THE DESIGN OF THE NEW SUN-REFLECTION-DIAL IN HEILIGENKREUZ/AUSTRIA

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Abstract

A few years ago, the author was involved in the design of a sundial in connection with a big mosaic-work on a cylindrical wall. The original plan of the artist was based on the shadow of a slim vertical tower; but this was not realizable for geometrical reasons. The only solution was to use suitable stripes on the East- and Westface of this tower as a mirror such that reflected sunlight became suitable for a sundial.

In the lecture the geometric background for this sun-reflection-dial, which is unique in Austria, will be analyzed. This is also a good opportunity to explain why sundials cannot show the exact time all over the year.

Key words: Sundial, reflection, Equation of Time

1. INTRODUCTION

In the years 2011-12 the author came in contact with the *„Verein Moderner Sakralbau"*, an Austrian organization which promotes modern art for Christian churches. This organization planned a monument in the area of the Heiligenkreuz monastry**[2]**. This monument (see Fig. 4, left) should consist of a big mosaic-work (8 x 3.5 m) on a

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[2] a Cistercian monastery, approx. 25 km south-west of Vienna, the eldest continuously occupied Cistercian monastery in the world. It is also famous because of its Gregorian Chant. <http://www.stift-heiligenkreuz.org/english>

cylindrical wall and − at the center of the cylinder − of an 8 m high slim tower, called *'Gnomon'*, in form of a three-sided pyramid made from reflecting steel. According to the design of the French artist Philippe Lejeune (*1924), the shadow of this tower falling onto the cylindrical wall Ψ should be utilized for a time telling device.

The exact position of this monument (Fig. 1) is as follows: Eastern longitude 16.132° and Northern latitude 48.049°.

It is well-known that the shadow casted from a vertical tower on a vertical wall has not the necessary property that its position depends only on the local time but is independent from variations of the season (see, e.g., [1, p. 393]). On the other hand, meeting the necessities for a sundial, i.e., inclining the 'Gnomon' until it is parallel to the earth's axis, would totally destroy the appearance of the artist's design. The only compromise was to use suitable reflecting stripes on the East- and Westface of the central tower such that the reflected sunlight works like a sundial.

In the following, the geometric background for this sun-reflection-dial, which is unique in Austria [4], will be presented. This offers also the opportunity to explain with the help of Descriptive Geometry [3, p.50] why (surprisingly) sundials cannot show the exact time all over the year – because of the 'Equation of Time' (for further details see [1, 392-402] or [2]).

Figure 1*. Heiligenkreuz Abbey with the marked monument „Epiphanie“*

2. WHY NO SUNDIAL WITH SHADOWS ?

We expect from a usual sundial that – independent from the season – at any given day-time t (e.g., 10 a.m.) the shadow of the style falls onto the same line. These so-called *hour-lines* are usually marked on the sundial. Of course, our requirement has consequences for the position of the *gnomon*, i.e., the style which causes the shadow. Let us first have a look on the shadow of one single point.

For the purpose of understanding sundials, it is convenient to adopt the geocentric view. This means we consider the earth to be fixed while the sun is moving relative to the earth. This motion is exactly inverse to the motion displayed in Fig. 6, the composition of the rotation of the earth about its axis and the translational movement along an elliptic path around the sun.

During the run of a year the line connecting the center M of the earth with the center S of the sun changes its inclination with respect to ('w.r.t.' in short) the equator plane. When the true local day-time[3] t is kept fixed over the year, the connecting lines SM vary within a plane through the earth's axis. Now we replace the center M by an arbitrary point P on the earth. In this respect, we may assume an infinite distance to the sun. Therefore we apply the translation $M \mapsto P$ and conclude: All sun rays which pass every day at the same true local time t through point P belong to a plane ε_t parallel to the original meridian plane. We call this translated plane a *hour-plane* ε_t. Which instant t we ever choose, the corresponding hour-plane ε_t contains the line a which is parallel to the earth's axis and passes through P. All hour-planes belong to a pencil with axis a, and when the date t increases by one hour the corresponding hour-plane rotates through an angle of $360/24 = 15^\circ$.

The sun casts a shadow from point P onto a given surface Ψ. The shadow originating from this single point at the same true local day-

[3] For any position X on the earth, the *true local time* is defined in the following way: Noon is fixed by the fact that the sun reaches its daily culmination relative to X, i.e., the sun passes exactly the meridian plane of X (= plane connecting X with the axis of the earth). *One hour* (in true local time) equals the 24^{th} part of the period between consecutive noons. For details like the deviation from the mean time see Section 4.
Historically, the need for an international *mean time* started only when a railway-network has been established.

time t varies from day to day along a hour-line which is the intersection of the surface Ψ with the hour-plane ε_t.

The gnomon must be chosen in such a way that the shadows of all its points fall onto the same hour-line. Therefore all its points must be located in the same hour-plane ε_t, and this must hold for all t. We can summarize (see also [1, p. 393 ff]):

Lemma 1: *The shadow casted from a style (=gnomon) at given local time t is for each t placed on the same hour-line independently from the season if and only if the gnomon is parallel to the earth's axis.*

Concerning the initial plan for Heiligenkreuz (Fig. 4, left), what are the conclusions ? A vertical tower used as a gnomon contradicts Lemma 1. Only at noon we would obtain a correct result as the shadow points North. In order to obtain a classical sundial, the gnomon must be slanted with an inclination of about 48° against the North direction. However, this would totally destroy the optical appearance of the artist's design. This was the reason why we started to pay attention to the reflecting properties of the faces at the original 'Gnomon'.

3. SUNDIAL BASED ON THE REFLECTION OF SUNLIGHT

Suppose, the sunbeam which is reflected at any point Q of the reflecting face μ meets a given surface Ψ at the point *Q'*. Then due to Fig. 2 (left), the luminous point *Q'* coincides with the shadow of point Q w.r.t. a *virtual sun* $\overline{S}$ which is the mirror of the original sun S w.r.t. μ. The daily movement of this virtual sun $\overline{S}$ around the earth is a rotation about an axis which is the mirror of the earth's axis w.r.t. μ. Therefore, for any given true local day-time t, the luminous point *Q'* in Ψ varies over a year along a hour-line which is the trace of the reflected hour-plane $\overline{\varepsilon_t}$, and $\overline{\varepsilon_t}$ includes the mirror $\overline{a}$ of line a. Now, Lemma 1 implies the following:

Lemma 2: *Suppose, the reflection of sunbeams along a line segment* ℓ *in the reflecting plane* μ *generates at given local time t on a surface* Ψ *a luminous curve segment* ℓ_t. *This spot* ℓ_t *is for each t a subset of a corresponding 'hour-line' all over the year if and only if the reflecting segment* ℓ *is parallel to the mirror of the earth's axis* *w.r.t.* μ*. Because of* $\ell \subset \mu$ *the reflecting plane* μ *must be parallel to the earth's axis.*

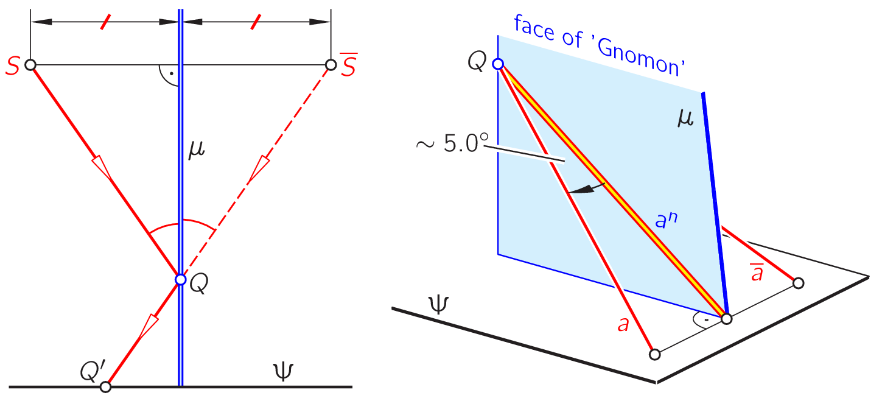

Figure 2. *The luminous point Q' caused by reflection of a sunbeam in the plane* μ *at point Q equals the 'shadow' of Q w.r.t. the reflected sun.*

Now we face a problem: According to the design of the 'Gnomon' in Heiligenkreuz (Fig. 4, left), no face is parallel to the earth's axis; neither any line *a* (Lemma 1) nor its mirror $\overline{a}$ (Lemma 2) is located in one of the reflecting faces μ of the 'Gnomon'. [4]

Consequently, we have to confine ourselves to an approximation: We replace $\overline{a}$ by its orthogonal projection a^n in μ, which at the same time is the orthogonal projection of line *a* (Fig.2, right). The angle between *a* and the reflecting face μ is smaller than 5°. Therefore this approximation seems to be admissable. Fig. 3 reveals that in fact the reflection of a stripe ℓ along a^n gives luminous stripes ℓ_t in Ψ which for each *t* follow almost a 'hour-line' over the seasons.

Remark: We note in Fig. 3 that for *t* = 7 a.m. the monthly luminous stripes ℓ_t fit better to a hour-line than for *t* = 11 a.m. This results from the fact that the exact hour-plane ε_6 for 6 a.m. is orthogonal to the meridian plane and therefore very close to the plane which connects the lines *a* and a^n. So, the error of our approximation by

[4] After reflection in μ the sunlight would cast from a style parallel to $\overline{a}$ a shadow which satisfies the requirements of a classical sundial.

choosing a^n instead of $\bar{a}$ is smallest at 6 o'clock in the morning and in the afternoon.

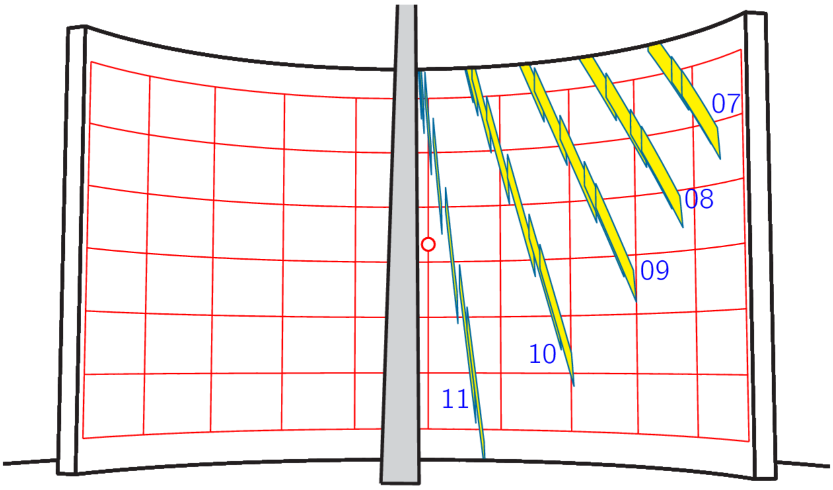

Figure 3. *The luminous stripes* ℓ_t on the wall Ψ *vary from month to month, but can be combined to hour-lines*

The result of our approximation was the plan displayed in Fig. 4 (right). The East-face of the 'Gnomon' serves as a mirror in the morning, the West-face in the afternoon. Only around noon the reflection fails; at this time only side-sunlight meets the two faces. This is the reason why the *noon line*, i.e., the hour-line for true local noon is missing at the final status (Fig. 10). Fortunately, just at noon the shadow of the 'Gnomon' shows the correct time.

The altitude of the reflecting stripes on the 'Gnomon' (Fig. 4, right) and the position of the 'Gnomon' w.r.t. the curved wall Ψ resulted from the fact that even at winter and summer solstices a luminous stripe should be visible on Ψ. While at classical sundials on the Northern hemisphere the shadow moves during each day from left to right, i.e., from West to East, at a sun-reflection-dial the luminous spots move in the opposite direction. The hour-lines on the monument in Heiligenkreuz are portions of (almost) ellipses since they are the intersections of the cylindrical wall Ψ with the (almost planar) reflected 'hour-planes' $\overline{\varepsilon_t}$.

Figure 5 shows the original plan for the workmen. The hour-lines were drawn only outside the mosaic-work. Since on the left and right hand side of the mosaic the portions of the hour-lines are rather short, they were extended over the vertical edge of the wall. These extensions were defined in such a way that for a visitor who stands exactly on the middle-axis a few meters in front of the 'Gnomon', the extensions look like straight elongations of the hour-lines (note the photo in Fig. 10 and compare the right and left hand side extensions).

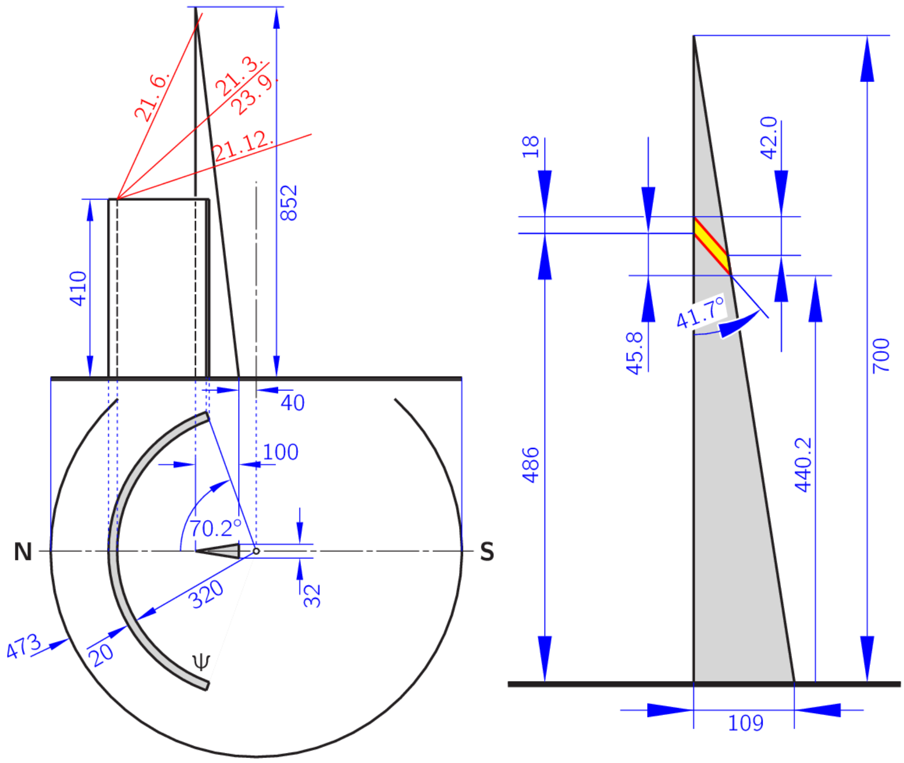

Figure 4. *Left: The initial plan of the monument; Right: the final position of the reflecting stripes on the East- and Westface of the 'Gnomon'.*

Figure 5 shows also that the hour-lines for morning and afternoon are not totally symmetric. This is caused by the fact that the sun-reflection-dial shows not the local time for Heiligenkreuz but CET, i.e., Middle European Time (wintertime). Because of the longitude ~16° of Heiligenkreuz, the sun reaches its daily culmination here at approx. 11:56 p.m. – apart from the general deviation according to the Equation of Time (see Section 4).

4. ON THE PRECISION OF SUNDIALS

Now we skip the geocentric view and turn over to the heliocentric view. From now on, our fixed frame includes the sun and the directions to the fixed stars.

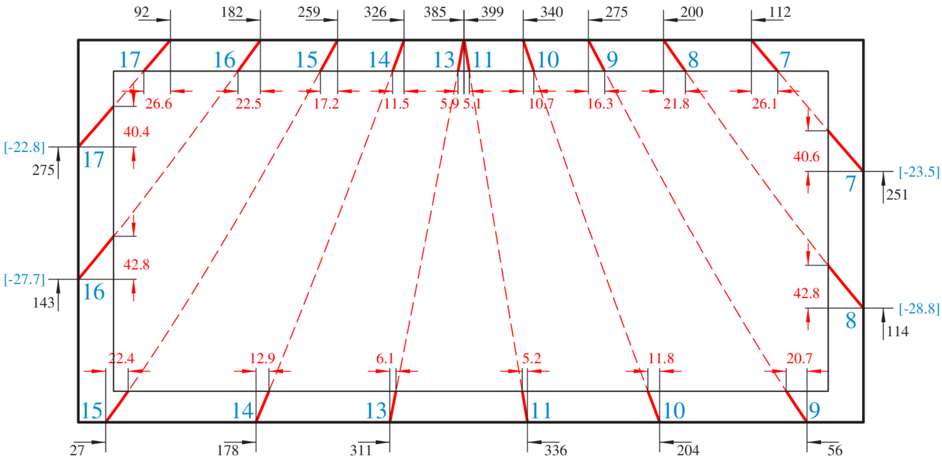

Figure 5. *The original plan for workmen in order to paint the hour-lines*

During the movement of the earth along the *ecliptic* around the sun (see Fig. 6) the direction of the earth's axis remains fixed (in first order approximation). The angle ε between the planes of the equator and the ecliptic is called *obliquity of the ecliptic*.

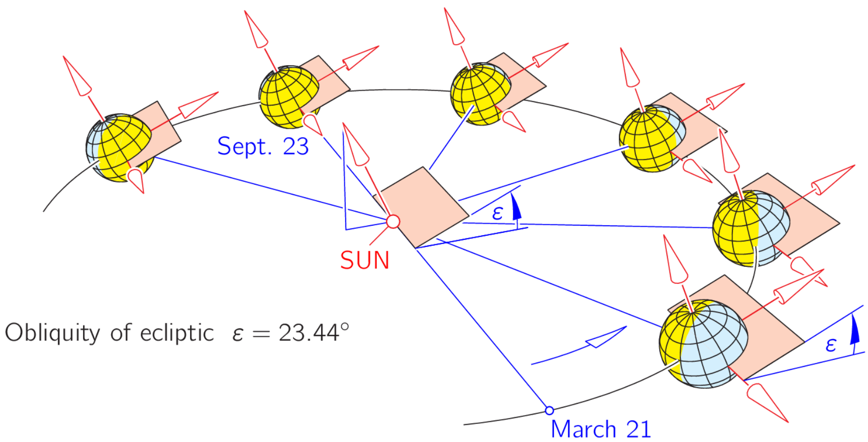

Figure 6. *The earth travels around the sun along the ecliptic*

In order to inspect the rotation of the earth about its axis in true size, we use an auxiliary view in direction of the earth's axis (Fig. 7). This view reveals: During consecutive sun culminations, i.e., from noon until noon of the next day (true local time), the earth must rotate about its axis through an angle of $360^\circ + \varphi$ where φ is the center angle swept during this time interval (~360°/365). However, even when the earth would move along a circular path with constant velocity, the center angle φ varies because of the affine distortion of this circle in our auxiliary view. Consequently, the duration of a day (w.r.t. true local time) measured in mean time is not constant.

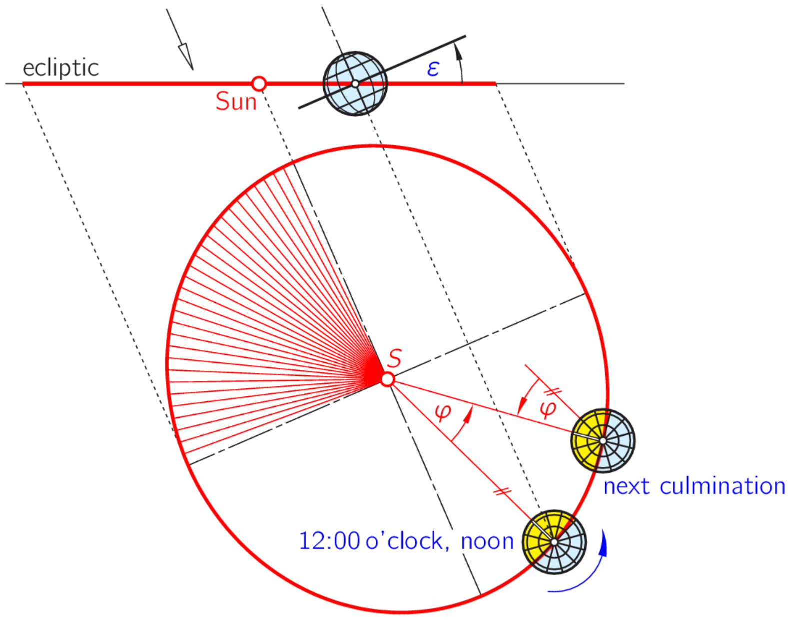

Figure 7*. Auxiliary view in direction of the earth's axis*

However, the obliquity of ecliptic is not the only reason for the variable duration of the period between consecutive sun culminations. Due to Kepler's First and Second Law, the ecliptic is an ellipse with focal point S, and the earth travels along this ellipse (with very small numeric eccentricity $e/a = 0.0167$) with constant areal velocity. This means, in periods of equal duration the segment SM connecting the centers of the sun and the earth sweeps sectors of equal area (Fig. 8).

Note that affine transformations preserve the ratio of areas; therefore also in our auxiliary view the earth moves with constant areal velocity.

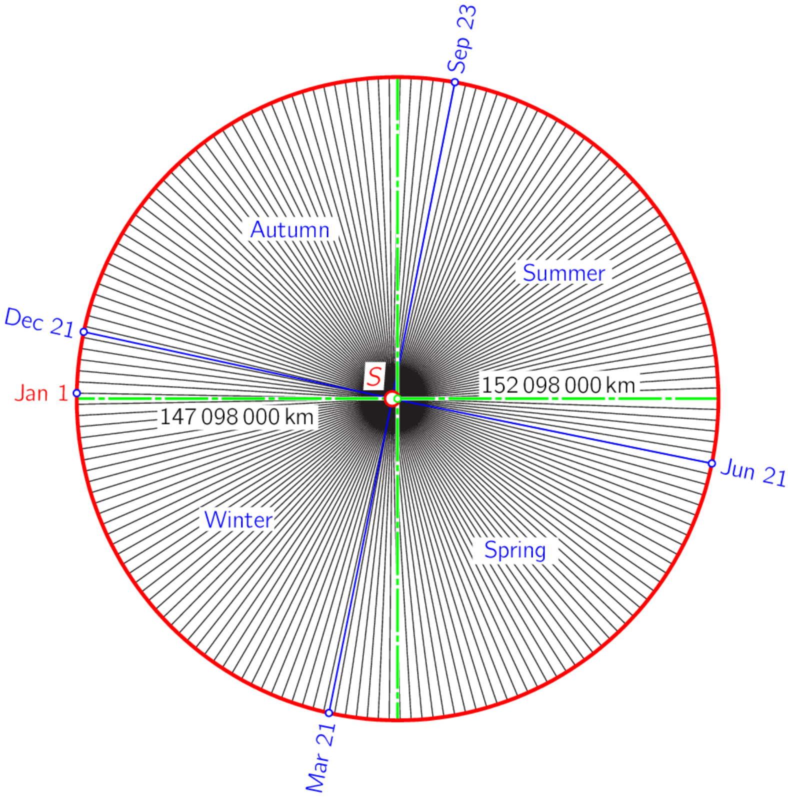

***Figure 8**. According to Kepler's First and Second Law the earth moves around the sun with constant areal velocity along an ellipse with focal point S*

Both effects, the affine distortion in our auxiliary view and the elliptic shape of the ecliptic, influence the *'Equation of Time'* (Fig. 9), which shows the difference *z* between true local time (reduced to CET) and *mean time* on our clocks. This deviation lies between approx. +15 and -15 minutes. The dotted line in Fig. 9 shows the pure influence of the obliquity of ecliptic, i.e., the deviation for a circular path. The dashed line in Fig. 9 indicates the influence of Kepler's Laws in the case of a vanishing obliquity.

The biggest deviations happen at the beginning of November – the sun is about 16 minutes before the median time (it is „getting dark earlier") – and at February 10 when the sun is approx. 14 minutes delayed (days „last already longer"). On the other hand, there is a high conformity between true time and mean time at midth of April and June, at the beginning of September and at (catholic) Christmas time. Of course, we cannot expect that a sundial pays attention to the change from CET to CET summertime every year.

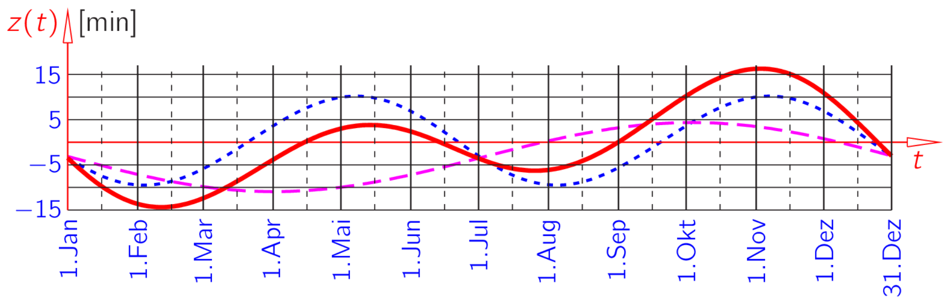

Figure 9. *Equation of Time: z = true time – mean time*

5. HOW TO READ THE TIME ?

Figure 10. *The final appearance of 'Epiphanie', a declared monument „for Freedom of Conscience and Religion as a Foundation for Peace"*

Figure 10 shows how the monument 'Epiphanie' with the hour-lines in the final status looks like. The photo was taken on August 14 and shows in the upper right corner the (additionally marked) luminous stripe. It indicates approx. 8:05 a.m. However, we must pay attention to the summertime. And, in addition, for midth of August the Equation of Time shows 5 minutes delay of the sun against CET. Hence, the result is approx. 9:10 a.m. which comes very close to the date stored in the camera.

The photo in Fig. 10 shows near the bottom some irregular light concentrations which can be confusing for visitors as they have nothing to do with the sundial. These strange looking spots are caused by the fact that the lower parts of the East- and West-face are slightly bent (due to production errors) but still reflecting. One can actually note that the reflection in the total East-face, which apart from the reflecting stripe and the lowest portion is unpolished, produces a vertical shine on the wall; this shine includes the marked luminous stripe above as well as the light concentrations at the bottom.

Literature

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