

Flexible Kokotsakis Meshes

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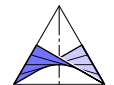


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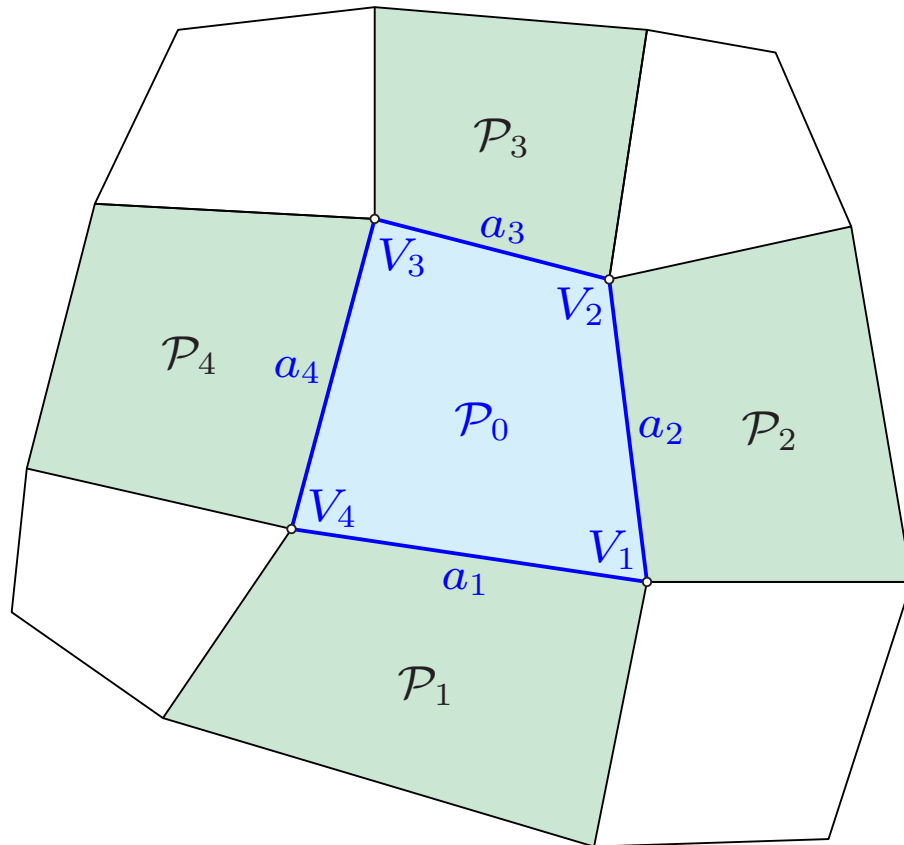


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1. Definition of Kokotsakis meshes



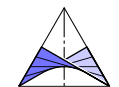
Special case: $n = 4$

A **Kokotsakis mesh** is a polyhedral structure consisting of an n -sided central polygon \mathcal{P}_0 surrounded by a belt of polygons.

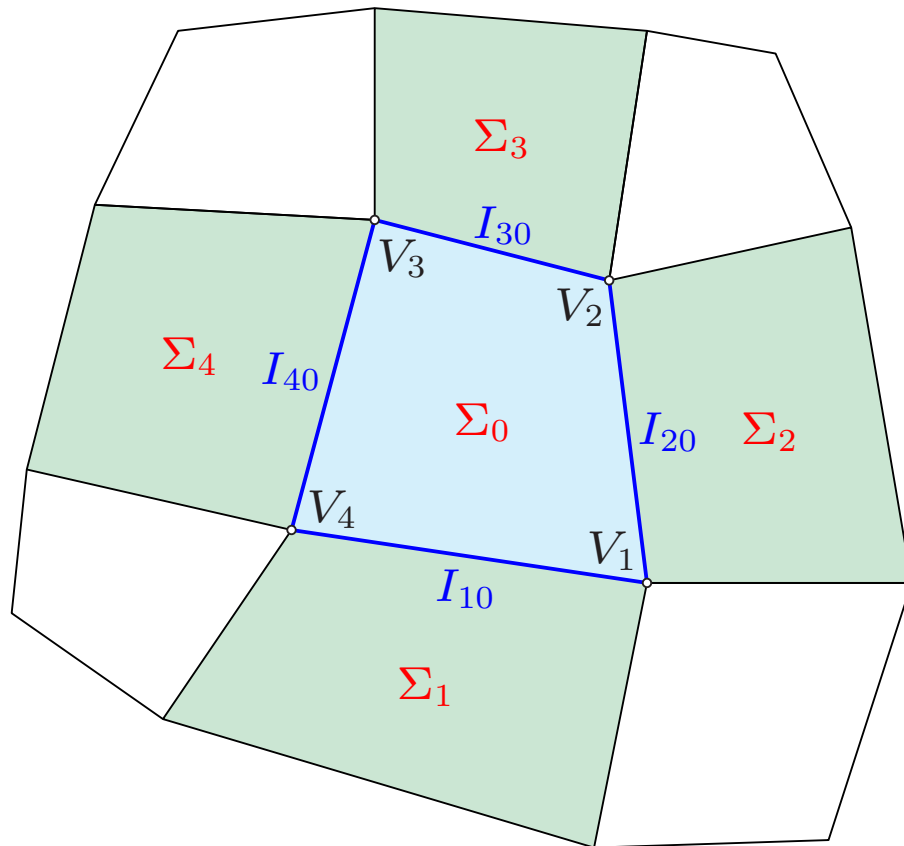
(A. KOKOTSAKIS, Athens, 1932)

Each **side** a_i , $i = 1, \dots, n$, of \mathcal{P}_0 is shared with a polygon \mathcal{P}_i .

Each **vertex** V_i of \mathcal{P}_0 is the meeting point of four faces.



1. Definition of Kokotsakis meshes



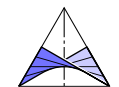
The polygons need **not** be planar

Kinematic interpretation:

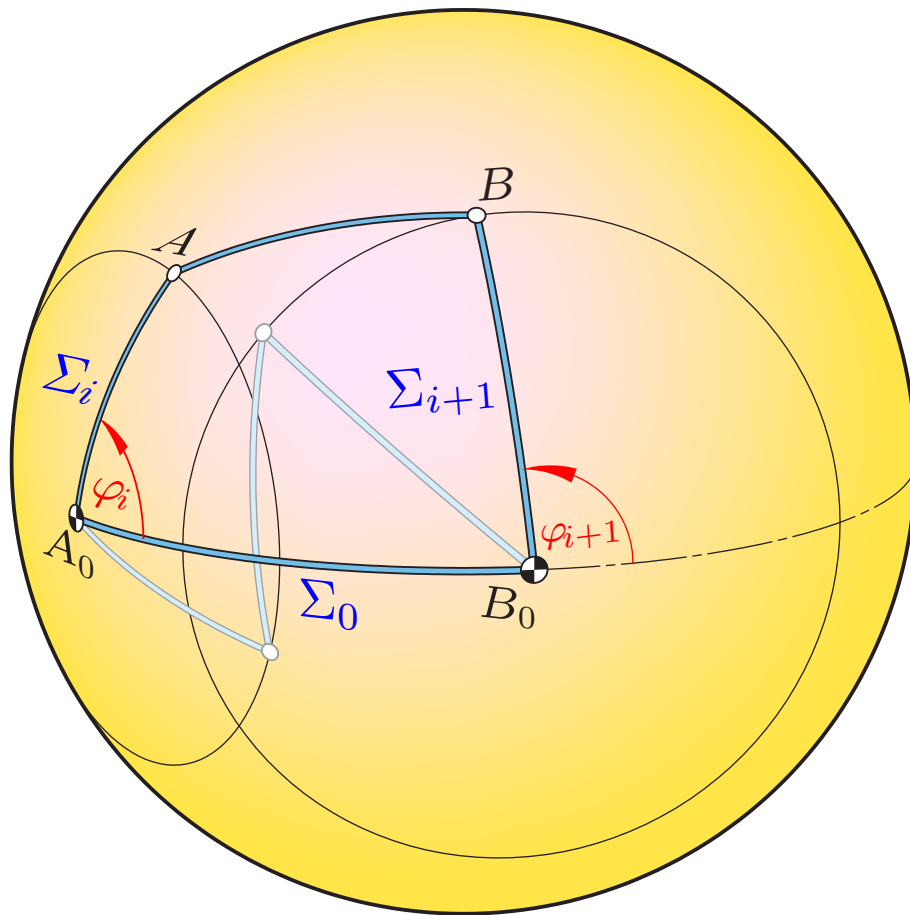
The polygons represent different systems $\Sigma_0, \dots, \Sigma_n$.

The sides a_i of \mathcal{P}_0 are instantaneous axes I_{i0} of the relative motions Σ_i/Σ_0 .

The relative motions Σ_{i+1}/Σ_i between consecutive systems are **spherical coupler motions**.



1. Definition of Kokotsakis meshes

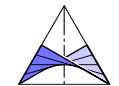


To recall:

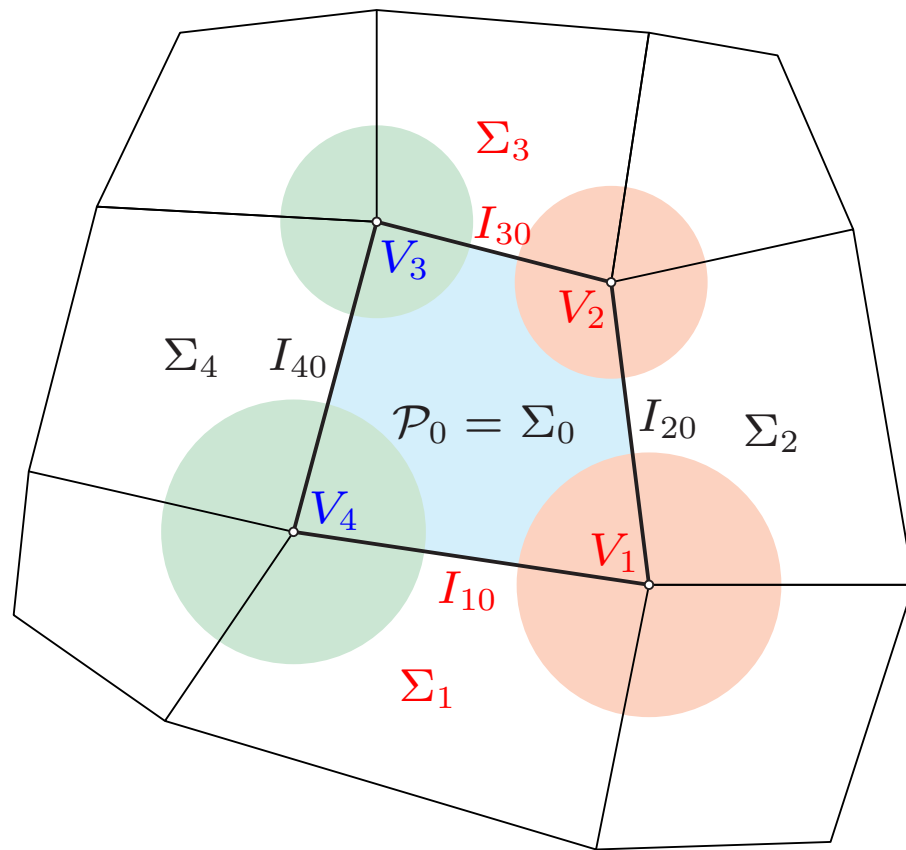
A **spherical coupler motion** transmits the rotation about the center A_0 by the **coupler AB** non-uniformly to the rotation about B_0 .

The arms A_0A and B_0B represent consecutive frames Σ_i, Σ_{i+1} .

$n=3$ is equivalent to an **octahedron** ('flexible Bricard octahedra') (R. BRICARD, 1897).



1. Definition of Kokotsakis meshes

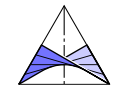


the planar case with $n = 4$ is called **Vierflach** (KOKOTSAKIS, SAUER)

E.g., flexibility of the ‘Vierflach’ means:

The transmission of the rotation Σ_1/Σ_0 to Σ_3/Σ_0 by the two coupler motions (V_1, V_2) on the right hand side is the same as via (V_3, V_4) on the right hand side.

The edge lengths $\overline{V_1V_2}, \dots, \overline{V_4V_1}$ of the central polygon \mathcal{P}_0 have no influence on the flexibility.



1. Definition of Kokotsakis meshes

Theorem:

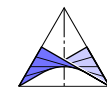
A **Kokotsakis mesh** is 1st or 2nd order infinitesimally or continuously **flexible** \iff its **spherical image** is 1st or 2nd order infinitesimally or continuously **flexible**, respectively.

1st order infinitesimal flexibility: KOKOTSAKIS 1932, KARPENKOV 2008. Due to WEGNER this is equivalent to that of a planar section.

2nd order infinitesimal flexibility: BOBENKO, HOFFMANN, SCHIEF 2008

In *discrete differential geometry* there is an interest on polyhedral structures composed of quadrilaterals, i.e., in **quadrilateral surfaces**.

Each quadrilateral is seen as a rigid body; only the dihedral angles can vary.



1. Definition of Kokotsakis meshes

If all quadrilaterals of a quadrilateral surface are **planar**, the edges form a *discrete conjugate net* ... as edges emanating from one parameter line to the same side constitute a *discrete torse*.

Theorem: [BOBENKO, HOFFMANN, SCHIEF 2008]

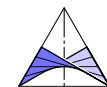
A discrete conjugate net is **continuously flexible** \iff all its 3×3 complexes are **continuously flexible**.

BOBENKO et al., 2008

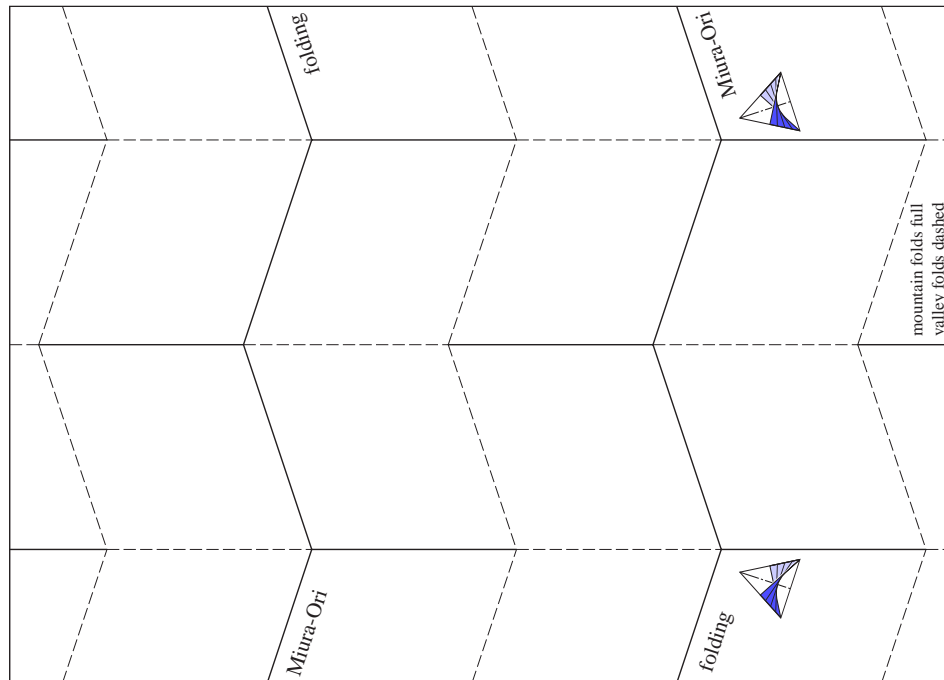
“... the complete classification of flexible discrete conjugate nets has not been achieved yet”

H. POTTMANN, Y. LIU,
J. WALLNER, A. BOBENKO,
W. WANG:

Geometry of Multi-layer Freeform Structures for Architecture
SIGGRAPH 2007



2. Particular flexible examples

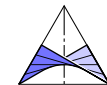


Unfolded miura-ori;
dashes are *valley folds*,
full lines are *mountain folds*

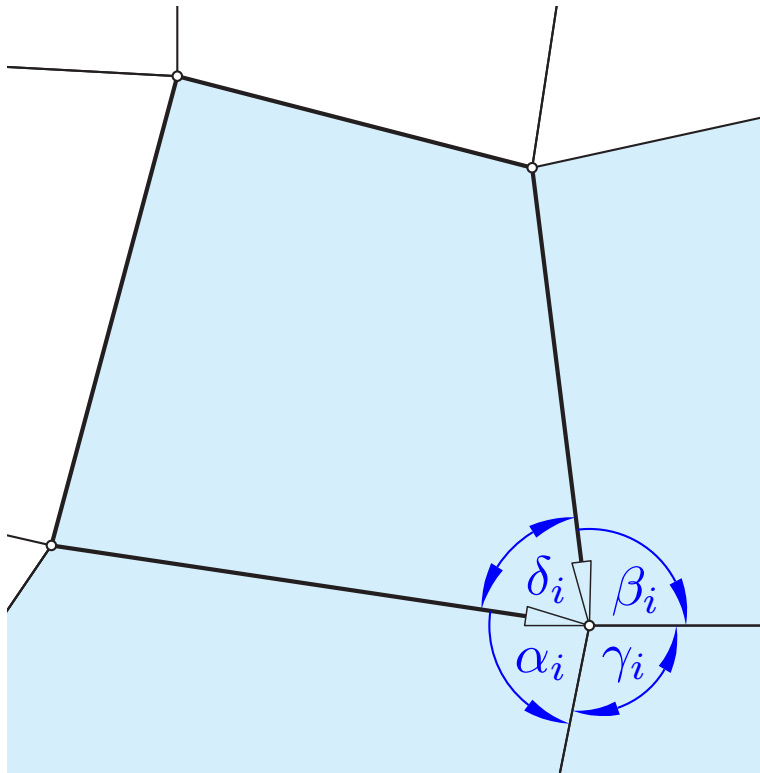
Miura-ori is a Japanese folding technique named after Prof. Koryo Miura, The University of Tokyo.

It is used for **solar panels** because it can be unfolded into its rectangular shape by pulling on one corner only.

On the other hand it is used as kernel to stiffen **sandwich structures**.



2. Particular flexible examples



Miura-ori is a special case of

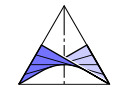
Theorem: [KOKOTSAKIS 1932]

A Kokotsakis mesh is flexible if at each vertex V_i **opposite angles** are either **equal** or **supplementary**, i.e.,

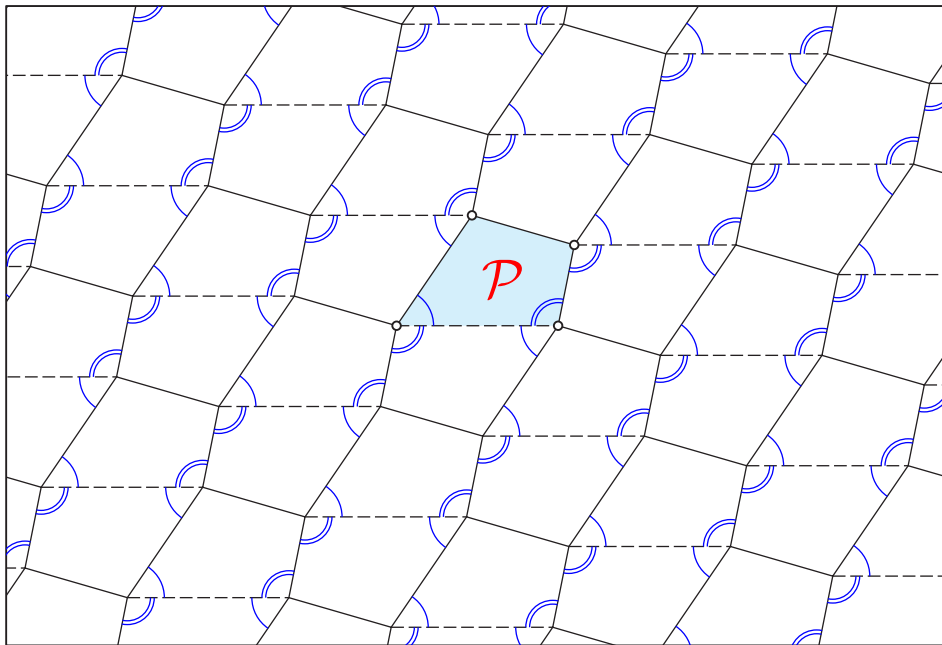
$$\alpha_i = \beta_i, \quad \gamma_i = \delta_i \quad \text{or} \\ \alpha_i = \pi - \beta_i, \quad \gamma_i = \pi - \delta_i.$$

A discrete conjugate net where all vertices are of this type is called **Voss surface**:

- Its folds are **geodesics**,
- it is **continuously flexible**.



A flexible tessellation

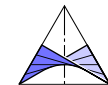


A. KOKOTSAKIS, 1932
Athens

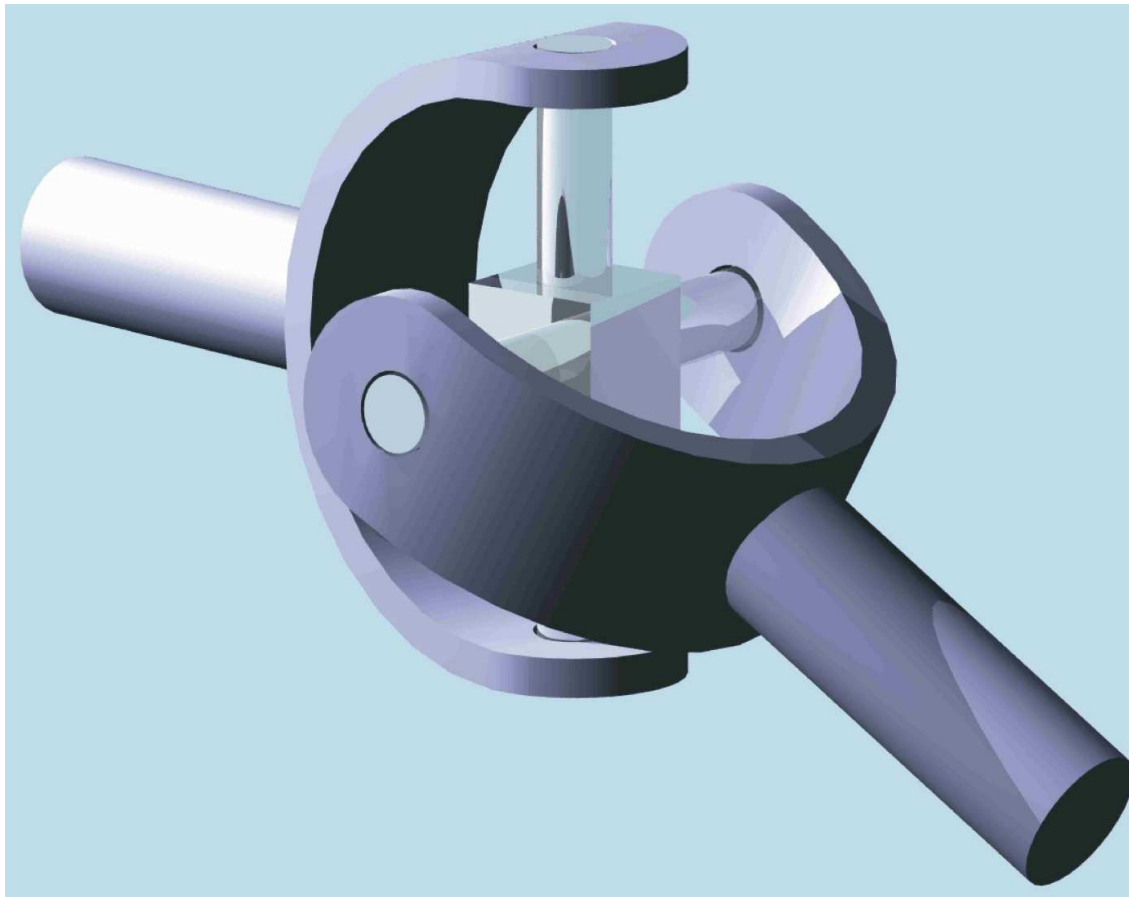
Any plane quadrangle is a tile for a **regular tessellation** of the plane.

It is obtained by applying **iterated 180°-rotations** about the midpoints of the sides of an initial quadrangle \mathcal{P} .

For convex \mathcal{P} this polyhedral structure is flexible

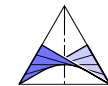


Cardan joints as flexible Kokotsakis meshes

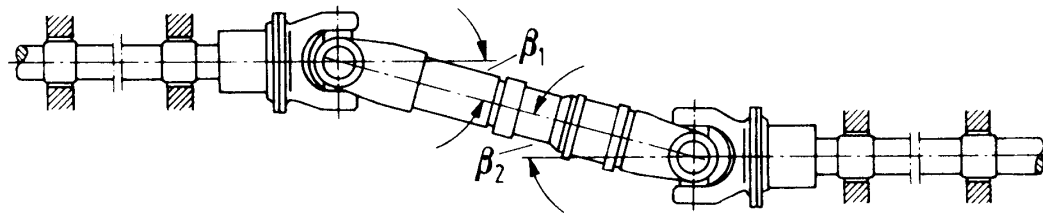


The transmission by one single cardan joint is **non-uniform** and obeys the rule

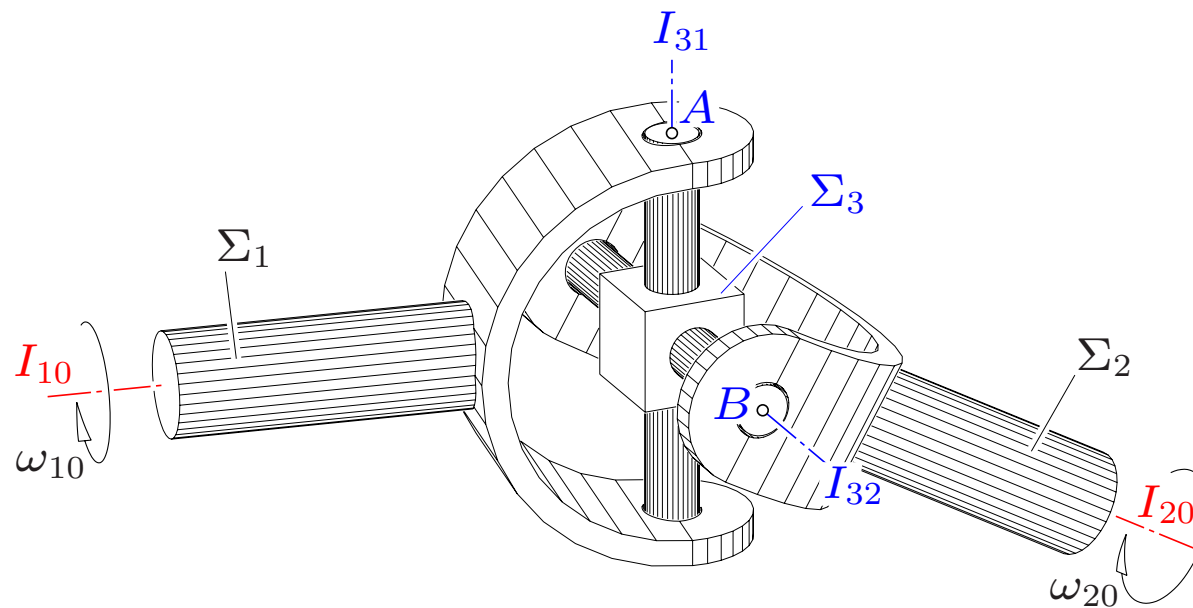
$$\tan \varphi_{20} = \frac{1}{\cos \delta} \tan \varphi_{10},$$



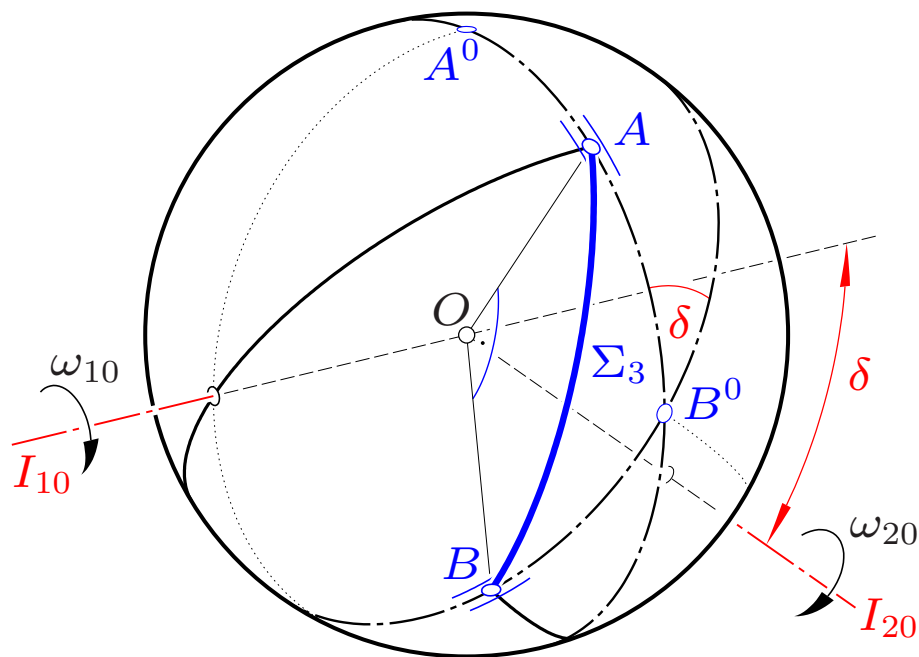
Cardan joints as flexible Kokotsakis meshes



However, two consecutive cardan joints in **Z-position**, i.e., under $\beta_2 = \beta_1$ produce a uniform transmission.



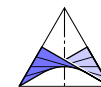
Cardan joints as flexible Kokotsakis meshes



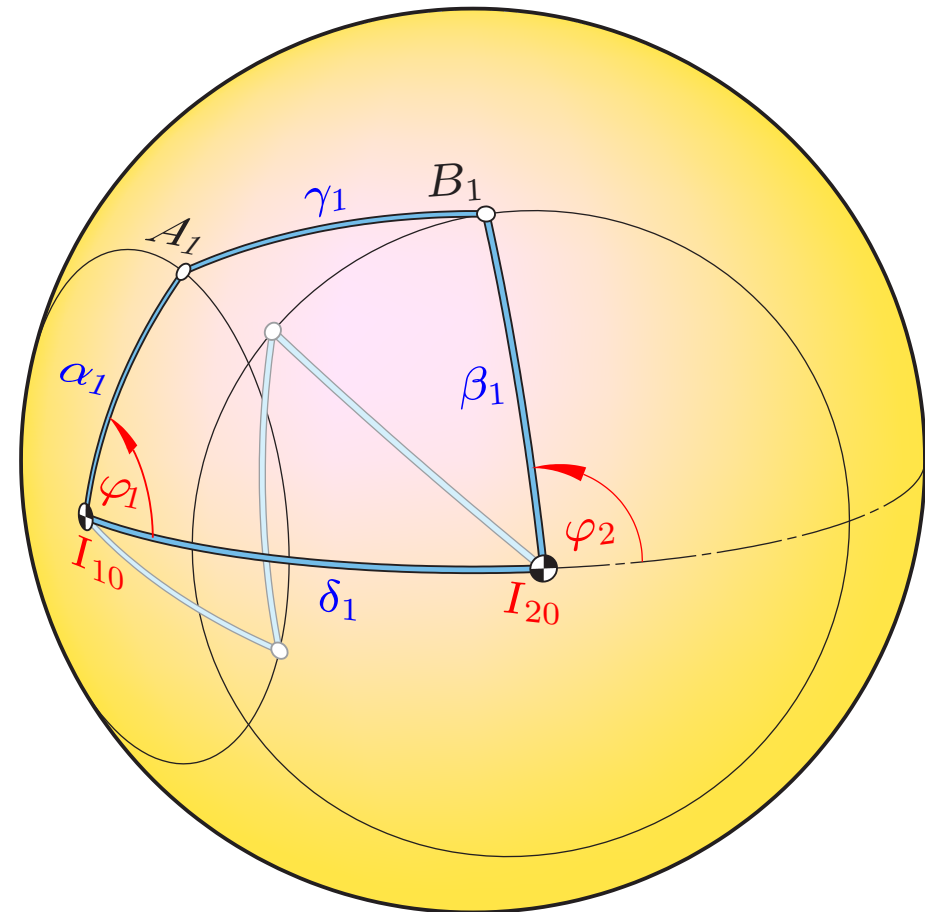
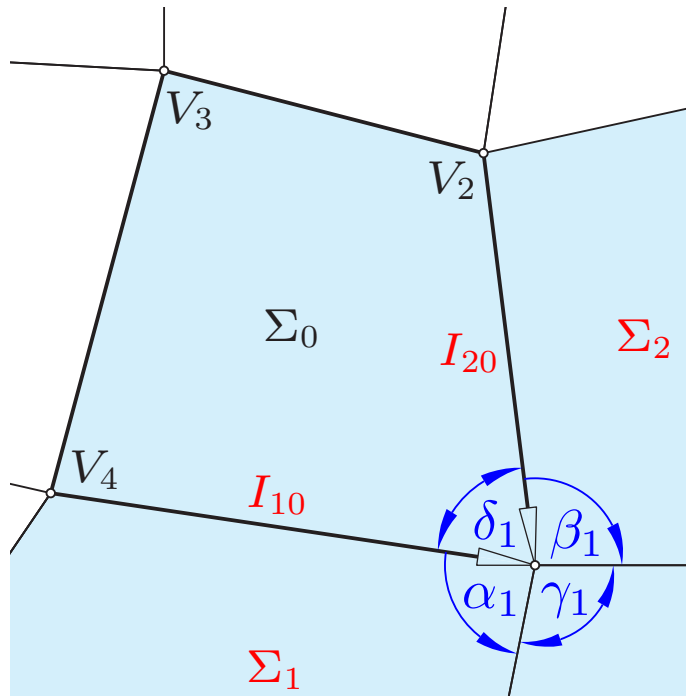
By $\overline{AB} = r\sqrt{2}$ and $\sphericalangle AOB = 90^\circ$ the bar $AB \in \Sigma_3$ performs a **spherical coupler motion** against Σ_0 . The length of the arms and of the frame links is $\pi/2$

This gives a flexible Kokotsakis mesh, e.g., for $n = 4$ and parallel axes I_{10} and I_{30} .

It is a special case of a flexible mesh detected by **GRAF and SAUER (1931)**.

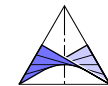


3. Transmission by one spherical coupler motion

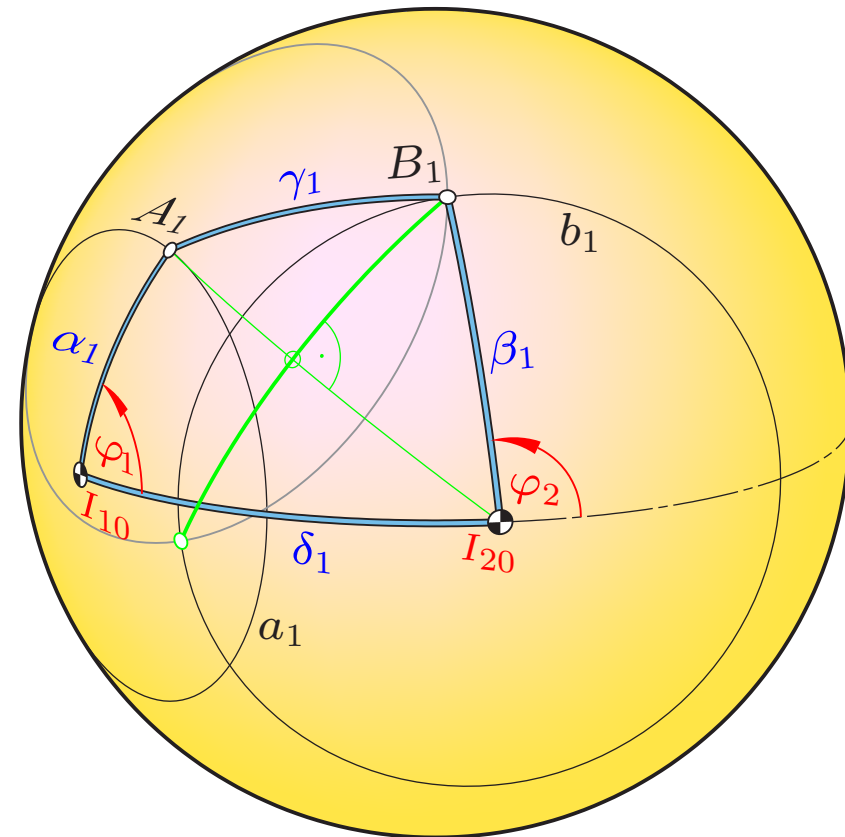
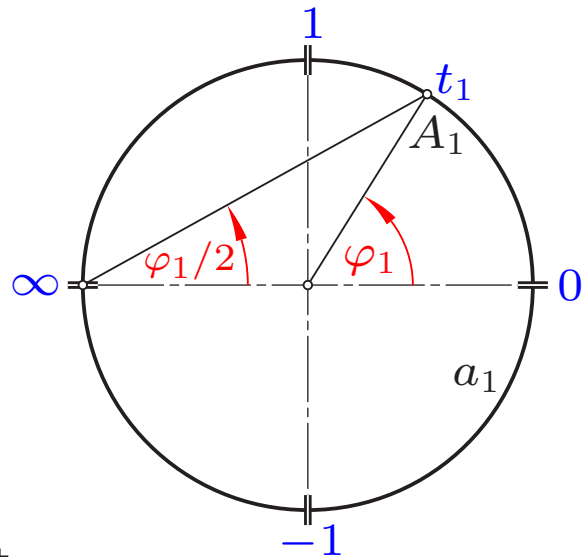


Coupler motion Σ_2/Σ_1 and its spherical image

$$0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < 180^\circ$$



3. Transmission by one spherical coupler motion

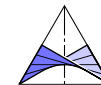


We set

$$t_1 := \tan \frac{\varphi_1}{2}, \quad t_2 := \tan \frac{\varphi_2}{2}.$$

t_1, t_2 are projective coordinates on the path circles a_1, b_1 of A_1 and B_1 , resp., and obtain

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0 \quad \text{with} \quad c_{ik} = f(\alpha_1, \dots, \delta_1)$$

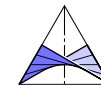
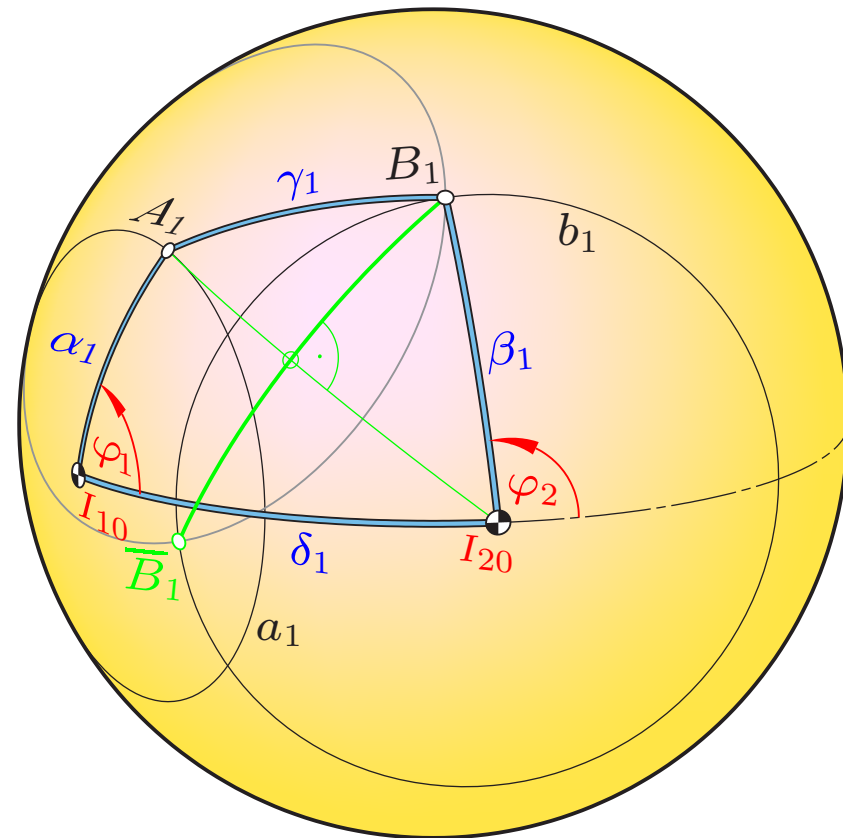


3. Transmission by one spherical coupler motion

The transmission by the coupler motion defines a 2-2-correspondance between the circles a_1 and b_1 :

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$

In general, the lines through the two points $B_1, \bar{B}_1 \in b_1$ corresponding to $A_1 \in a_1$ envelope a spherical conic.



3. Transmission by one spherical coupler motion

Coefficients in the biquadratic equation $c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$:

$$c_{22} = \sin \frac{\alpha_1 - \beta_1 + \gamma_1 + \delta_1}{2} \sin \frac{\alpha_1 - \beta_1 - \gamma_1 + \delta_1}{2},$$

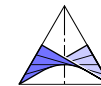
$$c_{20} = \sin \frac{\alpha_1 + \beta_1 + \gamma_1 + \delta_1}{2} \sin \frac{\alpha_1 + \beta_1 - \gamma_1 + \delta_1}{2},$$

$$c_{11} = -2 \sin \alpha_1 \sin \beta_1 \neq 0$$

$$c_{02} = \sin \frac{\alpha_1 + \beta_1 + \gamma_1 - \delta_1}{2} \sin \frac{\alpha_1 + \beta_1 - \gamma_1 - \delta_1}{2},$$

$$c_{00} = \sin \frac{\alpha_1 - \beta_1 + \gamma_1 - \delta_1}{2} \sin \frac{\alpha_1 - \beta_1 - \gamma_1 - \delta_1}{2}$$

Note that each vertex can be replaced by its antipode. E.g., replacing A_1 by \bar{A}_1 means replacing α_1 by $\alpha_1 - \pi$ and γ_1 by $\gamma_1 - \pi$.

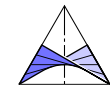


Particular cases of the transmission

The 2-2-correspondance between a_1 and b_1 splits into two projectivities \iff the quadrangle is a spherical isogram, i.e., $\beta_1 = \alpha_1$ and $\delta_1 = \gamma_1$ ($c_{00} = c_{22} = 0$).

In this case:

$$t \mapsto t_2 = \frac{\sin \alpha_1 \pm \sin \gamma_1}{\sin(\alpha_1 - \gamma_1)} t_1 \quad \text{with} \quad 0 \mapsto 0, \infty \mapsto \infty \quad \text{for} \quad \alpha_1 \neq \gamma_1, \pi - \gamma_1.$$



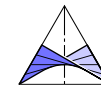
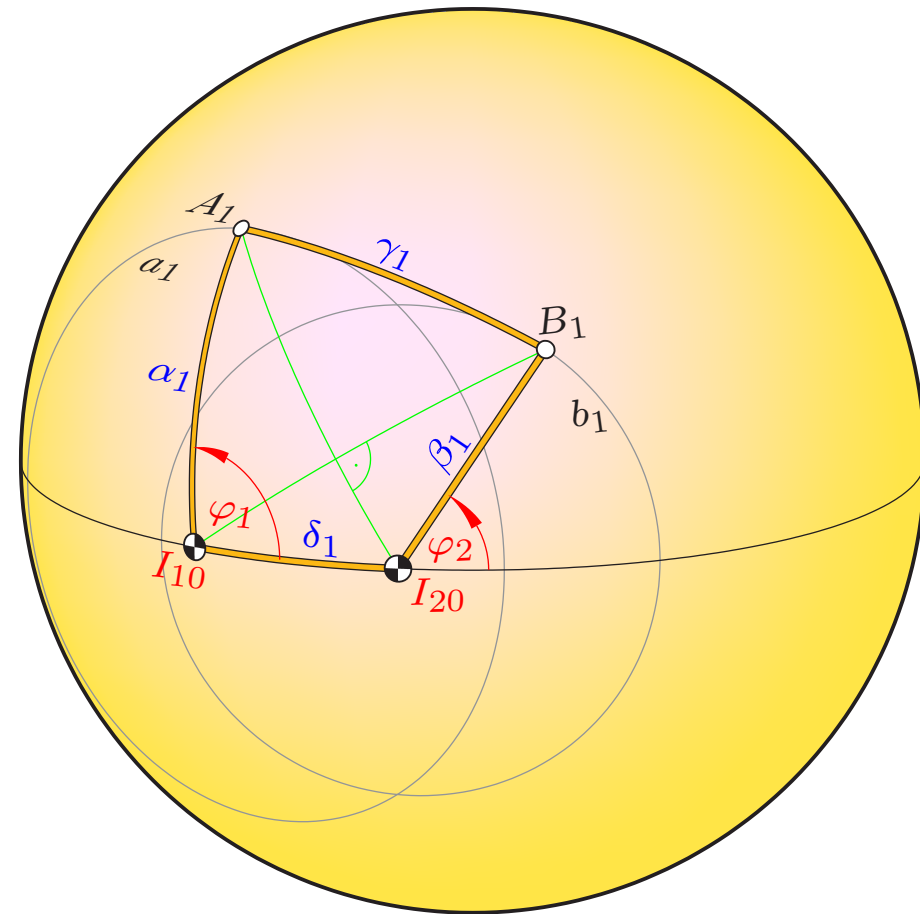
Particular cases of the transmission

Under the condition

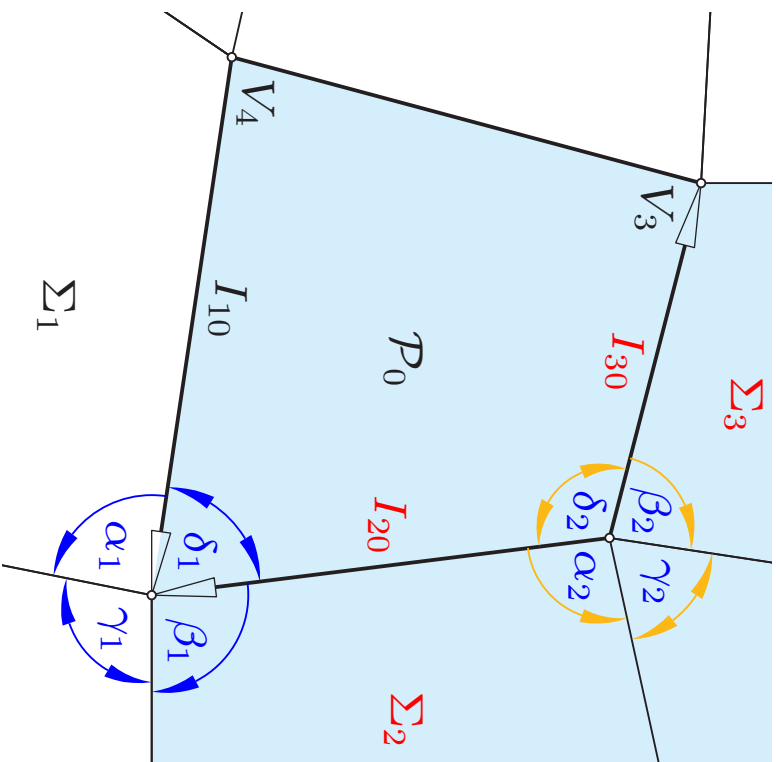
$$\cos \alpha_1 \cos \beta_1 = \cos \gamma_1 \cos \delta_1$$

(equivalent to $\det(c_{ik}) = 0$) each quadrangle has **orthogonal diagonals**.

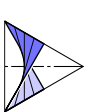
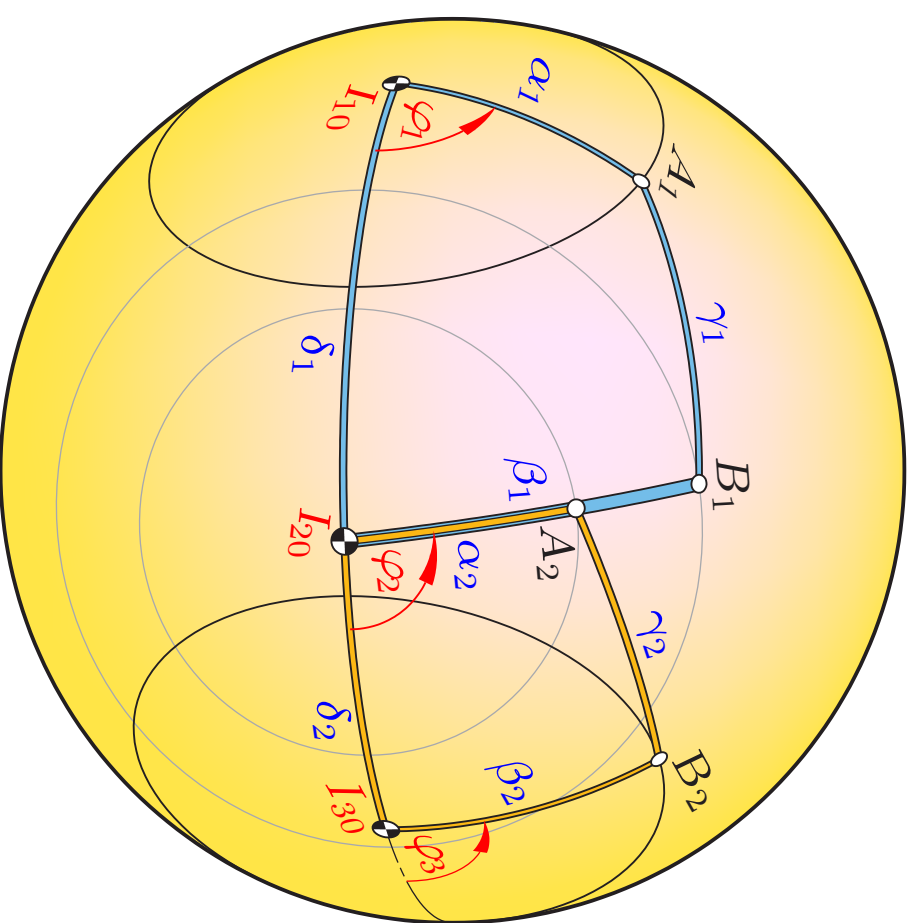
The 2-2-correspondance maps **pairs of points on a_1** aligned with B_0 onto **pairs of points on b_2** located on the orthogonal line through A_0 .



3. Transmission by two spherical coupler motions



coupler motions Σ_2/Σ_1 and Σ_3/Σ_1
and their spherical images



3. Transmission by two spherical coupler motions

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$

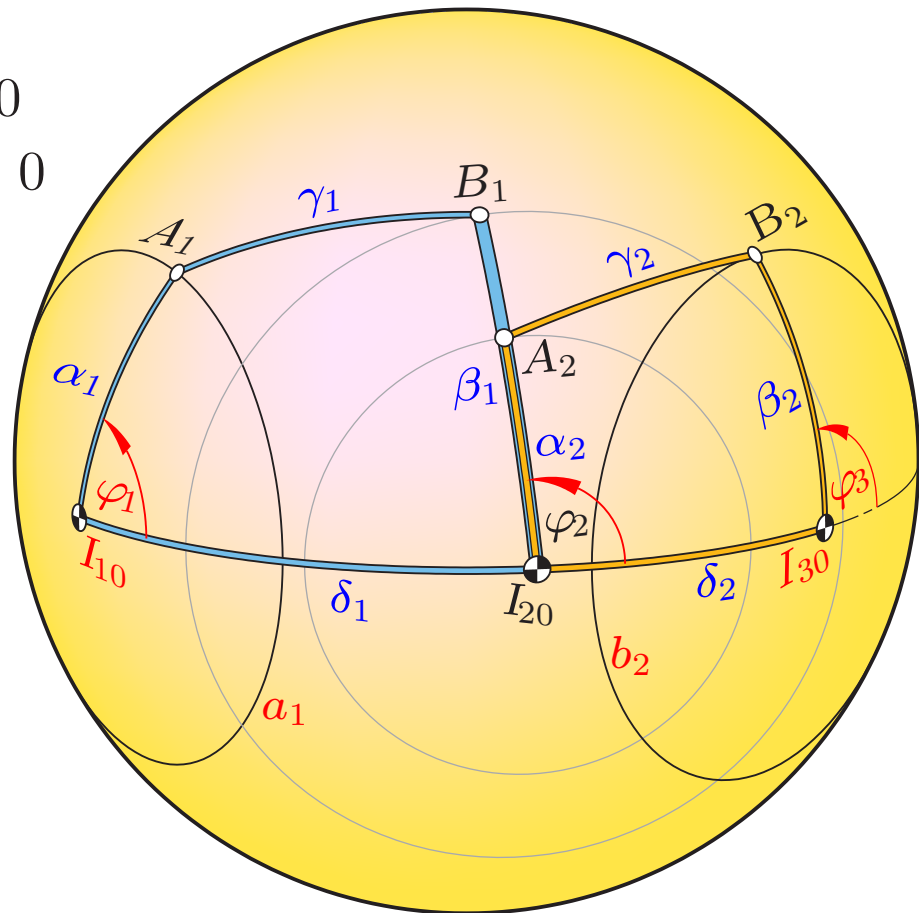
$$d_{22}t_1^2t_2^2 + d_{20}t_1^2 + d_{02}t_2^2 + d_{11}t_1t_2 + d_{00} = 0$$

The coupler transmissions are equivalent to these two bilinear equations.

We eliminate t_2 by computing the **resultant** with respect to t_2 . Thus we obtain a biquartic equation in

$$t_1 = \tan \frac{\varphi_1}{2} \text{ and } t_3 = \tan \frac{\varphi_3}{2},$$

i.e., a **4-4-correspondance** between the circles a_1 and b_2 .



Twofold decomposition of 4-4-correspondance

Continuous flexibility of a Kokotsakis mesh for $n = 4$ means:

The 4-4-correspondance or – in the reducible case – one component can be decomposed in two different ways.

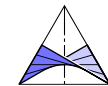
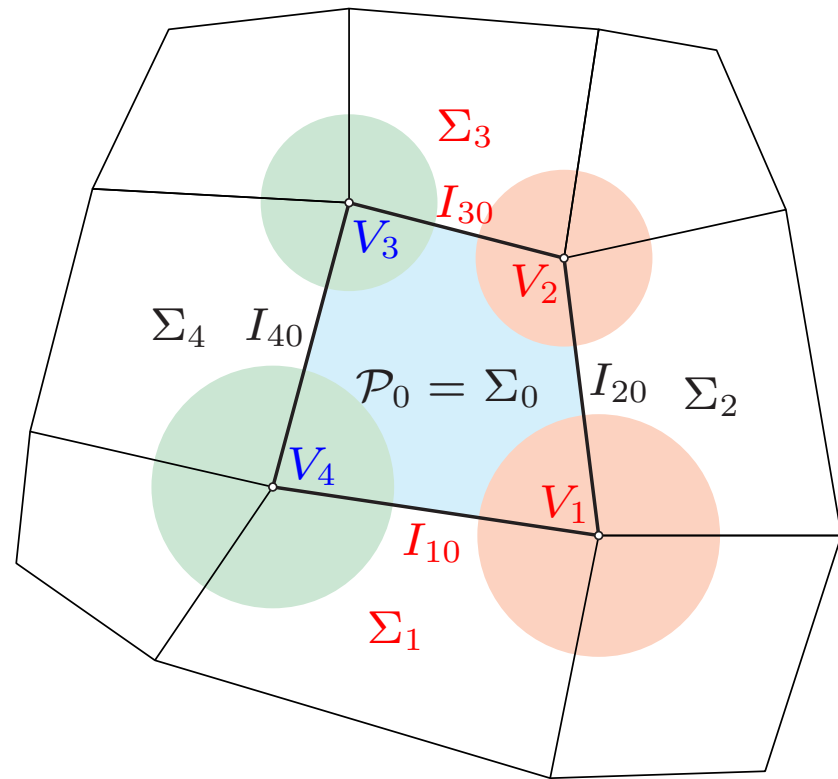
KOKOTSAKIS (GRAF, SAUER):

In the isogram case ($n \geq 4$)

$$\alpha_1 = \beta_1, \gamma_1 = \delta_1, \alpha_2 = \beta_2, \gamma_2 = \delta_2$$

the composition of two projectivities is a projectivity with $0 \mapsto 0$ and $\infty \mapsto \infty$.

⇒ Discrete Voss-surfaces admit continuous bendings.



Twofold decomposition of 4-4-correspondance

Under the conditions

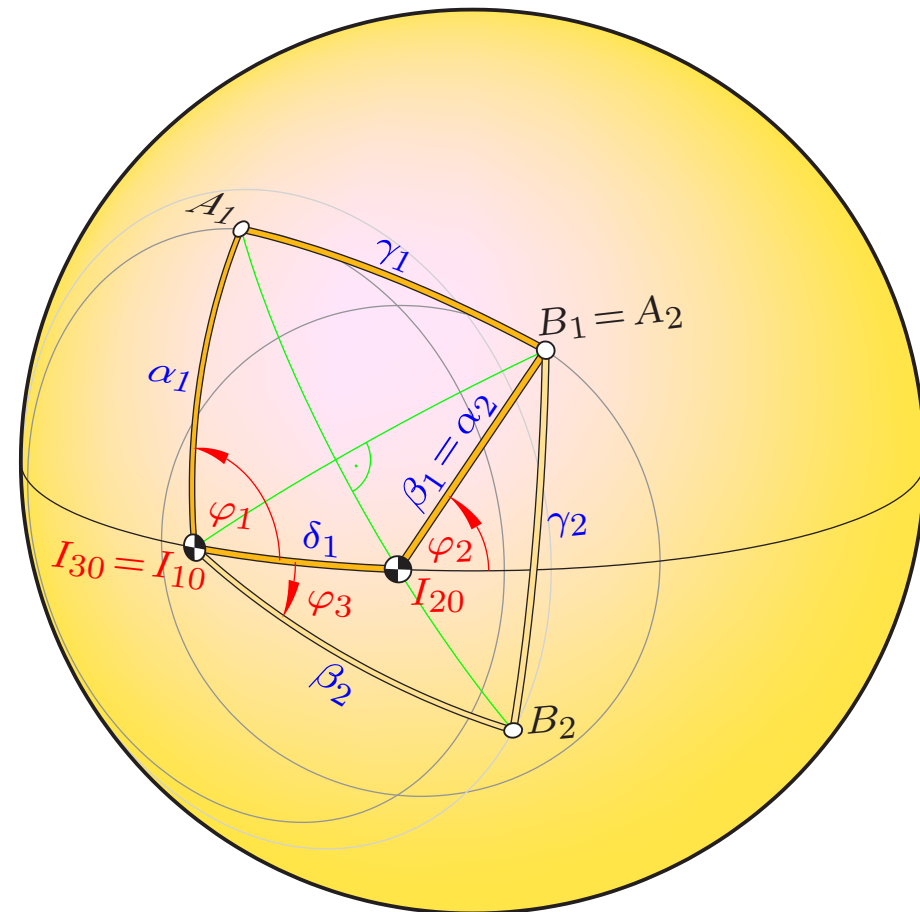
$$\cos \alpha_1 \cos \beta_1 = \cos \gamma_1 \cos \delta_1,$$

$$\cos \alpha_2 \cos \beta_2 = \cos \gamma_2 \cos \delta_2,$$

$$\alpha_2 = \beta_1, \quad \delta_2 = -\delta_1$$

both four-bars share the orthogonal diagonals.

Due to GRAF and SAUER (1931) there is a second decomposition of the same kind; all four-bars share one diagonal (spherical DIXON mechanism).



Twofold decomposition of 4-4-correspondance

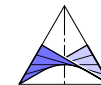
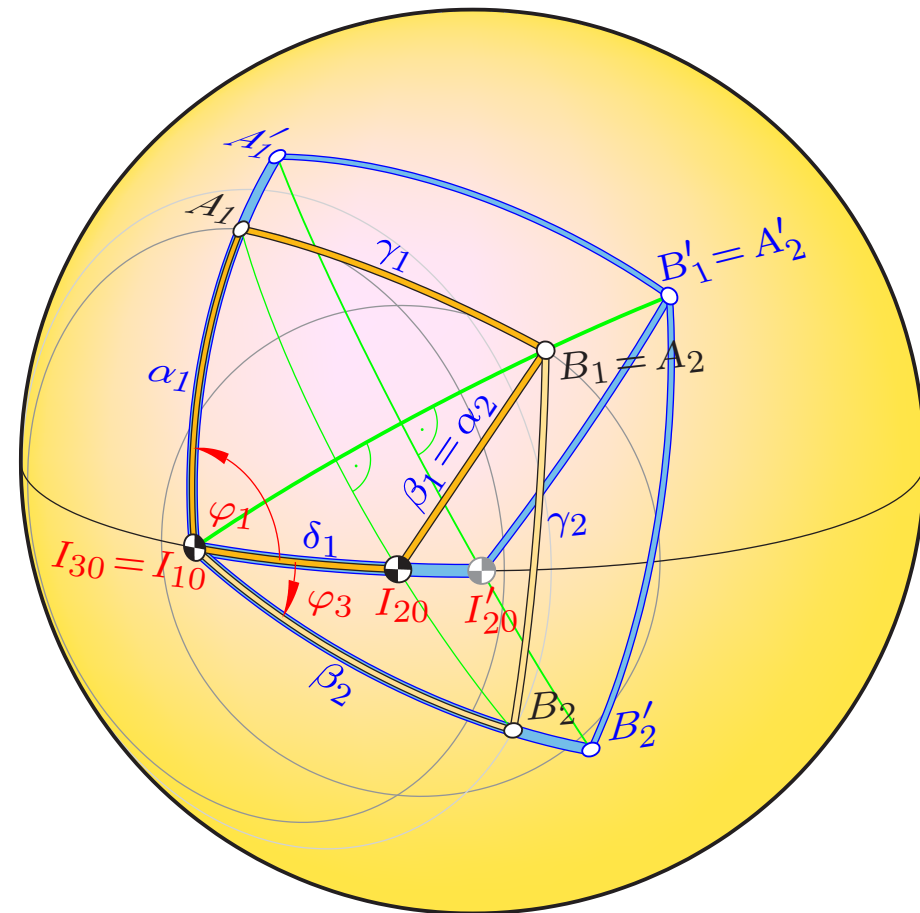
The 4-4-correspondance is the square of the 2-2-correspondance

$$c_{21}t_1^2t_3 + c_{12}t_1t_3^2 + c_{10}t_1 + c_{01}t_3 = 0$$

with coefficients depending on $\tan \alpha_1$, $\tan \delta_1$, $\tan \beta_2$, only.

The condition for the second decomposition reads

$$\begin{aligned} \tan \alpha'_1 : \tan \delta'_1 : \tan \beta'_2 \\ = \tan \alpha_1 : \tan \delta_1 : \tan \beta_2 . \end{aligned}$$



4. A new family of flexible Kokotsakis meshes

We focus on **opposite angles** in the spherical quadrangles. The spherical Cosine Theorem implies (c, s stand for sin and cos)

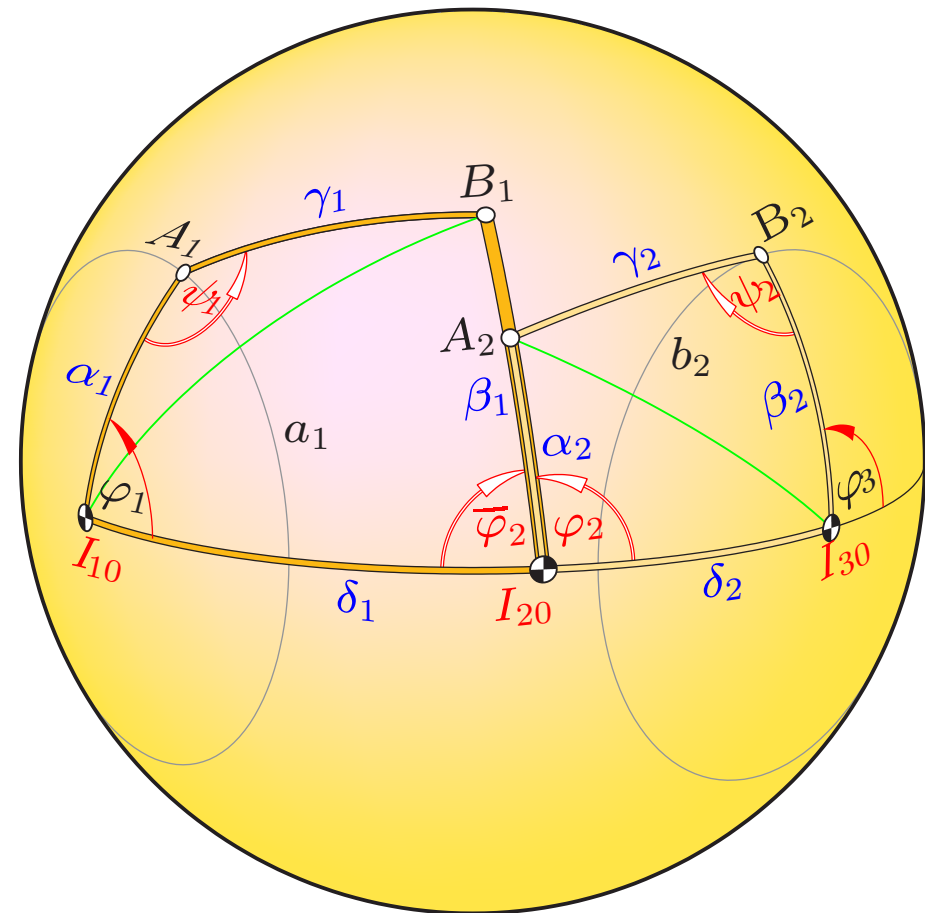
$$\begin{aligned}\cos I_{10}B_1 &= c\alpha_1c\gamma_1 + s\alpha_1c\gamma_1 c\psi_1 \\ &= c\beta_1c\delta_1 + s\beta_1c\delta_1 c\bar{\varphi}_2\end{aligned}$$

$$\begin{aligned}\cos I_{30}A_2 &= c\beta_2c\gamma_2 + s\beta_2c\gamma_2 c\psi_2 \\ &= c\alpha_2c\delta_2 + s\alpha_2c\delta_2 c\varphi_2\end{aligned}$$

\Rightarrow there are constants $k_i, l_i \in \mathbb{R}$

$$\cos \psi_1 = k_1 + l_1 \cos \bar{\varphi}_2,$$

$$\cos \psi_2 = k_2 + l_2 \cos \varphi_2.$$



4. A new family of flexible Kokotsakis meshes

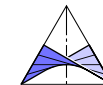
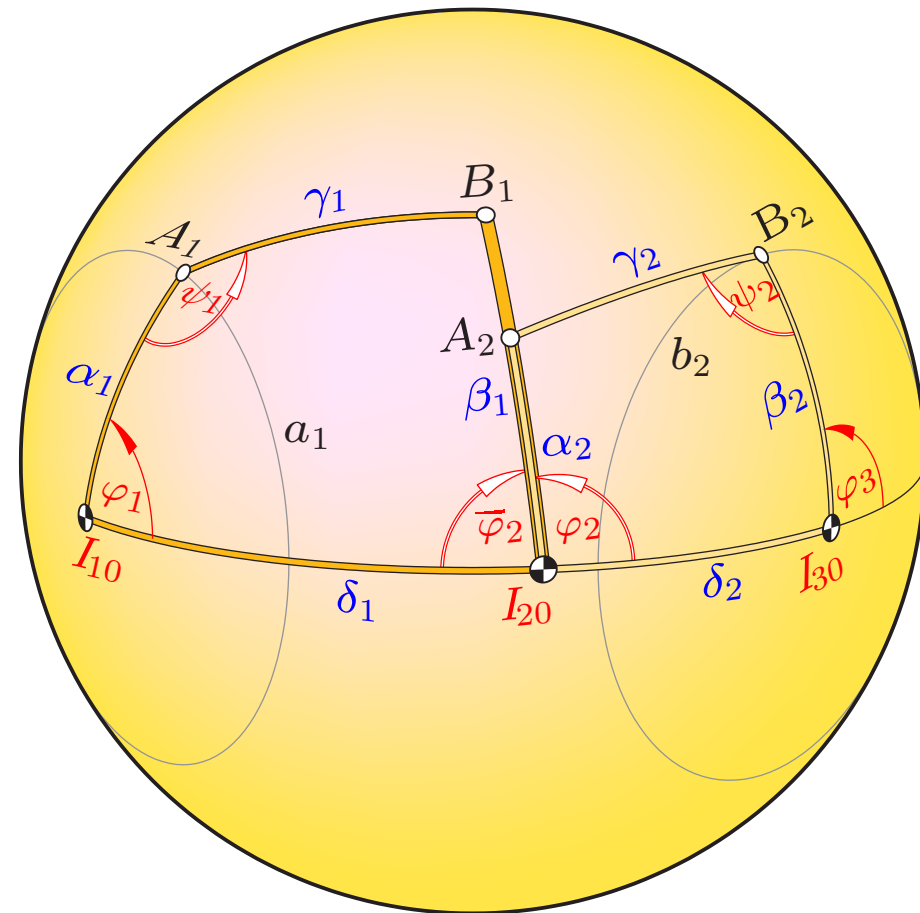
$$\cos \psi_1 = k_1 + l_1 \cos \bar{\varphi}_2$$

$$\cos \psi_2 = k_2 + l_2 \cos \varphi_2$$

When the equivalence

$$\varphi_2 - \bar{\varphi}_2 = \pi \iff \psi_1 - \psi_2 = \pi$$

holds, i.e., for $(k_2, l_2) = (\mp k_1, \pm l_1)$, we can **reflect** the quadrangles such that $I_{10}A_1$ and $I_{30}B_2$ become the frame links. When their sum of lengths equals $\delta_1 + \delta_2$, this gives a new composition with the same φ_1 and φ_3 .

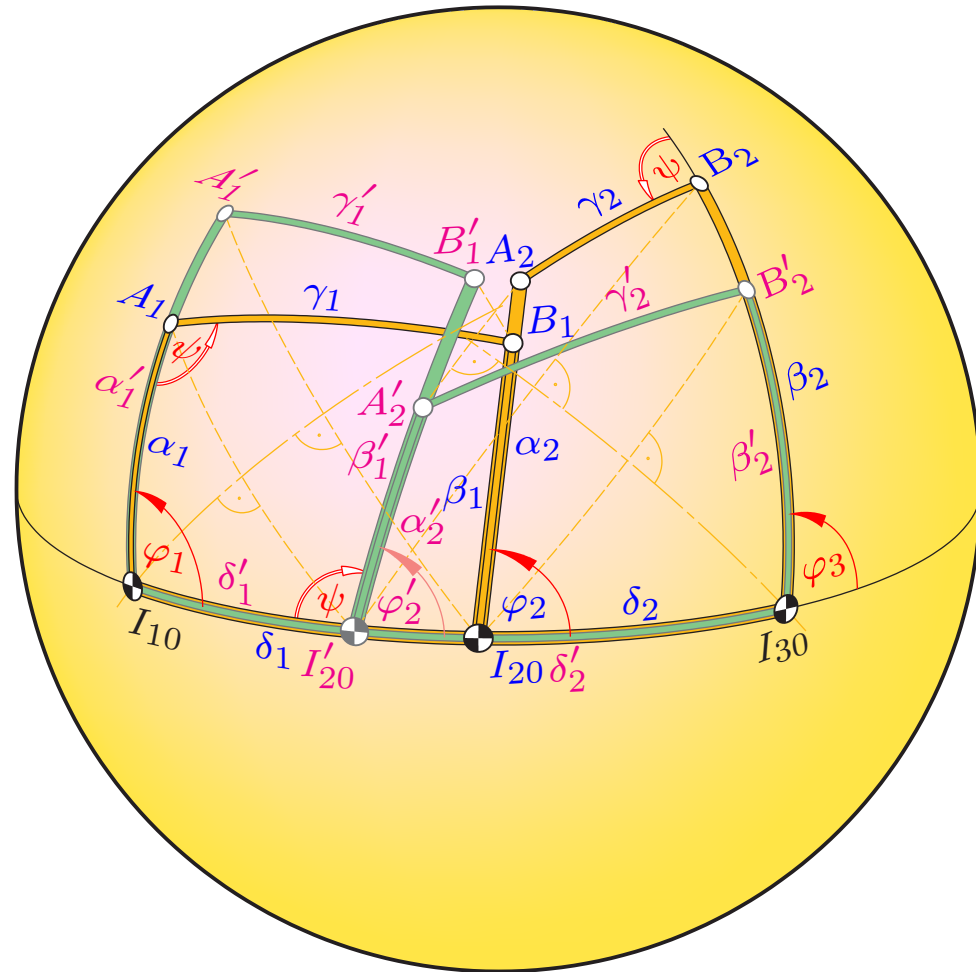


4. A new family of flexible Kokotsakis meshes

$$(k_2, l_2) = (\mp k_1, \pm l_1) \iff$$

Theorem: Under the conditions
 $\alpha_1 + \beta_2 = \delta_1 + \delta_2$ and
 $\pm s\alpha_1 s\gamma_1 : s\beta_2 s\gamma_2 = s\beta_1 s\delta_1 : s\alpha_2 s\delta_2$
 $= (c\alpha_1 c\gamma_1 - c\beta_1 c\delta_1) : (c\alpha_2 c\delta_2 - c\beta_2 c\gamma_2)$
 the transmission $\varphi_1 \rightarrow \varphi_3$ can be decomposed in a second way in a composition of two coupler motions.

One can choose, e.g., $\alpha_1, \beta_1, \gamma_1, \delta_1$ and β_2 freely. Then all other parameters are determined.



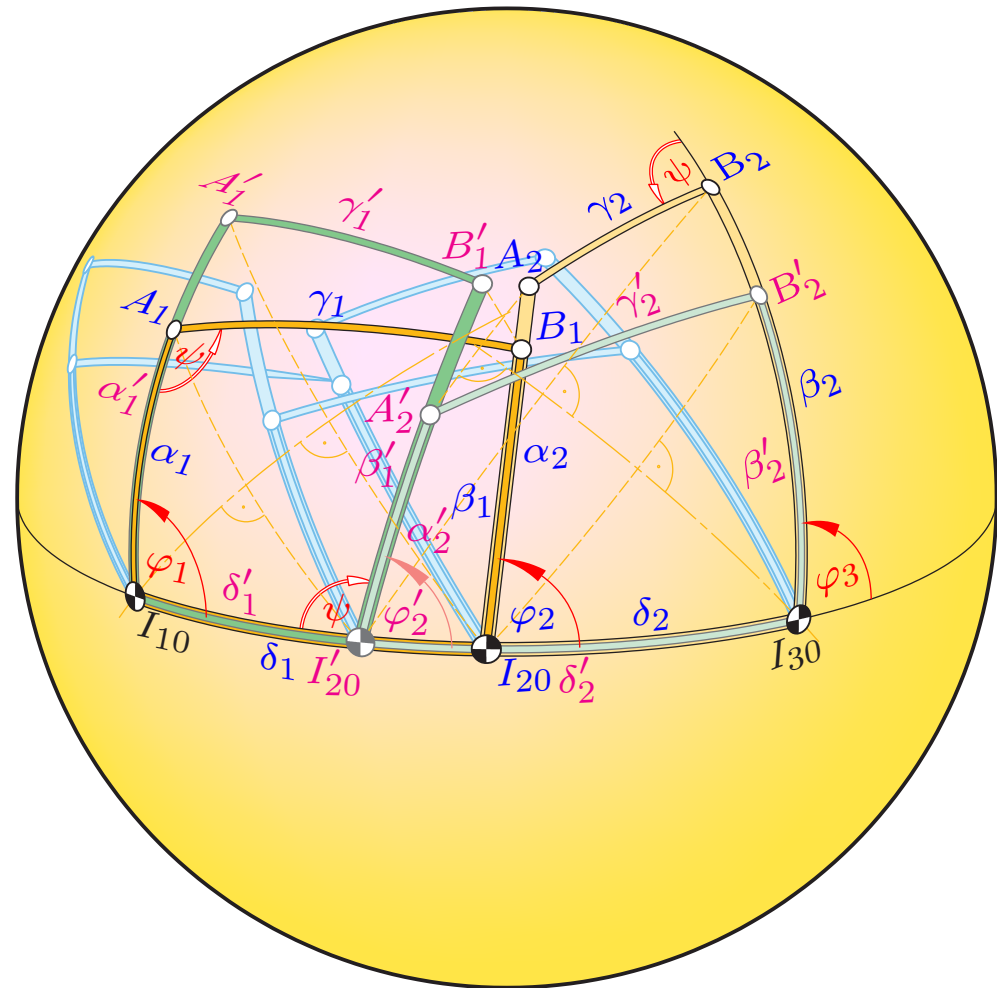
4. A new family of flexible Kokotsakis meshes

Additionally, for given φ_1 only two of the four possible φ_3 guarantee the correct signs of ψ_1 and ψ_2 .

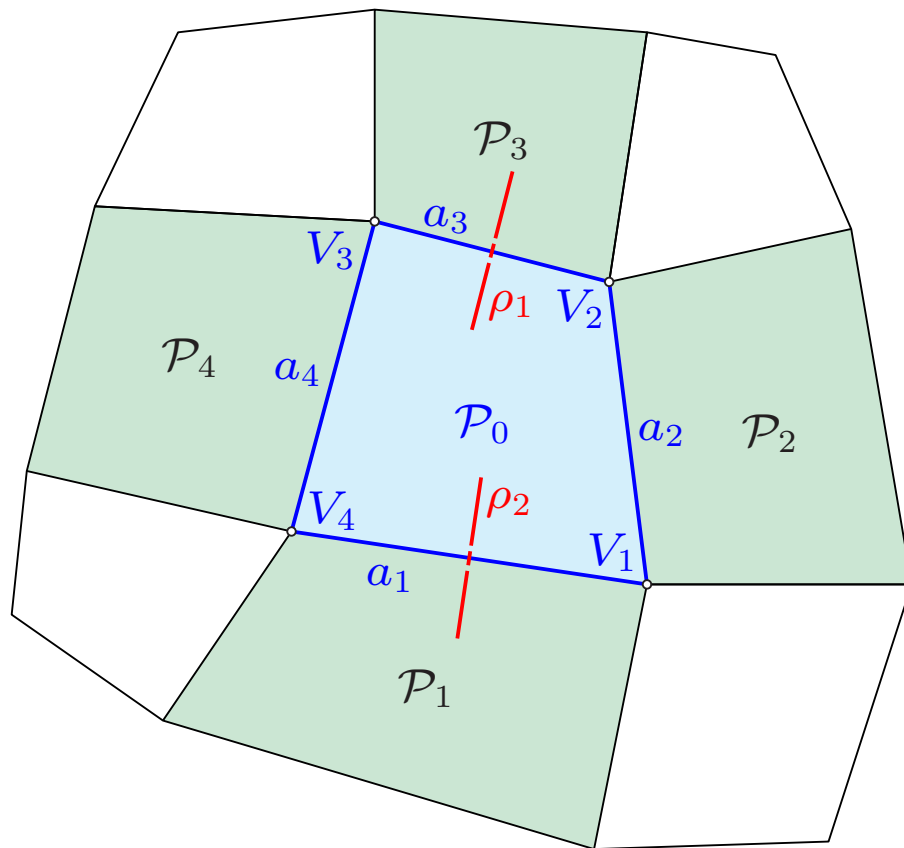
Hence, the two 4-4-correspondances share a 2-2-correspondance, which implies:

Theorem: Under the conditions above the 4-4-correspondance is reducible.

The above mentioned tessellation as well as the Cardan example are particular cases.

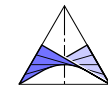


4. A new family of flexible Kokotsakis meshes



This implies for the corresponding Kokotsakis-mesh: The pyramids at the vertices are **pairwise congruent**.

The **half-rotation** ρ_1 maps the pyramid with vertex V_1 onto the pyramid with vertex V_4 . Analogously, ρ_2 maps the pyramid at V_2 onto the pyramid at V_4 .

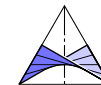
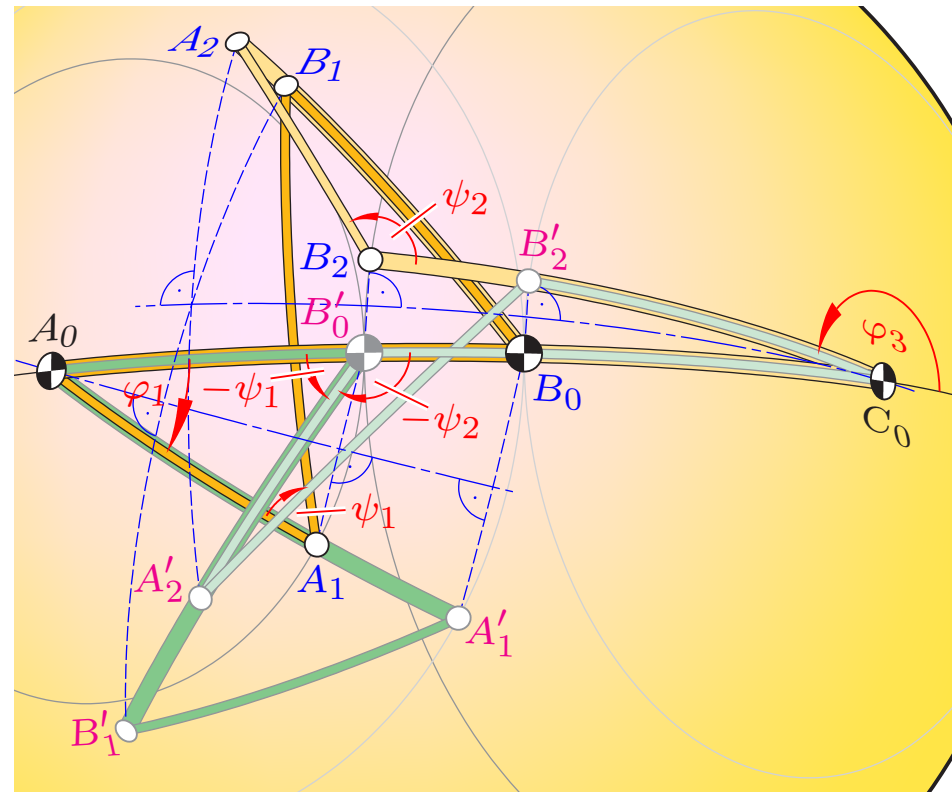


4. A new family of flexible Kokotsakis meshes

Data:

α	β	γ	δ
first decomposition:			
32.00°	35.90°	45.50°	47.00°
43.92°	54.00°	28.39°	39.00°
second decomposition:			
47.00°	45.50°	35.90°	32.00°
28.39°	39.00°	43.92°	54.00°

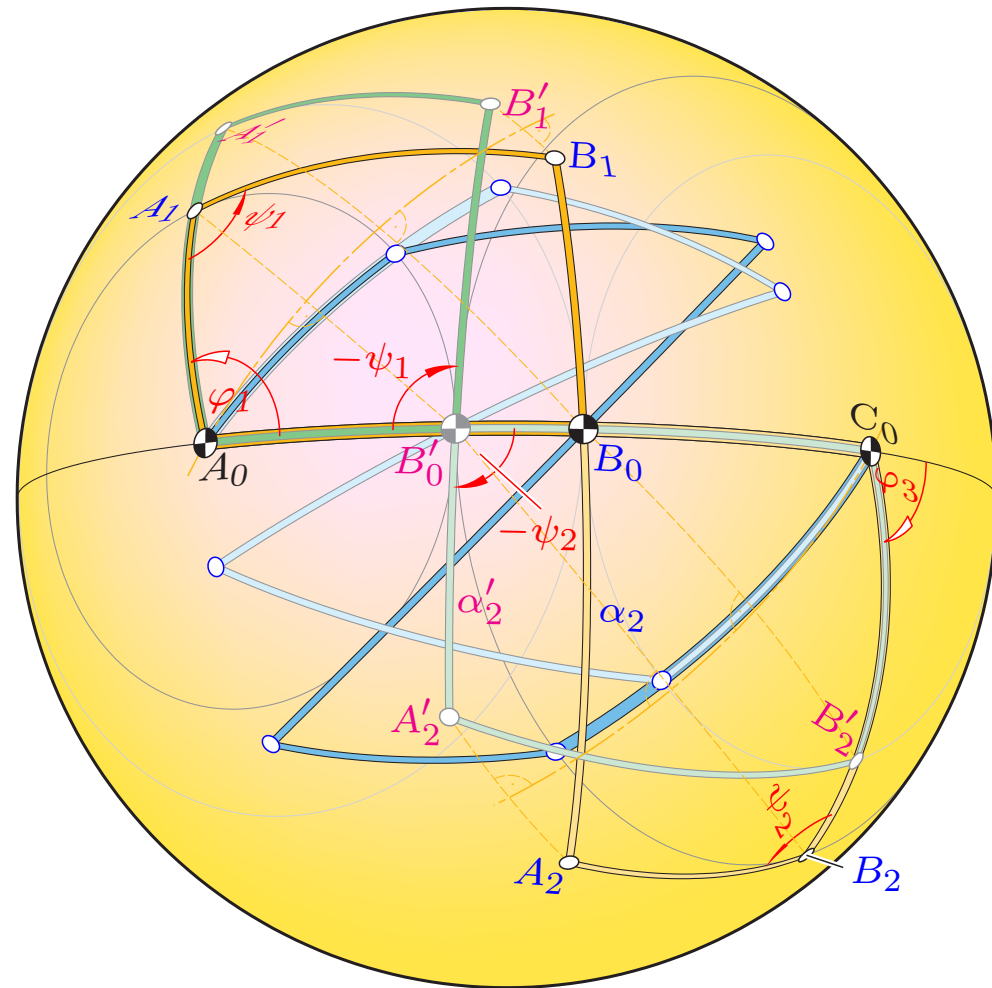
driving angle: $\varphi_1 = -35.50^\circ$,
 output angles, 1st decomposition: $\varphi_3 = 168.94^\circ, 121.73^\circ, -79.41^\circ, -113.01^\circ$
 output angle, 2nd decomposition: $\varphi_3 = 168.94^\circ, -121.73^\circ, -79.41^\circ, 113.01^\circ$.



4. A new family of flexible Kokotsakis meshes

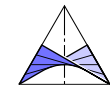
Theorem: Under the above conditions the shared 2-2-correspondance between φ_1 and φ_3 equals that of another coupler motion. However, the length of the frame link differs from $\delta_1 + \delta_2$.

Therefore we can compose with a particular third four-bar such that also this product has a second decomposition.



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