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REMARKS ON MIURA-ORI, A JAPANESE FOLDING METHOD

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Abstract: *Miura-ori is a Japanese folding technique named after Prof. Koryo Miura, The University of Tokyo. It is used for solar panels because it can be unfolded into its rectangular shape by pulling on one corner only. On the other hand it is used as kernel to stiffen sandwich structures. In this paper some insight will be given into the geometric structure of this folding method combined with an outlook to analogues and generalizations.*
Key words: *Miura-ori, Kokotsakis meshes, flexible polyedra.*

1. THE FLEXIBILITY OF MIURA-ORI

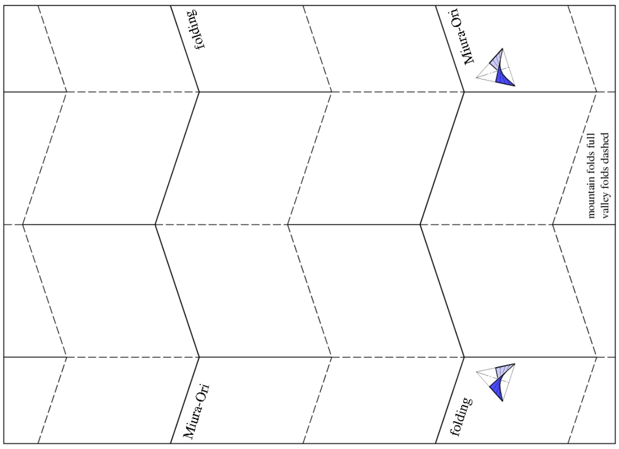

Fig. 1. Unfolded miura-ori; dashs are valley folds, full lines are mountain folds

Let us analyze the process of folding the sheet of paper depicted in Fig. 1 with given valley and mountain folds.
We start with two coplanar parallelograms with aligned upper and lower sides (Fig. 2). Then we rotate the right parallelogram against the left one about the common side through the angle $2\delta \neq 0^0, \pm 180^0$.

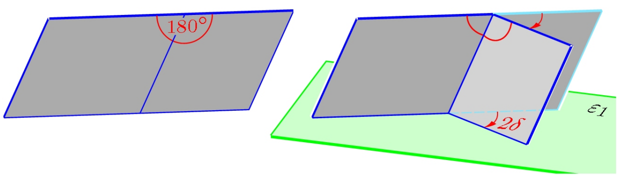

Fig. 2. We rotate the right parallelogram with respect to the left one

Then the lower sides span a plane ε_1 and the upper sides span a plane ε_2 parallel to ε_1. Now we extend the two parallelograms to a zig-zag strip by adding alternately parallelograms translatory congruent to the left or to the right initial parallelogram. After this the complete strip has its upper zig-zag boundary still placed in ε_1 and the lower one in ε_2 (see Fig. 3).

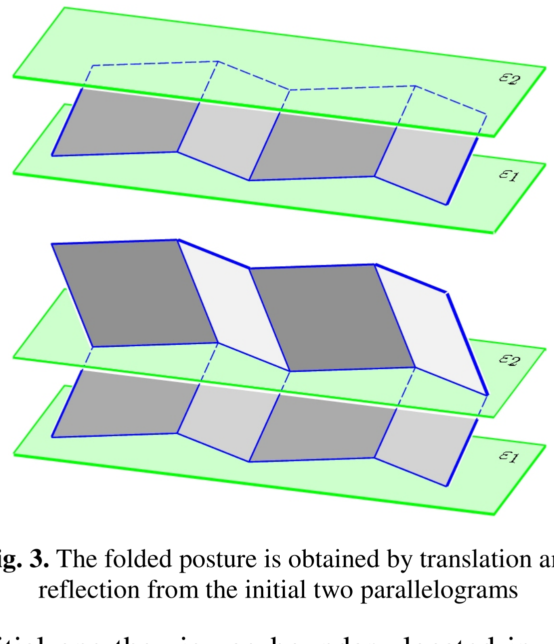

Fig. 3. The folded posture is obtained by translation and reflection from the initial two parallelograms

initial one the zig-zag boundary located in ε_2, and it does not restrict the movement when the bending angle 2δ varies continously. When δ tends to 0^0, the two strips become coplanar and remain connected as at the meeting point of four parallelograms the sum of interior angles

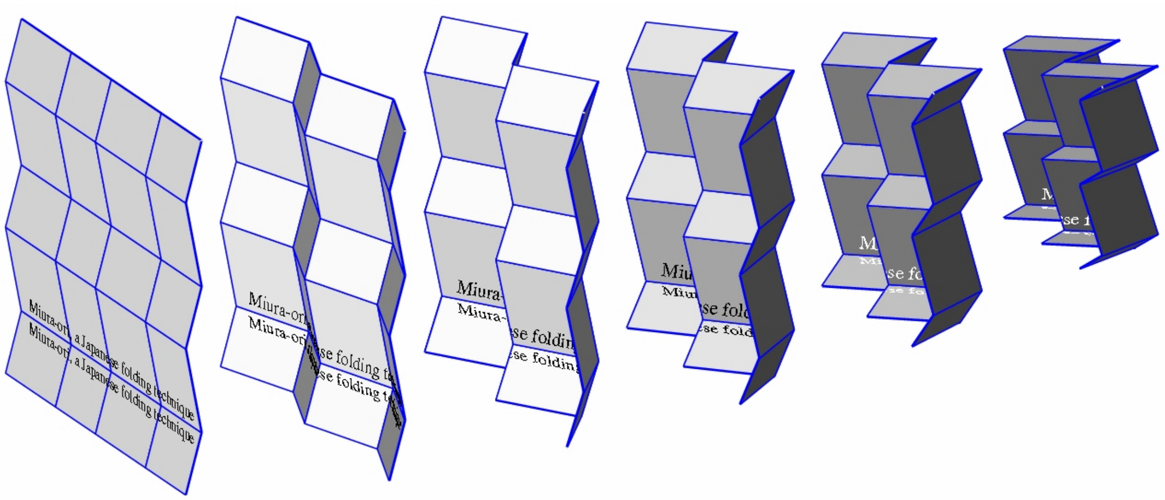

Fig. 4. Snapshots of the folding procedure of miura-ori

equals 360^0. This means that at these polyhedral vertices we have a vanishing *Gaussian curvature*, which is defined the "angle deficit" 360^0 minus the sum of adjacent interior angles (see, e.g., [1], p. 303).

After iterated reflection in planes ε_i parallel to ε_1 or after translation orthogonal to ε_1 the complete miura folding is obtained as depicted in Fig. 4.

Remarks: 1. The folding is still flexible when the angle between the upper sides of the two inital parallelograms differs from the choice 180^0 of Fig. 1. However, then the Gaussian curvature of the vertices is $\neq 0^0$. There would be no coplanar stretched position for $\delta = 0^0$. Any interior parallelogram together with its eight neighbor-parallelograms constitutes an example of a flexible *Kokotsakis mesh* (see [2,3]).

2. More general, the first zig-zag strip between ε_1 and ε_2 can be combined with another zig-zag strip placed between parallel planes ε_2 and ε_3, provided the boundaries in ε_2 are identical.

This can be iterated so that the parallelograms in different strips are incongruent. In this way again examples of flexible Kokotsakis meshes are obtained.

2. THE NET OF EDGES AT MIURA-ORI

The edges of miura-ori constitute two sets of folds on the flexible polygonal structure. For better orientation we assume that the planes ε_1, ε_2 are horizontal. Then the zig-zag lines placed in the horizontal planes are the lines of the first set and called *horizontal*. They are aligned in the flat position (Fig. 1) and the compounds of alternate valley and mountain folds.

The transversal folds, called the *vertical* zig-zag lines, are either pure valley folds or mountain folds. The segments of the vertical lines can be obtained from their initial part in the starting strip by reflections in the horizontal planes ε_1, ε_2, ... Hence these vertical zig-zag lines are located in vertical planes.

For the sake of simplicity we assume that all edges of our folding have unit length. Now we keep the planes of the horizontal and vertical fold with crossing point V fixed and concentrate on one parallelogram P_1 of the four parallelograms $P_1 ... P_4$ meeting at V: Two sides of P_1 can rotate within the fixed planes (see Fig. 5) such that the included interior angle, say α, remains constant at V. A second parallelogram P_2 is the mirror of P_1 with respect to the horizontal plane ε_2. It has the same interior angle α at V and moves like P_1. We may assume $\alpha < 90^0$.

Let us we elongate the horizontal sides of the other two parallelograms P_3 and P_4 (with interior angle $180^0 - \alpha$) by unit length beyond the fixed vertical plane. This gives two additional parallelograms P_3^* and P_4^* with interior angle α at V (Fig. 5). Each shares a 'vertical' edge with P_1 or P_2. Hence, P_3^* and P_4^* are the mirrors of P_1 and P_2 with respect to the fixed vertical plane.

This reveals a hidden local symmetry of miura-ori: When at each vertex V two adjacent parallelograms P_3, P_4 with congruent interior angles at V are replaced by their 'horizontal elongations' P_3^* and P_4^*, respectively, we obtain a pyramide consisting of four congruent parallelograms P_1, P_2, P_3^*, P_4^* with apex V. This pyramide flexes such that it remains symmetrical with respect to the planes spanned by the horizontal and vertical folds passing trough V.

2.1 Angles

Next we study the relations between angles: We use a coordinate frame with origin V, with the x- and y-axis in the horizonal plane ε_2, and with the $[xy]$-plane spanned by the vertical fold passing through V. Let 2φ and 2ψ be the bending angles between consecutive segments of the horizontal and vertical folds (red lines in Fig. 5), respectively. Thus, the sides of P_1 have the direction vectors

$$\mathbf{h} = \begin{pmatrix} \sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ \sin\psi \\ \cos\psi \end{pmatrix}$$

with the constant dot product

$$\mathbf{v}\cdot\mathbf{h} = \cos\varphi\,\sin\psi = \cos\alpha \qquad (1)$$

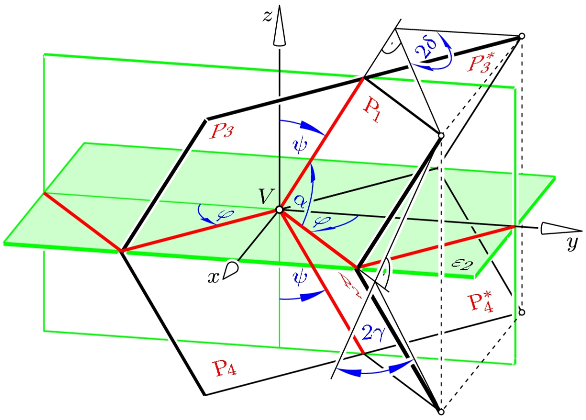

Fig. 5. There is a hidden local symmetry at each vertex

In the stretched position the parallelogram P_1 is located in the vertical $[yz]$-plane; the corresponding (half) bending angles are $\varphi = 0^0$ and $\psi = 90^0 - \alpha$. The other limit is the totally folded position with P_1 in the $[xy]$-plane, with $\varphi = \alpha$, and $\psi = 90^0$. In order to obtain formulas for the dihedral angles 2γ and 2δ along edges of the horizontal and vertical folds (see Fig. 5) we need the unit vector

$$\mathbf{n} = \frac{1}{\sin\alpha}(\mathbf{h}\times\mathbf{v}) = \frac{1}{\sin\alpha}\begin{pmatrix} \cos\varphi\cos\psi \\ -\sin\varphi\cos\psi \\ \sin\varphi\sin\psi \end{pmatrix}$$

perpendicular to the plane of P_1. Its dot products with the unit vectors along the x- and z-axis give

$$\begin{cases} \cos\delta\sin\alpha = \cos\varphi\cos\psi \\ \sin\gamma\sin\alpha = \sin\varphi\sin\psi \end{cases} \qquad (2)$$

We have $\delta = \gamma = 0^0$ in the stretched position and $\delta = \gamma = 90^0$ when miura-ori is completely folded.

Remark: Miura-ori admits more flexions than the one-parameter bending explained above. It is trivial to bend the stretched position about its (aligned) horizontal folds independently from each other. In addition, one can fold some adjacent horizontal strips one behind the other and treat them like one single strip at the one-parameter miura-ori as mentioned above.

3. A FLEXIBLE TESSELATION

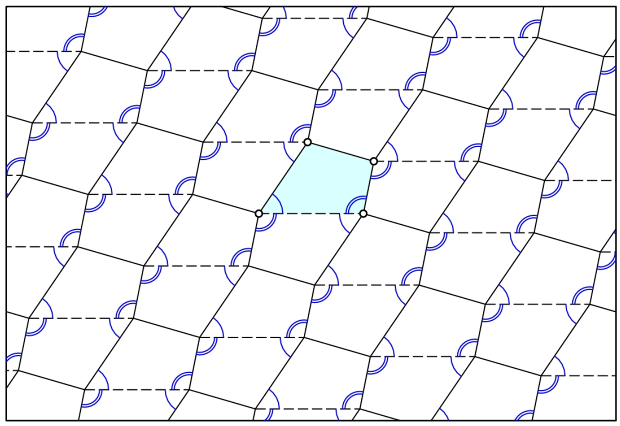

Fig. 6. The tesselation with any plane convex quadrangles gives also a flexible polyhedral structure

Among several generalizations of miura-ori there is one remarkable case which dates back zu Kokotsakis [3, p. 647]: Take any arbitrary plane convex quadrangle. By iterated 180^0-rotations about the midpoints of the sides we obtain a wellknown regular tesselation of the whole plane (Fig. 6). If the quadrangles are seen as planar faces of a polyhedral structure with an initial flat position, but changeable dihedral angles, then this polyhedron is flexible.

Proof: First we extract four pairwise congruent faces $P_1,\ldots,P_4$ adjacent to the vertex V_1 from our

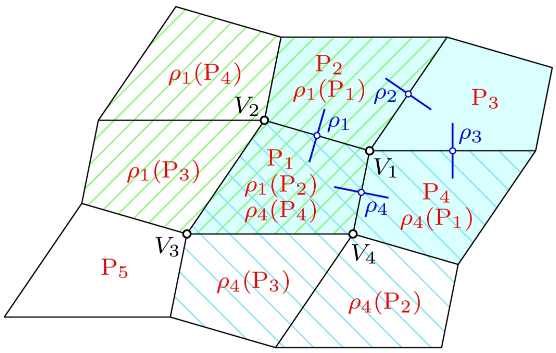

Fig. 7. How to continue the flexion of one pyramide to the whole structure

tesselation (note the shaded area in Fig. 7). These faces form a four-sided pyramide which is flexible, provided the fundamental quadrangle is convex. We start with any nonplanar flexion.

For any pair $(P_1, P_2), \ldots, (P_4, P_1)$ of neighbouring faces there is a respective 180^0-rotation ρ_1, …, ρ_4 which swaps the two faces. So, e.g., $P_2 = \rho_1(P_1)$ and $P_1 = \rho_1(P_2)$. The axis of ρ_1 (see Fig. 7) is perpendicular to the common edge V_1V_2, and it is located in the plane which bisects the dihedral angle between P_1 and P_2.

After applying all four 180^0-rotations consecutively in ascending order to the quadrangle P_1, this is mapped via P_2, P_3, P_4 onto itself, hence $\rho_1\,\rho_2\,\rho_3\,\rho_4 = \mathrm{id}$. Because of $\rho_i^{-1} = \rho_i$ we obtain

$$\rho_4\,\rho_3 = \rho_1\,\rho_2\,. \tag{3}$$

After that we extend this flexible structure stepwise by adding congruent copies of the initial pyramide without restricting the flexibility: The rotation ρ_1 exchanges P_1 with P_2 and transforms the pyramide with vertex V_1 into a congruent copy with vertex V_2 sharing two faces with its preimage. Analogously, ρ_4 generates a pyramide with vertex V_4 including the faces P_1 and P_4.

Finally there are two ways to generate a pyramide with vertex V_3 (see Fig. 7). Either, we transform ρ_2 with ρ_1 and use $\rho_1\,\rho_2\,\rho_1$, which swaps V_2 and V_3. Or we proceed with $\rho_4\,\rho_3\,\rho_4$, which exchanges V_4 and V_3. Hence, the product $(\rho_1\,\rho_2\,\rho_1)\rho_1$ as well as $(\rho_4\,\rho_3\,\rho_4)\rho_4$ maps the original paramide into a pyramide with vertex V_3. Fortunately, both displacements are equal by (3), and we get $\rho_1\,\rho_2 = \rho_4\,\rho_3$: $P_3 \mapsto P_1$, $P_2 \mapsto \rho_1(P_3)$, $P_4 \mapsto \rho_4(P_3)$, and $P_1 \mapsto P_5$. Hence each compound of 3×3 quadrangles like that schematically displayed in Fig. 7, is flexible.

4. CONCLUSION

We presented two examples of flexible polyhedral structures which can be produced from a sheet of paper. The proofs for their continuous flexibility are given by pure geometric reasoning thus demonstrating the power of this kind of argumentation.

5. REFERENCES

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BEMERKUNGEN ZU MIURA-ORI, EINER JAPANISCHEN FALTTECHNIK:

Abstract: *Miura-ori ist eine Japanische Falttechnik, benannt nach Prof. Koryo Miura von der University of Tokyo. Diese Technik wird z.B. in der Satellitentechnik zum Falten der Sonnensegel verwendet, denn diese lassen sich entfalten, indem lediglich an einer Ecke angezogen wird. Außerdem wird diese Faltung in der Leichtbautechnik eingesetzt als Kern zum Versteifen von Platten in Sandwich-Bauweise. Ziel dieses Beitrages ist eine geometrische Analyse von Miura-ori und gewisser Verallgemeinerungen.*

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