The Influence of Geometry on the Rigidity or Flexibility of Structures

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Abstract—This is an overview on flexible frameworks. After the definition of continuous flexibility as well as infinitesimal flexibility of different order, the paper focuses on geometric characterizations of flexible cases, thus revealing that structures which look rigid can still be flexible due to their particular geometric properties. Some of the basic theorems are presented together with their proofs.[1]

Keywords - rigidity; flexibility; frameworks; polyhedra.

I. Introduction

This lecture presents an overview on flexible structures like polyhedra and frameworks: Which conditions are necessary and sufficient for flexibility, which metric or combinatorial properties must change or remain constant under flexing?

The first important result in this field claims that every convex polyhedron is rigid; this is due to A. L. Cauchy 1813 [7]. On the other hand, R. Bricard classified 1897 [5] all flexible octahedra. But these flexible polyhedra have self-intersections.

Since then, questions around this topic attracted many prominent mathematicians like R. Bricard, H. Lebesgue, M. Dehn, W. Blaschke, N. V. Efimov, W. Wunderlich, A. D. Alexandrov, and A. V. Pogorelov. Nevertheless, a few outstanding results were proved rather recently:

1977 R. Connelly [8] constructed a flexible polyhedron without self-intersection, topologically a *'flexing sphere'*. 1985 R. Alexander proved that every flexible polyhedron preserves its total mean curvature during the flex [1]. 1995 I. Sabitov proved the famous *Bellows Conjecture* stating that for every flexible polyhedron the volume keeps constant during the flex [14,9]. According to V. Alexandrov [3] Sabitov's result is of algebraic nature, while R. Alexander's result stems from analysis. Recent problems around this topic are addressed in [4].

The question whether the edge lengths of a polyhedron or – more generally – of a framework determine its planar or spatial shape uniquely, is also important for many engineering applications, e.g., for mechanical or constructional engineers, for biologists in protein modelling or for the analysis of isomers in chemistry.

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II. Definition of Rigidity and Flexibility

Let us return to Cauchy's basic result on the rigidity of any *convex* polyhedron. Here we think of a polyhedron made from cardbord, with planar faces, but with variable dihedral angles between any two faces sharing a common edge. It is quite natural to call a polyhedron *flexible*, if its shape can continuously vary without changing the metric of its faces. Otherwise the polyhedron is called *rigid*. It turns out that the borderline between flexibility and rigidity is not as strict as one might conjecture. There are different *degrees of flexibility* to distinguish.

On the other hand, we can also distinguish between global and local rigidity. We call a polyhedron *globally rigid* when its shape is uniquely defined by its unfolding. With the term 'shape' we mean its spatial form – apart from movements in space. A polyhedron is called *locally rigid*, when it is not flexible, but their unfolding can admit mutually incongruent *realizations*.

Any tetrahedron, i.e., any three-sided (non-flat) pyramide is globally rigid. However, a regular octahedron, i.e., a double-pyramide erected over a square with 8 congruent triangular faces is no more globally rigid. There is a convex form, which is locally rigid. But besides, we can re-assemble the structure by erecting both four-sided pyramides to the same side. Then it is no more convex, each face is two-fold covered by originally different faces. Thus we come up with only one four-sided pyramide without basis, and this is even flexible. It turns out that the computation of the spatial from of any four-sided double-pyramide, i.e., of any general octahedron with given unfolding is an algebraic problem of degree 8. Hence, up to 8 different realizations are possible. Apart from particular cases, each of these realizations is locally rigid.

This demonstrates already that one has to be careful with the terminology. Hence we start with some definitions, and in the sequel we see polyhedra not as piecewise linear surfaces, but as wireframes, i.e., as *frameworks*. This means we concentrate on its edges only. If not all faces are triangular, we must insert additional edges to keep their faces planar. When only checking the rigidity of any polyhedron, we can replace a face with more than 3 edges by a pyramide erected over this face. Finally, it should be noted that we don't care about technical problems like stiffness of edges and clearances along the hinges. We just focus on the geometry.

Definition 1: A *framework* F in $\mathbb{R}^d$ consists of a set $V = \{\mathbf{x}_1, ..., \mathbf{x}_v\} \subset \mathbb{R}^d$ of vertices and a set E of edges, $E = \{(i, j) \mid i < j, 1 \le i, j \le v\}$. For any edge $\mathbf{x}_i\mathbf{x}_j$ of our framework F, i.e., for $(i, j) \in E$, the length is denoted by $l_{ij} > 0$, and we set $f_{ij}(\mathbf{x}_i, \mathbf{x}_j) := \|\mathbf{x}_i - \mathbf{x}_j\|^2 - l_{ij}^2$.
This framework is called *flexible*, if there is a continuous family F_t of frameworks for $0 \le t \le 1$ with $F_0 = F$ and $f_{ij}(\mathbf{x}_i(t), \mathbf{x}_j(t)) = 0$ for all $(i, j) \in E$, provided there are at least two vertices which do not keep their distance constant. We call the family F_t, $0 \le t \le 1$, a *flex* of F.

We say, the edge set E defines the *combinatorial structure* of F. By the request that at least one distance between vertices does not remain constant during the flex, we exclude *trivial* flexes, i.e., pure motions of the framework as a rigid body, expressable in matrix form by $\mathbf{x}_i(t) = \mathbf{a}(t) + \mathbf{A}(t)\mathbf{x}_i$ with $\mathbf{a}(t) \in \mathbb{R}^d$ and an orthogonal $d \times d$-matrix $\mathbf{A}(t)$, i.e., $\mathbf{A}^T = \mathbf{A}^{-1}$. The conditions for keeping the lengths of edges constant are of algebraic nature. Hence, the flex is not only continuous but *analytic*. Therefore the $\mathbf{x}_i(t)$ can be expanded into Taylor series. This is the basis for the following definition.

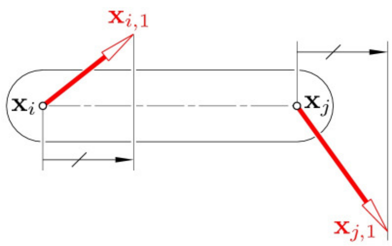

Figure 1. Projection Theorem

Definition 2: A framework F in $\mathbb{R}^d$ is called *infinitesimally flexible of order n* if there is a polynomial function $\mathbf{z}_i(t) = \mathbf{x}_i + \mathbf{x}_{i,1}t + ... + \mathbf{x}_{i,n}t^n$ for each vertex $\mathbf{x}_i$ such that the replacement of $\mathbf{x}_i$ by $\mathbf{z}_i(t)$ in the distance function $f_{ij}(\mathbf{x}_i, \mathbf{x}_j)$ gives a function with a zero at $t = 0$ of multiplicity $> n$ for all $(i, j) \in E$.
The first derivative $\mathbf{x}_{i,1}$ of $\mathbf{z}_i(t)$ at $t = 0$ is called *velocity vector*, the second derivative $2\mathbf{x}_{i,2}$ *acceleration vector* of vertex $\mathbf{x}_i$. In order to exclude trivial flexes, the coefficients of t in the polynomial functions, i.e., the *velocity vectors* $\mathbf{x}_{1,1}, ..., \mathbf{x}_{v,1}$ must not originate from any motion of F as a rigid body, i.e., by an instantaneous motion with $\mathbf{x}_{i,1} = \mathbf{s} + \mathbf{S}\mathbf{x}_i$ with $\mathbf{s} \in \mathbb{R}^d$ and a skew-symmetric $d \times d$-matrix $\mathbf{S}$, i.e., $\mathbf{S}^T = -\mathbf{S}$. Each continuously flexible framework admits a nontrivial analytic flex and is therefore also infinitesimally flexible of any order.

The conditions for a framework of given combinatorial structure to be infinitesimally flexible of given order can be obtained by substituting the polynomial functions $\mathbf{z}_i(t)$ in the distance functions f_{ij} and comparing the coefficients of all powers of t up to t^n. This results in a series of systems of linear equations. So, checking a given framework whether it is rigid or higher-order flexible is reduced to inspect the solvability of these systems of linear equations step by step.

The converse, i.e., finding the geometric meaning of these conditions, is not as straight forward as one might expect. The system for first order flexibility is homogeneous. Therefore the existence of a nontrivial first-order flex is equivalent to a sufficiently high rank-deficiency of the coefficient matrix, the socalled *ridity matrix* of F. The solution of the first system defines the absolute values in the inhomogeneous system for second-order flexibility. If this system is solvable, its solution defines the absolute values for the third system, and so on. Instead of presenting details of this more or less technical method, we focus on the underlying geometric conditions.

It should be noted that a real-world model of such an infinitesimally flexible framework is really flexible because of the clearances at the vertices. The difference to a rigid framework is apparent as well as that to a continuously flexible framework since the flexes remain limited within a small neighborhood.

There are several applications of infinitesimally flexible frameworks. In robotics, such poses are called *singular* and usually avoided since at least one degree of freedom is missing. When in surveying the relative position of points is determined by measuring some of the mutual distances and when the corresponding framework is infinitesimally flexible, then this position is called *critical* as it results in numerical instability.

III. EXAMPLES OF INFINITESIMALLY FLEXIBLE FRAMEWORKS

A. *First order flexibility*

The first derivative of $f_{ij}(\mathbf{z}_i, \mathbf{z}_j)$ at $t = 0$ vanishes if and only if

$$(\mathbf{x}_i - \mathbf{x}_j) \, . \, (\mathbf{x}_{i,1} - \mathbf{x}_{j,1}) = 0 \text{ for all } (i, j) \in E. \quad (1)$$

This vanishing scalar product means that for each edge $\mathbf{x}_i\mathbf{x}_j$ of F the components of the velocity vectors of $\mathbf{x}_i$ and $\mathbf{x}_j$ in direction of the edge $\mathbf{x}_i\mathbf{x}_j$ are equal. This is called *Projection Theorem* (Fig. 1). We summarize:

Theorem 1: *A framework F is infinitesimally flexible if to each vertex $\mathbf{x}_i$ a velocity vector $\mathbf{x}_{i,1}$ can be assigned such that for all edges of F the Projection Theorem* (1) *is fulfilled.*

The first example in Fig. 2 shows a planar *bipartite* framework. Bipartite means that the vertices can be divided into two sets and each edge connects points from different sets. In our case there are six vertices $\mathbf{x}_i$ and $\mathbf{y}_j$, $i, j \in \{1, 2, 3\}$, and 9 edges $\mathbf{x}_i\mathbf{y}_j$. It has been well-known at least for one century that this framework *is* *infinitesimally flexible if and only if the vertices are placed on a curve of degree 2*, i.e., either on a conic c or on two lines. This is still true when more than 6 points $\mathbf{x}_i$ and $\mathbf{y}_j$ are chosen on the same conic. And it is also valid in higher dimensions when the conic is replaced by any *quadric* in $\mathbb{R}^d$. The following short proof due to W. Whiteley [25] reveals that this condition is sufficient and that the velocity vectors can be chosen perpendicular to the quadric – as shown in Fig. 2 for the two-dimensional case.

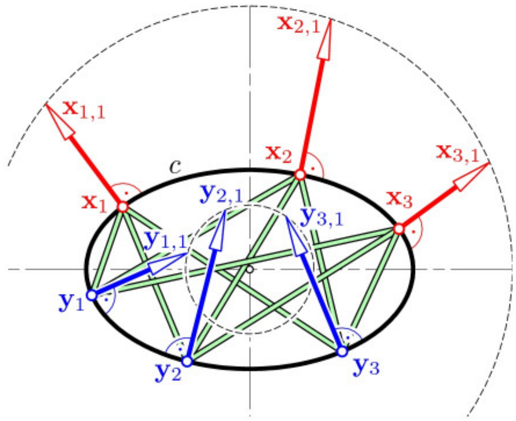

Figure 2. A planar bipartite framework is infinitesimally flexible if the vertices are located on a 2nd-order curve

Proof: We write the coordinate vectors in columns and set up the equation of the quadric in matrix form by $\mathbf{x}^T \mathbf{Q}\,\mathbf{x} = 0$ with a symmetric $d \times d$ -matrix $\mathbf{Q}$. Since $\mathbf{x}_i$ and $\mathbf{y}_j$ are specified as points on the quadric, we have $\mathbf{x}_i^T \mathbf{Q}\,\mathbf{x}_i = \mathbf{y}_j^T \mathbf{Q}\,\mathbf{y}_j = 0$. Now we specify the velocity vectors by $\mathbf{x}_{i,1} = \mathbf{Q}\,\mathbf{x}_i$ and $\mathbf{y}_{j,1} = -\mathbf{Q}\,\mathbf{y}_j$, and we verify that for the edge $\mathbf{x}_i\mathbf{y}_j$ the Projection Theorem is fullfilled. For this purpose we write the scalar product in matrix form and obtain

$$(\mathbf{x}_i - \mathbf{y}_j)^T (\mathbf{x}_{i,1} - \mathbf{y}_{j,1}) = (\mathbf{x}_i - \mathbf{y}_j)^T (\mathbf{Q}\,\mathbf{x}_i + \mathbf{Q}\,\mathbf{y}_j)$$
$$= \mathbf{x}_i^T \mathbf{Q}\,\mathbf{x}_i - \mathbf{y}_j^T \mathbf{Q}\,\mathbf{x}_i + \mathbf{x}_i^T \mathbf{Q}\,\mathbf{y}_j - \mathbf{y}_j^T \mathbf{Q}\,\mathbf{y}_j = 0$$

since $\mathbf{x}_i$ and $\mathbf{y}_j$ obey the quadric equation and the real number $\mathbf{y}_j^T \mathbf{Q}\,\mathbf{x}_i$ is the transpose of $\mathbf{x}_i^T \mathbf{Q}\,\mathbf{y}_j$. On the other hand, $(\mathbf{x}_i^T \mathbf{Q})\,\mathbf{x} = 0$ is the equation of the tangent plane of the quadric at point $\mathbf{x}_i$, and hence the vector $\mathbf{x}_{i,1} = \mathbf{Q}\,\mathbf{x}_i$ is perpendicular to the quadric. The same holds for $\mathbf{y}_{j,1}$ (Fig. 2). ♦

The assignment of velocity vectors to an infinitesimally flexible framework F is not unique. Apart from a scaling we can additionally impose an instantanous motion. This means we can add $\mathbf{s} + \mathbf{S}\,\mathbf{x}_i$ to each $\mathbf{x}_{i,1}$ without disturbing eq. (1) since $(\mathbf{x}_i - \mathbf{x}_j)^T \mathbf{S}\,(\mathbf{x}_i - \mathbf{x}_j)$ is the nullform when $\mathbf{S}$ is skew-symmetric.

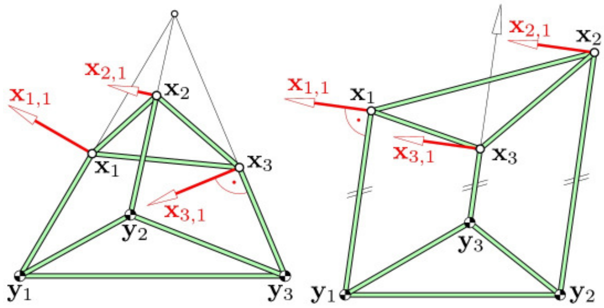

Figure 3. Another infinitesimally flexible framework with 6 vertices and 9 edges

The next example displayed in Fig. 3 is again a planar framework with 6 vertices and 9 edges, but not bipartite. It is a *pinned* framework, i.e., vertices indicated by the black-and-white points in Fig. 3 are fixed. This framework is *infinitesimally flexible if and only if the three lines* $\mathbf{x}_i\mathbf{y}_i$ *have one point in common or are parallel.*

In the last two examples the geometric characterizations are of projective nature. If we transform a conic by a collineation, the image is again a conic. The same holds for the condition in Fig. 3. This is surprising since rigidity deals with lengths of edges, i.e., is based on a metrical definition, and distances, angles and even parallelities change under collinear transformations. The following theorem has first been proved by H. Liebmann (1920) [13] (for an alternative proof and historical remarks see I. Izmestiev [11]).

Theorem 2: *If an infinitesimally flexible framework is transformed by a collinear transformation without mapping any vertex onto a point at infinity, then the obtained framework is still first-order flexible.*

Proof: We follow B. Wegner (1984) [24] and formulate the proof for a planar framework though the same idea can immediatly be used in higher dimensions.

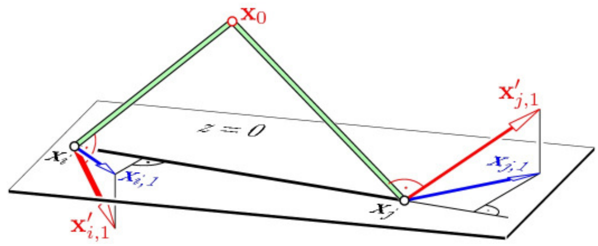

Figure 4. Proof of the projective invariance of first-order flexibility by the principle of 'coning'

Suppose, the framework F with vertices $\{\mathbf{x}_1, ..., \mathbf{x}_v\}$ and edges $\mathbf{x}_i\mathbf{x}_j$ is located in the plane $z = 0$. Now we extend F to a framework F' in $\mathbb{R}^3$ by adding an additional vertex $\mathbf{x}_0$ outside of $z = 0$ and by including the v edges $\mathbf{x}_0\mathbf{x}_i$. We call F' a *conical* framework as it actually consists of triangular plates $\mathbf{x}_0\mathbf{x}_i\mathbf{x}_j$. Now we prove *that* F' *is flexible if and only if* F *is flexible*. This principle has been called *'coning'* by W. Whiteley [26].

Suppose, F is flexible, i.e., there are velocity vectors $\mathbf{x}_{i,1}$ such that the Projection Theorem is fulfilled for each edge of F. Now we show that there are also velocity vectors $\mathbf{x}'_{j,1}$ for all vertices $\{\mathbf{x}_0, ..., \mathbf{x}_v\}$ of F' such that they are compatible with all edge lengths of F':

We set $\mathbf{x}'_{0,1} = \mathbf{0}$. And we specify $\mathbf{x}'_{i,1}$ such that it is perpendicular to the edge $\mathbf{x}_0\mathbf{x}_i$ and its top view onto $z = 0$ coincides with $\mathbf{x}_{i,1}$ (see Fig. 4). We immediately see that for all edges $\mathbf{x}_0\mathbf{x}_i$ the Projection Theorem is fulfilled; the orthogonal projections of $\mathbf{x}'_{0,1}$ and $\mathbf{x}'_{i,1}$ onto this edge are zero. But also for the edges $\mathbf{x}_i\mathbf{x}_j$ of the original framework F the Projection Theorem is fulfilled, as the orthgonal projections of $\mathbf{x}_{i,1}$ and of $\mathbf{x}'_{i,1}$ onto $\mathbf{x}_i\mathbf{x}_j$ are equal.

Conversely, if the conical framework F' is infinitesimally flexible with given velocity vectors, we apply an instantaneous translation by adding to all velocity vectors the vector $-\mathbf{x}'_{0,1}$ so that $\mathbf{x}_0$ gets zero velocity like before. Then, by defining the top views of the other velocity vectors as the velocity vectors at the vertices of the framework F in the plane $z = 0$, we can immediately verify the infinitesimal flexibility of F.

These arguments are true for each planar section of the conical framework – as long as the plane does not pass through $\mathbf{x}_0$. Since any collinear transformation between planes can be obtained by intersecting a conical structure with different planes, we have thus proved Liebman's theorem. ♦

B. *Higher-order infinitesimal flexibility*

Figure 5 characterizes 2nd-order flexible frameworks of the combinatorial type of Fig. 3. This reveals that this is no more projectively invariant as the congruence of angles wouldn't be preserved under a proper collineation. In the last picture of Fig. 5 a continuously flexible framework of this type is displayed.

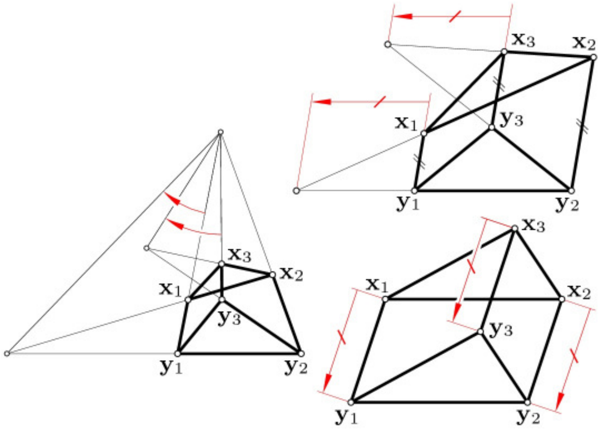

Figure 5. 2nd-order flexible framework [16]

Figure 6 shows that 3rd-order flexibility is given if and only if the points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are specified on a certain cubic or conic.

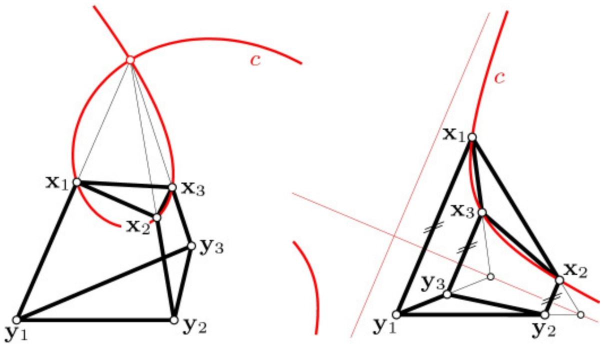

Figure 6. 3rd-order flexible framework [16]

Figure 7 shows an example of a 2nd-order flexible bipartite planar framework. Due to A. C. Dixon [10] there are two types of continuously flexible planar bipartite frameworks (see Fig. 8). Even their spherical analoga are continuously flexible.

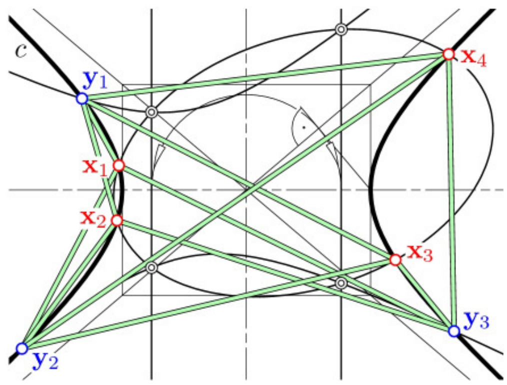

Figure 7. 2nd-order flexible bipartite framework [17]

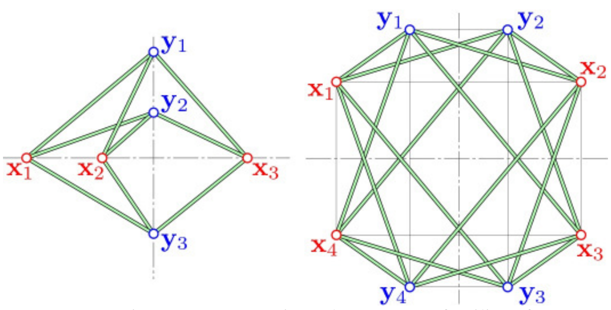

Figure 8. Due to Dixon there are two families of continuously flexible bipartite planar frameworks

T. Tarnai presented in [23] an example of an infinitesimally flexible framework of order $2^m - 1$ for arbitrary m. It is again a pinned framework. A similar structure can be found in a sketch of Leonardo da Vinci – as a device to multiply forces like a leverage or a tackle block. As the conditions for continuous flexibility are algebraic, it is understandable that due to V. Alexandrov [2] any framework which is infinitesimally flexible of sufficiently high order must be continuously flexible. In the case of a planar bipartite framework 8[th]-order flexibility must imply continuous flexibility.

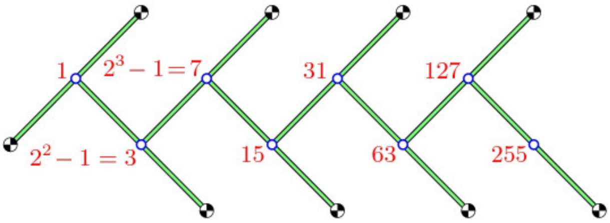

Figure 9. T. Tarnai's iterative example of an infinitesimally flexible framework of high order

C. *First-order flexibility vs. snapping frameworks*

Infinitesimal flexibility can be seen as the limiting case where two realizations of a framework coincide. This was the way how W. Wunderlich studied infinitesimal flexibility. There is a direct connection between pairs of realizations of any framework and an infinitesimally flexible framework of the same combinatorial type. W. Whiteley [26] calls this *'averaging'*; in I. Izmestiev [11] it is called *Pogorelov map*. W. Wunderlich [27] called such two sufficiently close realizations *'snapping'* since a real-world model can change from one realization into the other by applying slight force.

Theorem 3: *Let $\mathbf{y}_1,\ldots,\mathbf{y}_v$ and $\mathbf{y}'_1,\ldots,\mathbf{y}'_v$ be the vertices of two incongruent realizations of a framework F' with the same metric. Then the midpoints $\mathbf{x}_i=\frac{1}{2}(\mathbf{y}_i+\mathbf{y}'_i)$ of corresponding vertices make a framework F of the same combinatorial structure which is infinitesimally flexible with velocity vectors $\mathbf{x}_{i,1}=\frac{1}{2}(\mathbf{y}_i-\mathbf{y}'_i)$.*

Conversely, any infinitesimally flexible framework F with vertices $\mathbf{x}_1,\ldots,\mathbf{x}_v$ and velocity vectors $\mathbf{x}_{1,1},\ldots,\mathbf{x}_{v,1}$ gives rise to two incongruent realizations of a framework F' of the same combinatorial type. Its vertices are $\mathbf{y}_i=\mathbf{x}_i+\mathbf{x}_{i,1}$ and $\mathbf{y}'_i=\mathbf{x}_i-\mathbf{x}_{i,1}$, respectively.

Proof: The proof is unexpectedly short. The equation $(\mathbf{y}_i-\mathbf{y}_j)^2-(\mathbf{y}'_i-\mathbf{y}'_j)^2=0$ is equivalent to $(\mathbf{y}_i-\mathbf{y}_j+\mathbf{y}'_i-\mathbf{y}'_j).(\mathbf{y}_i-\mathbf{y}_j-\mathbf{y}'_i+\mathbf{y}'_j)=0$, and this is just the statement of the Projection Theorem (1) because of $2\mathbf{x}_i=\mathbf{y}_i+\mathbf{y}'_i$ and $2\mathbf{x}_{i,1}=\mathbf{y}_i-\mathbf{y}'_i$. ♦

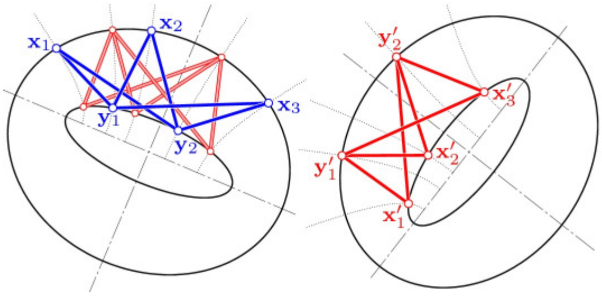

Figure 10. Snapping bipartite planar frameworks

As an example, we focus again on the bipartite planar framework of Fig. 2. In [19] it is proved that any two incongruent realizations of a bipartite framework F are related by Ivory's theorem; the vertices are located on two confocal conics (Fig. 10). Hence, averaging between corresponding points of two congruent pairs of confocal conics gives vertices located on one conic. Conversely, in the notation of W. Whiteley's proof above the points $\mathbf{x}_i+\mathbf{Q}\mathbf{x}_i$ and $\mathbf{y}_j-\mathbf{Q}\mathbf{y}_j$ as well as $\mathbf{x}_i-\mathbf{Q}\mathbf{x}_i$ and $\mathbf{y}_j+\mathbf{Q}\mathbf{y}_j$ are located on two pairs of confocal conics. In Fig. 2 these are two concentric circles (dashed lines) passing through the tips of the velocity vectors.

When F is rigid and the assignment of velocity vectors is trivial, i.e., $\mathbf{x}_{i,1}=\mathbf{s}+\mathbf{S}\mathbf{x}_i$, then the corresponding two 'snapping' realizations are congruent. This results from the fact that for skew-symmetric $\mathbf{S}$ the matrix product $(\mathbf{I}_d+\mathbf{S})^{-1}(\mathbf{I}_d-\mathbf{S})$ is orthogonal, provided $\mathbf{I}_d$ is the unit matrix and $\mathbf{S}$ has eigenvalues $\neq \pm 1$. This is known as Cayley's method to parametrize orthogonal matrices.

IV. FLEXIBLE POLGYGONAL STRUCTURES

We conclude with two types of polyhedra and start with octahedra, i.e., double-pyramids with a quadrangular basis $\mathbf{x}_1,\ldots,\mathbf{x}_4$. This can be seen as a spatial bipartite framework since beside the sides of the quadrangle the edges connect vertices $\mathbf{x}_1,\ldots,\mathbf{x}_4$ of the basis with the two apices $\mathbf{y}_1$ and $\mathbf{y}_2$. Hence, we can immediately apply our previous results on bipartite frameworks.

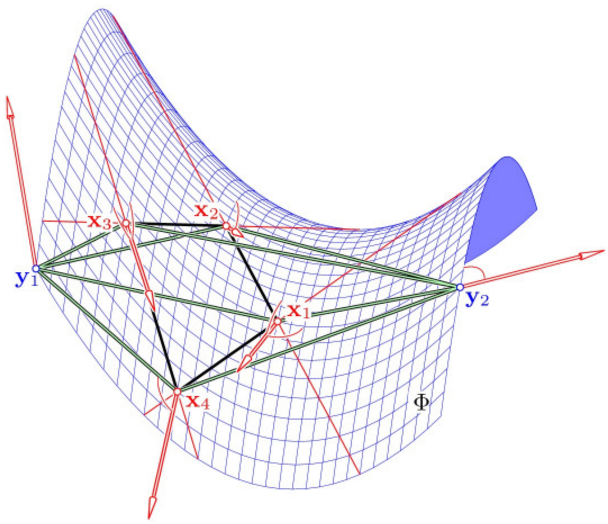

Figure 11. Infinitesimally flexible octahedron

An octahedron is *infinitesimally flexible if and only if the four sides of the quadrangle $\mathbf{x}_1,\ldots,\mathbf{x}_4$ are located on a quadric which at the same time passes through the two apices $\mathbf{y}_1$ and $\mathbf{y}_2$* (see Fig. 11). There are equivalent characterizations, obtained by different authors. However, the characterization based on quadrics can be generalized onto higher dimensions (see C. S. Borcea [6]). On the other hand, the snapping case, first studied by W. Wunderlich [27], is related to confocal quadrics. And this was the basis for reproving R. Bricard's classical result that there are exactly 3 families of continuously flexible octahedra in [15]. Also in the hyperbolic space the analogous cases are continuously flexible [20, 21]. It is an open problem whether these are the only ones. In a similar way it might be possible to prove that in the 4-space those presented in [18] are the only ones which are continuously flexible, and that there are no continuously flexible cross-polytopes in a space of dimension > 4.

Another family of polyhedral structures where the question of flexibility found interest, is that of Kokotsakis meshes, i.e., in the standard case the compounds of 3×3 quadrangles. The 1st-order infinitesimal flexibility has already been characterized by A. Kokotsakis [12] himself. Up to now 5 families of continuously flexible versions are known [22], and it is still open whether this list is complete.

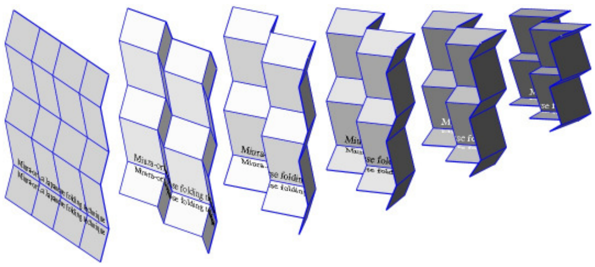

Figure 12. Miura-ori, a flexible polygonal structure; snapshots of the folding procedure

A famous example of a flexible polygonal structure which is composed from particular Kokotsakis meshes is *Miura-ori*, a folding technique originating from Japan (see Fig. 12). It should be noted that also Bricard's flexible octahedra are particular Kokotsakis meshes of general type.

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