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DUAL CONTINUATION OF CYCLOIDAL BEVEL GEARS SKEW GEARS WITH RULED SURFACES AS TOOTH FLANKS

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ABSTRACT: Study's principle of transference (German: Übertragungsprinzip) allows to transfer results from spherical geometry by "dualization" to the geometry of oriented lines in the Euclidean 3-space. In this paper the principle of Reuleaux (or Camus) for obtaining tooth profiles of bevel gears is dualized — thus following and extending results reported by Martin Disteli at the turn of the 20th century. The main goal is the design of ruled surfaces as tooth flanks for gears with skew axes such that at any instant the contact points are located on a straight line.

Keywords: Principle of transference, spatial gearing, Reuleaux principle

SUMMARY

Due to E. Study the *directed line* (spear) $g = \boldsymbol{p} + \mathbb{R}\,\boldsymbol{g}$, $\|\boldsymbol{g}\| = 1$, in the Euclidean 3-space can be identified with the three-dimensional dual vector $\underline{\boldsymbol{g}} = \boldsymbol{g} + \varepsilon\,\widehat{\boldsymbol{g}} \in \mathbb{D}^3$. Here $\mathbb{D}$ denotes the ring of dual numbers $r + \varepsilon\,\widehat{r}$, $(r, \widehat{r}) \in \mathbb{R}^2$, with the dual unit ε obeying $\varepsilon^2 = 0$; the dual part $\widehat{\boldsymbol{g}}$ of $\underline{\boldsymbol{g}}$ is defined by $\widehat{\boldsymbol{g}} = \boldsymbol{p} \times \boldsymbol{g}$. We call $\underline{\boldsymbol{g}}$ a *dual unit vector* because of $\underline{\boldsymbol{g}} \cdot \underline{\boldsymbol{g}} = 1$ provided the dot product of any two dual vectors $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$ is defined by $\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{v}} = \boldsymbol{u}\cdot\boldsymbol{v} + \varepsilon(\widehat{\boldsymbol{u}}\cdot\boldsymbol{v} + \boldsymbol{u}\cdot\widehat{\boldsymbol{v}})$.

Together with the analogous *dualization* of the vector product we obtain for any two dual unit vectors $\underline{\boldsymbol{g}}, \underline{\boldsymbol{h}}$ the equations

$$\underline{\boldsymbol{g}} \cdot \underline{\boldsymbol{h}} = \cos\varphi - \varepsilon\,\widehat{\varphi}\sin\varphi = \underline{\cos}\,\underline{\varphi}\,, \quad \underline{\boldsymbol{g}}\times\underline{\boldsymbol{h}} = (\sin\varphi + \varepsilon\widehat{\varphi}\cos\varphi)(\boldsymbol{n} + \varepsilon\,\widehat{\boldsymbol{n}}) = \underline{\sin}\,\underline{\varphi}\;\underline{\boldsymbol{n}}\,,$$

when the *dual angle* $\underline{\varphi} = \varphi + \varepsilon\,\widehat{\varphi}$ between $\underline{\boldsymbol{g}}$ and $\underline{\boldsymbol{h}}$ combines the dual angle φ and the distance $\widehat{\varphi}$ between the lines; $\underline{\boldsymbol{n}}$ is a common perpendicular of the two spears $\underline{\boldsymbol{g}}, \underline{\boldsymbol{h}}$. The dualization of any differentiable real function obeys $\underline{f}(\underline{x}) = \underline{f}(x + \varepsilon\,\widehat{x}) = f(x) + \varepsilon\,\widehat{x}\,f'(x)$.

There are other examples of formal agreement between formulas of spherical geometry and the geometry of lines. E.g., any *spatial motion* of spears can be expressed by

$$\underline{\boldsymbol{g}} \mapsto \underline{\boldsymbol{g}}' = \underline{A}\,\underline{\boldsymbol{g}} \quad \text{with} \quad \underline{A}^\top = \underline{A}^{-1}, \;\text{ i.e., }\; \underline{A} = A + \varepsilon\widehat{A},\; A^T = \underline{A}^{-1},\; \widehat{A} = SA\,,\; S^T = -S\,.$$

The *instantaneous motion*, which acts on points $\boldsymbol{x}$ by $\dot{\boldsymbol{x}} = \widehat{\boldsymbol{q}} + (\boldsymbol{q} \times \boldsymbol{x})$, maps the oriented line $\underline{\boldsymbol{g}}$ onto $\dot{\underline{\boldsymbol{g}}} = \underline{\boldsymbol{q}} \times \underline{\boldsymbol{g}}$ making use of the instantaneous *twist* $\underline{\boldsymbol{q}} = \boldsymbol{q} + \varepsilon\,\widehat{\boldsymbol{q}} \in \mathbb{D}^3$.

The dualization of the spherical *Euler-Savary equation* yields a result which dates back to M. Disteli (1914). It expresses the instantaneous *Disteli axis* $\underline{\boldsymbol{g}}^*$ of the ruled surface traced by line $\underline{\boldsymbol{g}}$ under a one-parameter motion: Suppose that $\underline{\boldsymbol{g}}$ has the *dual polar coordinates* $(\underline{\alpha}, \underline{\psi})$ with respect to the Frenet frame $(\underline{\boldsymbol{f}}_1, \underline{\boldsymbol{f}}_2, \underline{\boldsymbol{f}}_3)$ of the axodes, i.e.,

$$\underline{\boldsymbol{g}} := \underline{\cos}\,\underline{\alpha}\,\underline{\boldsymbol{f}_1} + \underline{\sin}\,\underline{\alpha}\,(\underline{\cos}\,\underline{\psi}\,\underline{\boldsymbol{f}_2} + \underline{\sin}\,\underline{\psi}\,\underline{\boldsymbol{f}_3}).$$

Then the dual polar coordinates $(\underline{\alpha}^*, \underline{\psi})$ of $\underline{\boldsymbol{g}}^*$ are given by

$$(\underline{\cot}\,\underline{\alpha}^* - \underline{\cot}\,\underline{\alpha})\,\underline{\sin}\,\underline{\psi} = \frac{\omega}{\underline{\lambda}} = \underline{\cot}\,\underline{\gamma}_0 - \underline{\cot}\,\underline{\gamma}$$

with cot $\underline{\gamma}_0$ and $\underline{\cot}\,\underline{\gamma}$ as *dual geodesic curvatures* of the axodes (see, e.g., [2]).

W. Blaschke [1] was one of the first who used this most natural way for an elegant approach to spatial kinematics. Here we address the dualization of Reuleaux's principle:

According to the spherical case we choose an *auxiliary system* Σ_4 which moves such that the instantaneous screws of the relative motions Σ_4/Σ_2 and Σ_4/Σ_3 against the two wheels Σ_2 and Σ_3 equal that of Σ_3/Σ_2. This holds if and only if the instantaneous axis $\underline{\boldsymbol{p}_{40}}$ of the movement of Σ_4 against the gear box Σ_0 is placed on Plücker's *cylindroid* defined by the gear axes $\underline{\boldsymbol{p}_{20}}$ and $\underline{\boldsymbol{p}_{30}}$ (see [3]). This condition implies

Theorem 1: *For any line $\underline{\boldsymbol{g}}$ attached to Σ_4 the ruled surfaces Φ_2, Φ_3 traced under Σ_4/Σ_2 and Σ_4/Σ_3, respectively, are conjugate tooth flanks which at any instant are in contact along a line.*

When $\underline{\boldsymbol{g}}$ is kept constant on the cylindroid, we obtain the particular case of *spatial cycloid gears*. Otherwise, i.e., for variable $\underline{\boldsymbol{g}}$ on the cylindroid, almost all conjugate tooth profiles with permanent line contact — and in the torsal case with coinciding cuspidal points — can be generated by this principle. This result is due to M. Disteli; it can be proved with the help of the spatial Euler-Savary formula.

Also the following generalization of Reulaux's principle is valid in 3-space:

Theorem 2: *For each surface Φ_4 attached to Σ_4 the envelopes Φ_2, Φ_3 of Φ_4 under Σ_4/Σ_2 and Σ_4/Σ_3, resp., are conjugate tooth flanks.*

The proof is based on the fact that the linear complexes of normals of the motions Σ_4/Σ_2, Σ_4/Σ_3 and Σ_3/Σ_2 are identical at each instant. Hence, each enveloping point P of Φ_4 under Σ_4/Σ_2 is at the same time an enveloping point of Φ_4 under Σ_4/Σ_3; hence P is a point of contact between the envelopes Φ_2 and Φ_3.

REFERENCES

[1] Blaschke, W. *Kinematik und Quaternionen.* VEB Deutscher Verlag der Wissenschaften, Berlin 1960.

[2] Stachel, H. *Instantaneous spatial kinematics and the invariants of the axodes.* Proc. Ball 2000 Symposium, Cambridge 2000, no. 23, 14 p.

[3] Figliolini, G., Stachel, H., and Angeles, J. *The Computational Fundamentals of Spatial Cycloidal Gearing.* In Kecskeméthy, A., and Müller, A. (eds.) Computational Kinematics, Proceedings of the 5th Internat. Workshop on Computational Kinematics, Dortmund/Germany 2009, pp. 375–384.

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