4 and its envelope Φ_i under Σ_4/Σ_i, $i = 2, 3$, if and only if the surface normal $\widehat{\boldsymbol{n}}$ of Φ_4 at C belongs to the linear line complex

$$\boldsymbol{q}_{4i0} \cdot \boldsymbol{n} + \boldsymbol{q}_{4i} \cdot \boldsymbol{n}_0 = 0.$$

Twists differing by a real factor define the same linear line complex. $\square$

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5. Conclusions

The main goal of this paper is to shed light on the geometry of the tooth flanks of gears with skew axes.

To this end, the authors follow and extend results reported by Martin DISTELI, thereby deriving ruled surfaces as conjugate tooth flanks in contact along a line.

A detailed analysis of the tooth flanks obtained for constant and for variable axis $\widehat{\boldsymbol{p}}_{41} \in \Psi$ is left for future research.

Conjecture: Any pair of conjugate ruled tooth flanks with contact along lines can be obtained by this principle.

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