Some news about flexible Kokotsakis meshes

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(joint work with Georg Nawratil)

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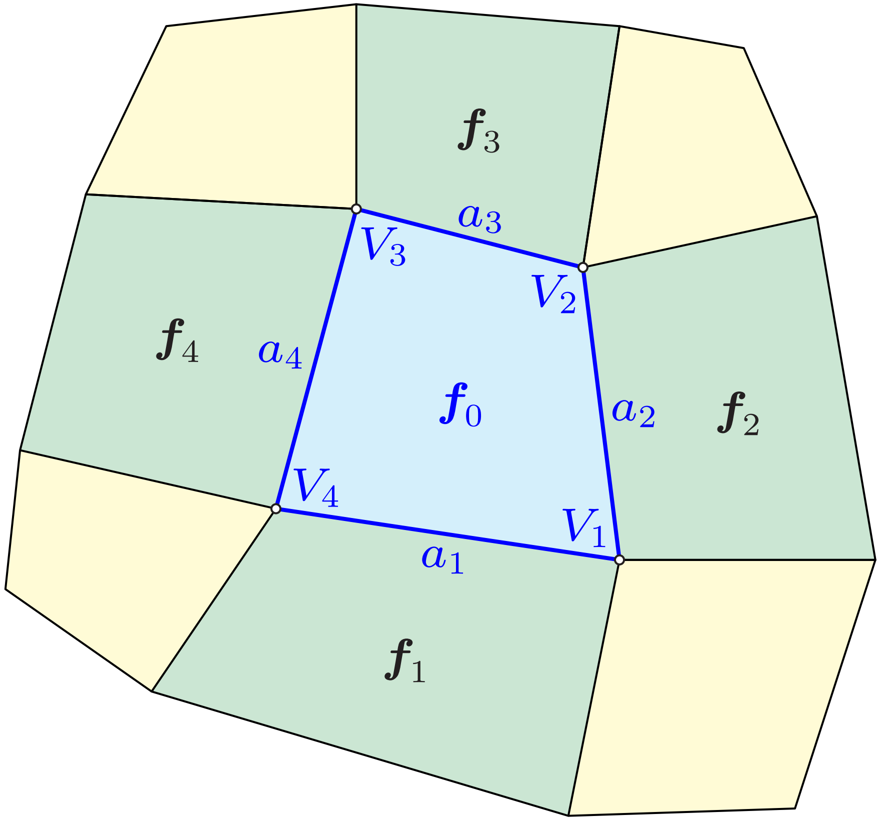
Outline

1. Introduction
2. Two examples of flexible quad meshes
3. What means flexible?
4. Flexible Kokotsakis meshes
5. Flexibility vs. reducibility of meshes

1. Introduction



Special case $n = 4$ (German: “Vierflach”)

A Kokotsakis mesh is a polyhedral structure consisting of an n-sided central polygon f_0 surrounded by a belt of polygons.

Each side a_i, $i = 1, \ldots, n$, of f_0 is shared by a polygon f_i. Each vertex V_i of f_0 is the meeting point of four faces.

Each face is seen as a rigid body; only the dihedral angles can vary. Under which conditions a Kokotsakis mesh is continuously flexible ?

1. Introduction

Antonios Kokotsakis
1899–1964

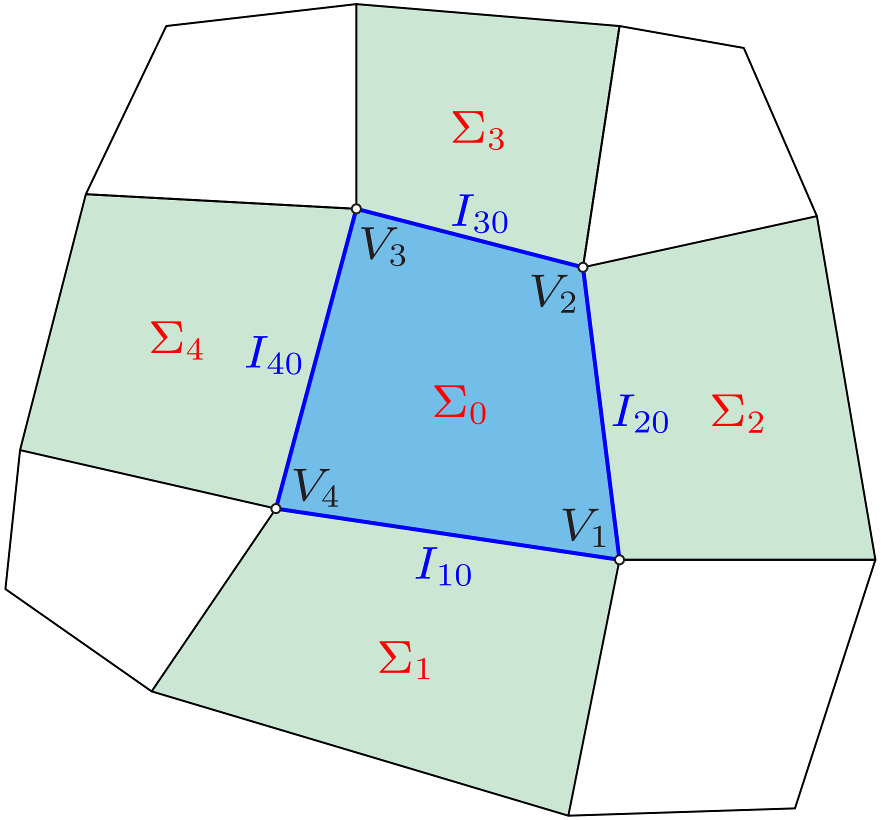
He was born on the island Crete in Greece. As a precocious child, he was accepted at the Department of Civil Engineering of Technical University of Athens already in the age of 16 .

After graduation he was appointed a lecturer in the Department of Descriptive and Projective Geometry. He finished his PhD-thesis entitled *“About flexible polyhedra”* under the supervision of K. Caratheodori in Munich/Germany.

After returning to Athens, Kokotsakis worked in many important construction projects, e.g., 1955–56 he supervised the restoration of the Holy Tomb in Jerusalem.

His list of publications contains not more than 5 titles.

1. Introduction



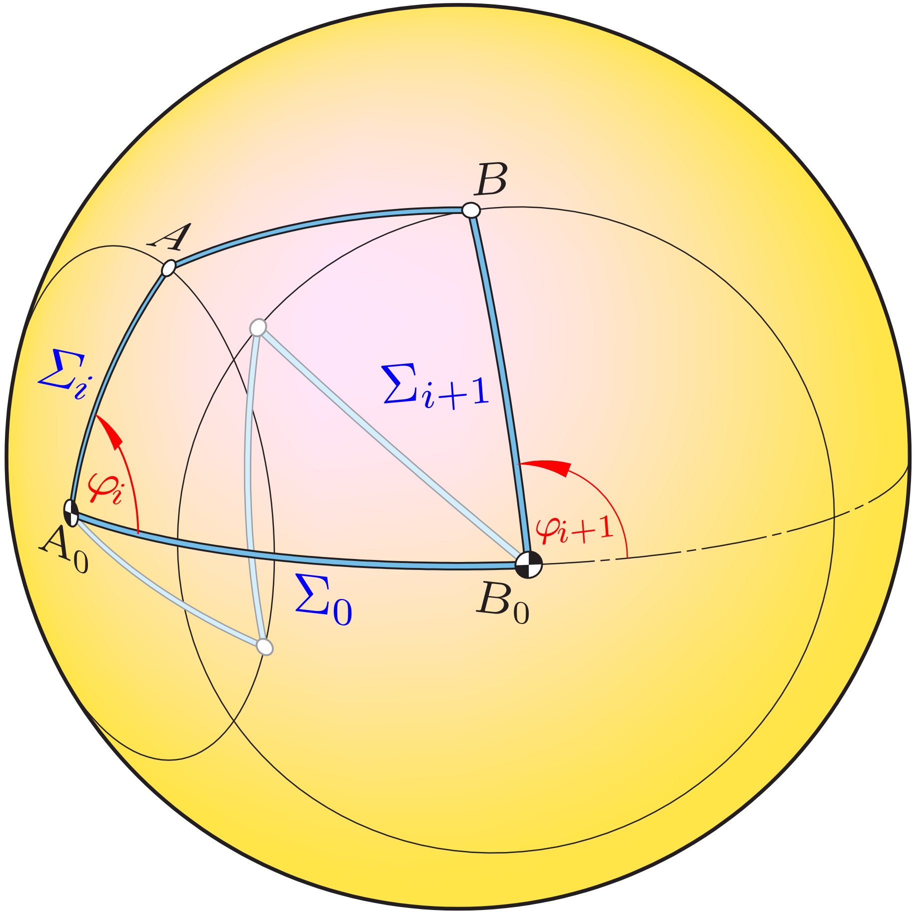
The polygons need not be planar

Kinematic interpretation:

The polygons represent different systems $\Sigma_0, \ldots, \Sigma_n$.
The sides a_i of $\boldsymbol{f}_0$ are instantaneous axes I_{i0} of the relative motions Σ_i/Σ_0.

The relative motions Σ_{i+1}/Σ_i between consecutive systems are spherical four-bars mechanisms.

1. Introduction



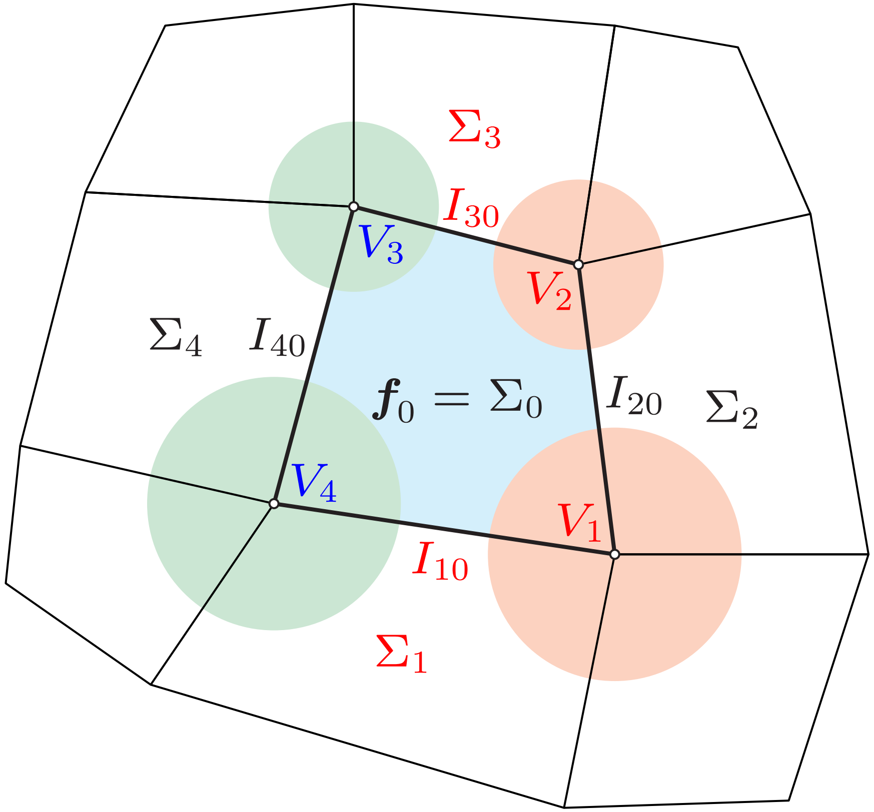
The transmission from Σ_i to the following Σ_{i+1}, $\varphi_i \mapsto \varphi_{i+1}$, is realized by a spherical four-bar:

To recall:

A spherical four-bar transmits the rotation about the center A_0 by the coupler AB non-uniformly to the rotation about B_0.

The arms A_0A and B_0B represent consecutive systems Σ_i, Σ_{i+1}.

1. Introduction



The edge lengths $\overline{V_1V_2}, \ldots, \overline{V_4V_1}$ of the central polygon $\boldsymbol{f}_0$ have no influence on the flexibility $\Longrightarrow$

Theorem: *A Kokotsakis-mesh for $n = 4$ is flexible if and only if the transmission $\varphi_1 \mapsto \varphi_3$ realized by the two four-bars (V_1, V_2) on the right hand side equals that via (V_3, V_4) on the left hand side.*

(we do not care about intersections between the involved quadrangles)

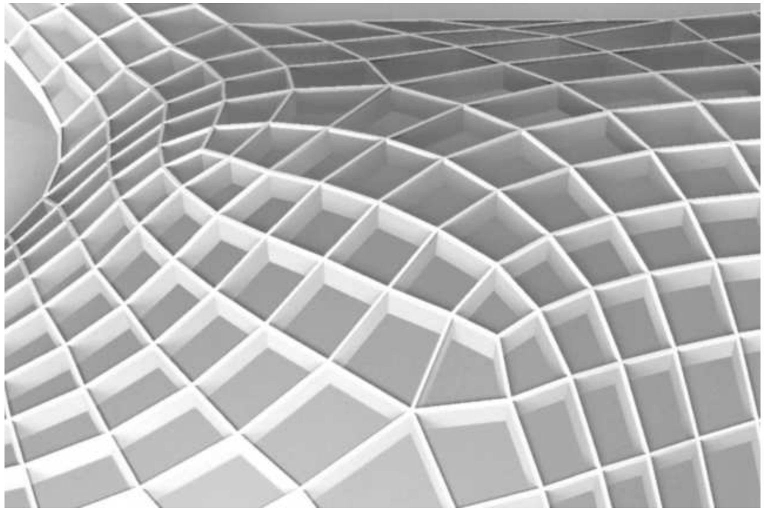
1. Introduction

In *discrete differential geometry* there is a interest on polyhedral structures composed of quadrilaterals (quadrilateral surfaces). If all quadrilaterals are planar, they form a *discrete conjugate net* = **quad mesh**.

Theorem: [Bobenko, Hoffmann, Schief 2008]

A discrete conjugate net in general position is continuously flexible $\iff$ *all its* 3×3 *complexes are continuously flexible.*

Bobenko et al., 2008:



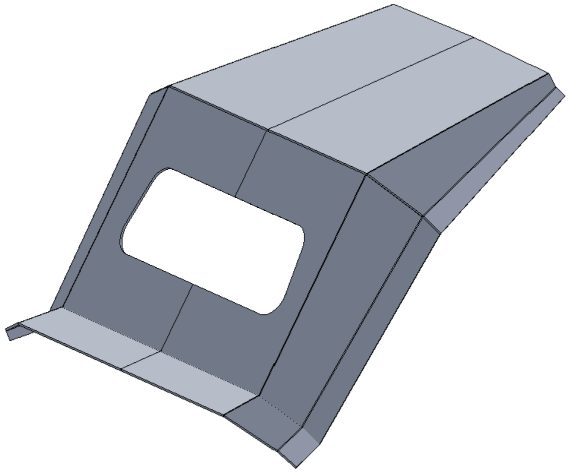
H. Pottmann, Y. Liu, J. Wallner, A. Bobenko, W. Wang: *Geometry of Multi-layer Freeform Structures for Architecture.* ACM Trans. Graphics **26** (3) (2007), SIGGRAPH 2007

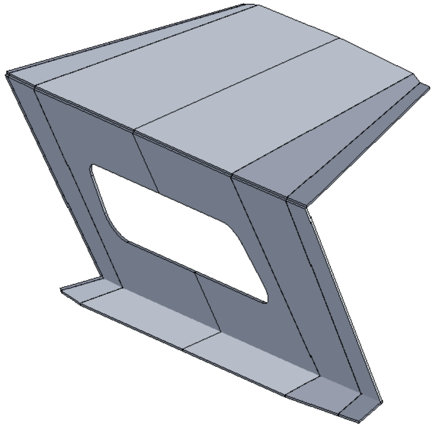
"... the complete classification of flexible discrete conjugate nets (*"quad meshes"*) has not been achieved yet"

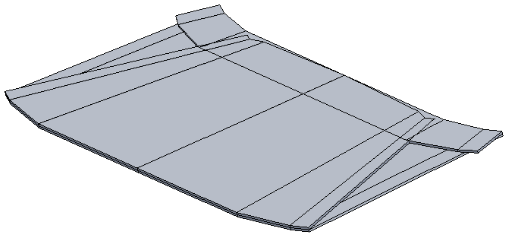
1. Introduction

Also the folding of the roof at cabrios is based on a flexible quad mesh

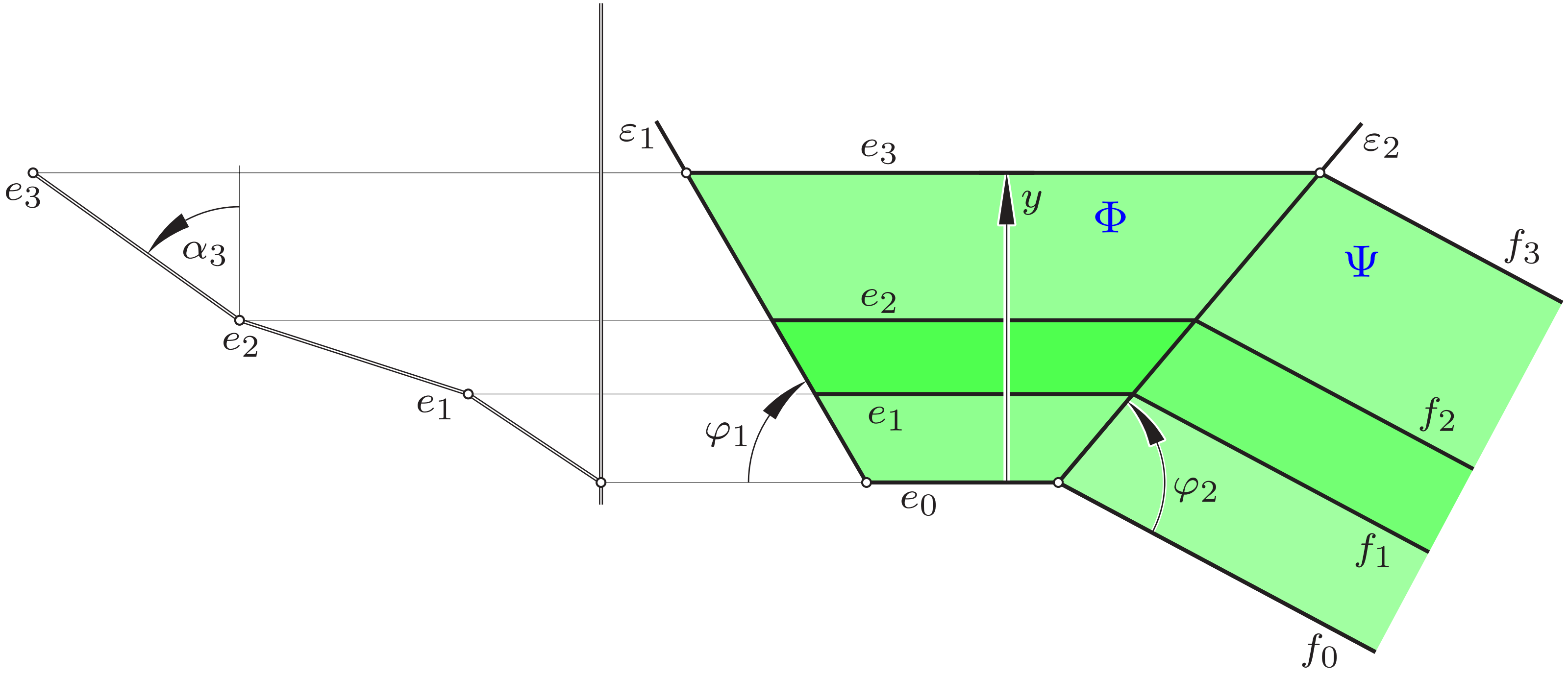
(Diploma thesis Nadja Posselt, TU Dresden 2010)







2. Two examples of flexible quad meshes



H. Graf, R. Sauer 1931: A T-flat is a compound of prisms $\Phi, \Psi, \ldots$ (see above: top view and side view. 'T' stands for 'trapezoid').

The horizontal folds $e_i, f_i, \ldots$ are located in horizontal planes, the vertical folds in vertical planes $\varepsilon_1, \varepsilon_2, \ldots$

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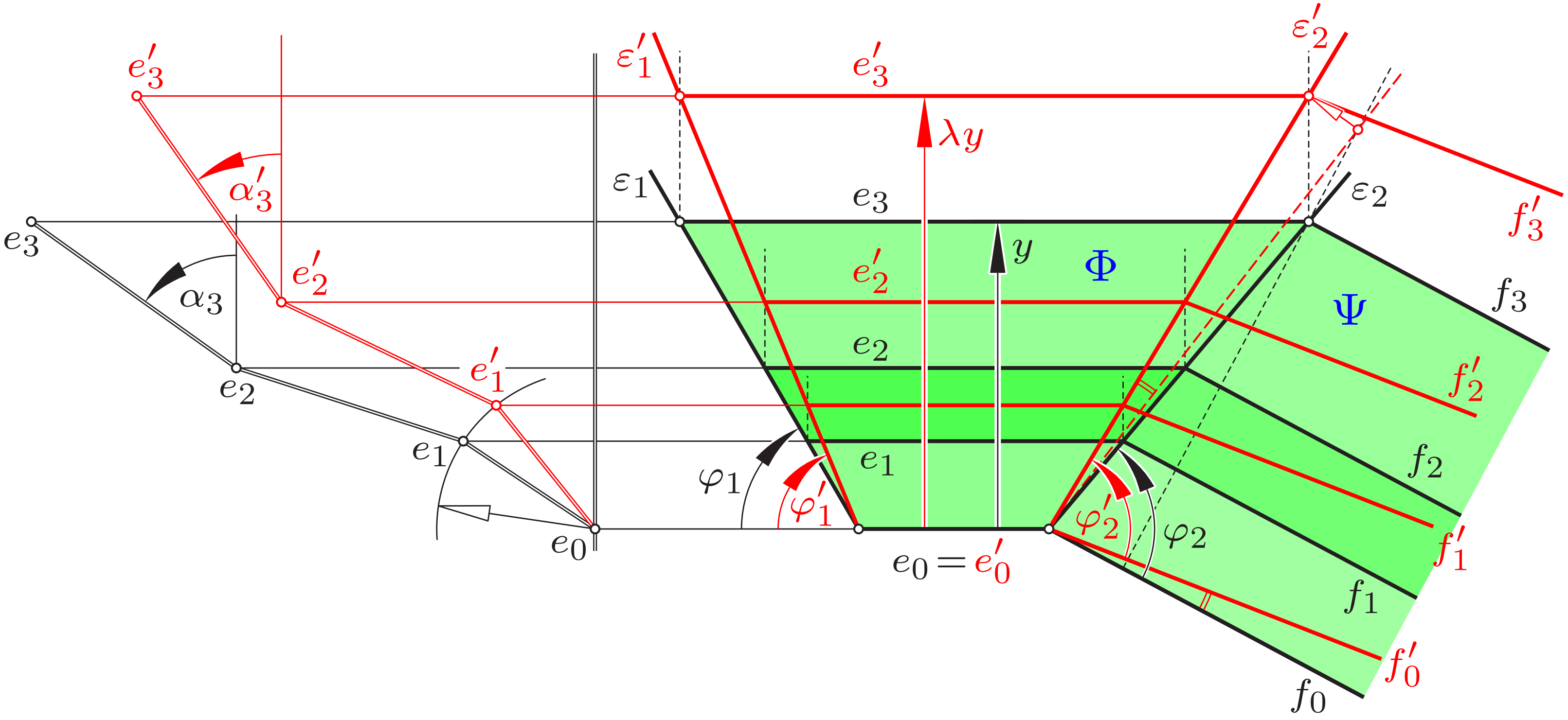
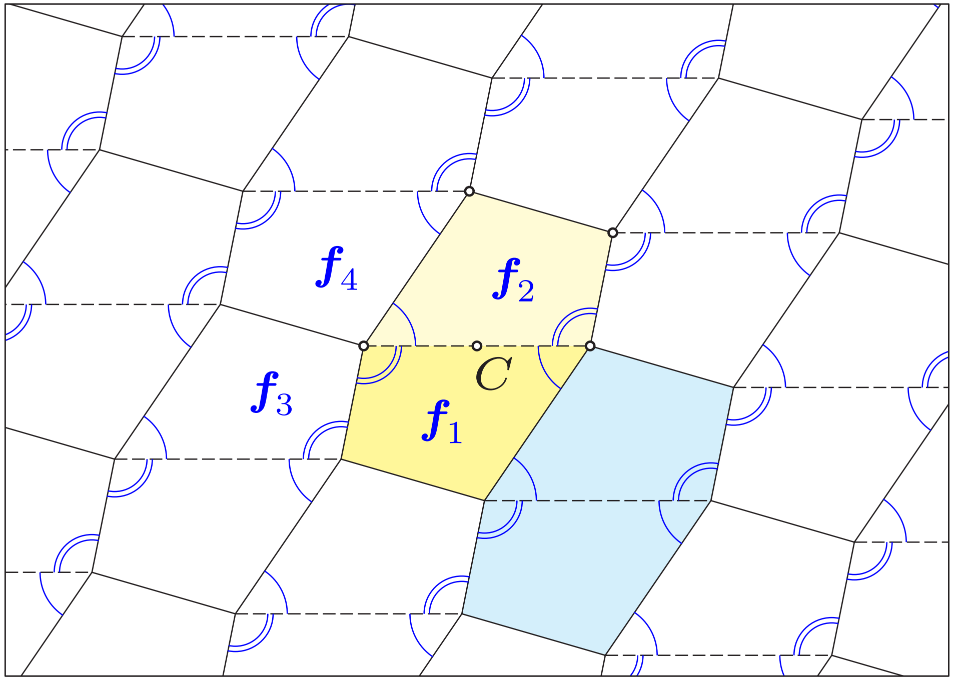


T-flat together with a flexion (in red).

The top view of Φ performs a scaling with factor λ orthogonal to e_0.
This implies analogous bendings of the other prisms $\Psi, \ldots$

2. Two examples of flexible quad meshes



A. Kokotsakis, 1932
Athens

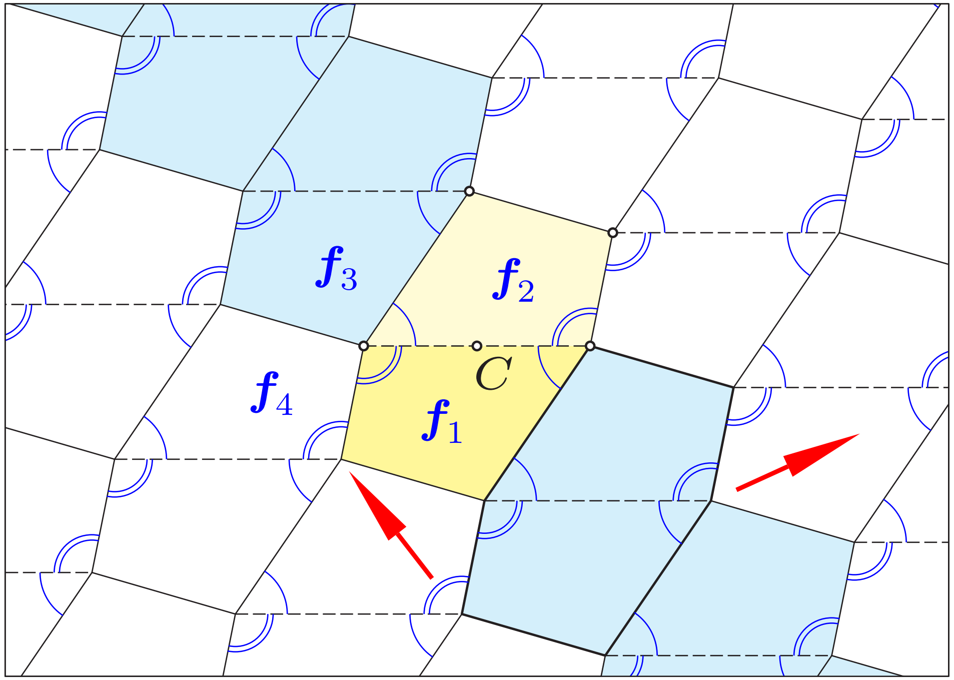
Any plane quadrangle is a tile for a regular tessellation of the plane.

It is obtained by applying

- iterated 180°-rotations about the midpoints of the sides of an initial quadrangle or
- by applying iterated translations on a centrally symmetric hexagon.

For a convex f_1 this polyhedral structure is continuously flexible.

2. Two examples of flexible quad meshes



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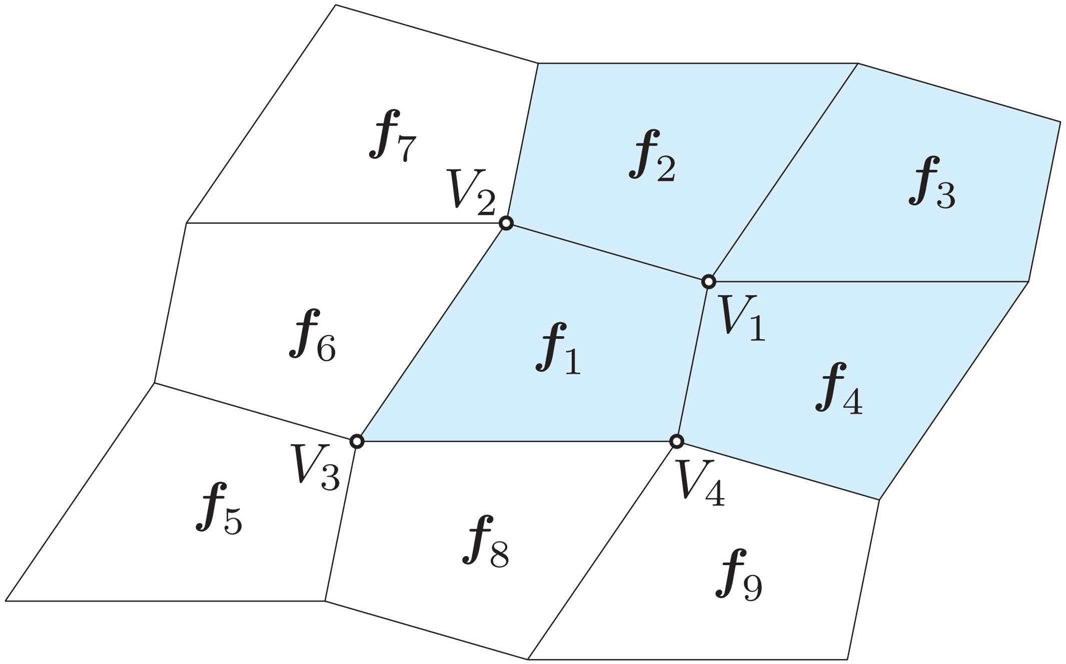
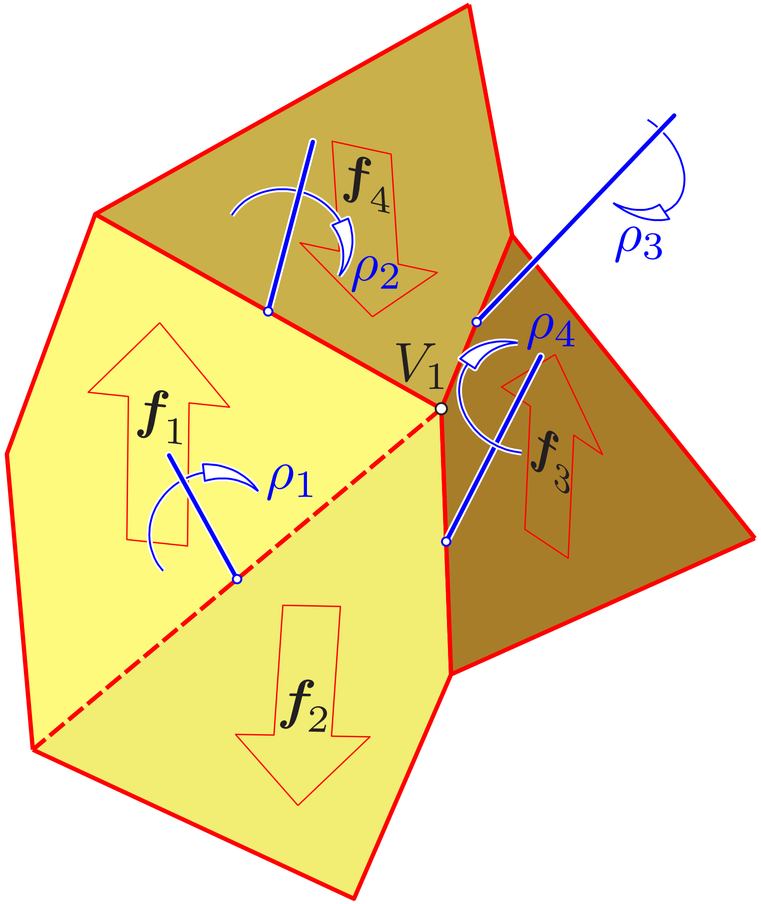


We first focus on the four-sided pyramide with vertex V_1.

Due to the required convexity no interior angle is $> \pi$. Therefore this pyramide is continuously flexible.

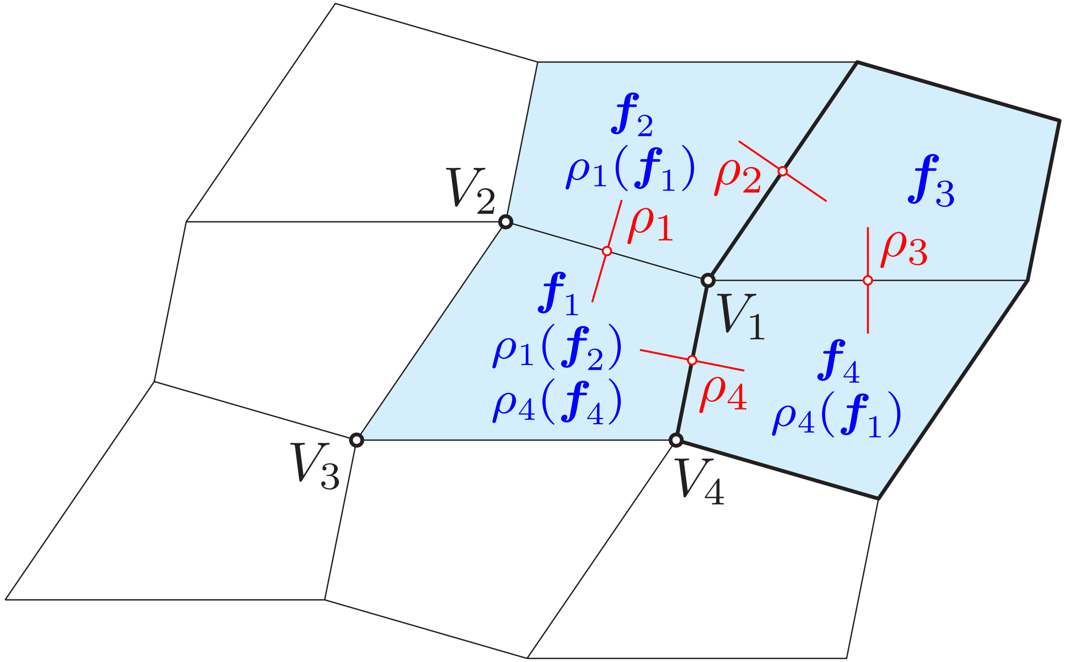
2. Two examples of flexible quad meshes



In each pose, for any two neighbouring faces there is a 180°-rotation ρ_i which interchanges these two faces.

The axis of ρ_i is located in a bisector plane and passes through the midpoint of the common edge.

2. Two examples of flexible quad meshes



We apply the four 180°-rotations (= half-rotations) consecutively to the quadrangle f_1. Then it is mapped via f_2, f_3, f_4 onto itself, hence

$$\rho_3 \circ \rho_4 = \rho_2 \circ \rho_1 .$$

The composition of two half-rotations is a helical motion about the common perpendicular of the axes.

All four vertices $V_1, \ldots, V_4$ have the same distance from the common perpendicular of the axes of $\rho_1, \ldots \rho_4$.

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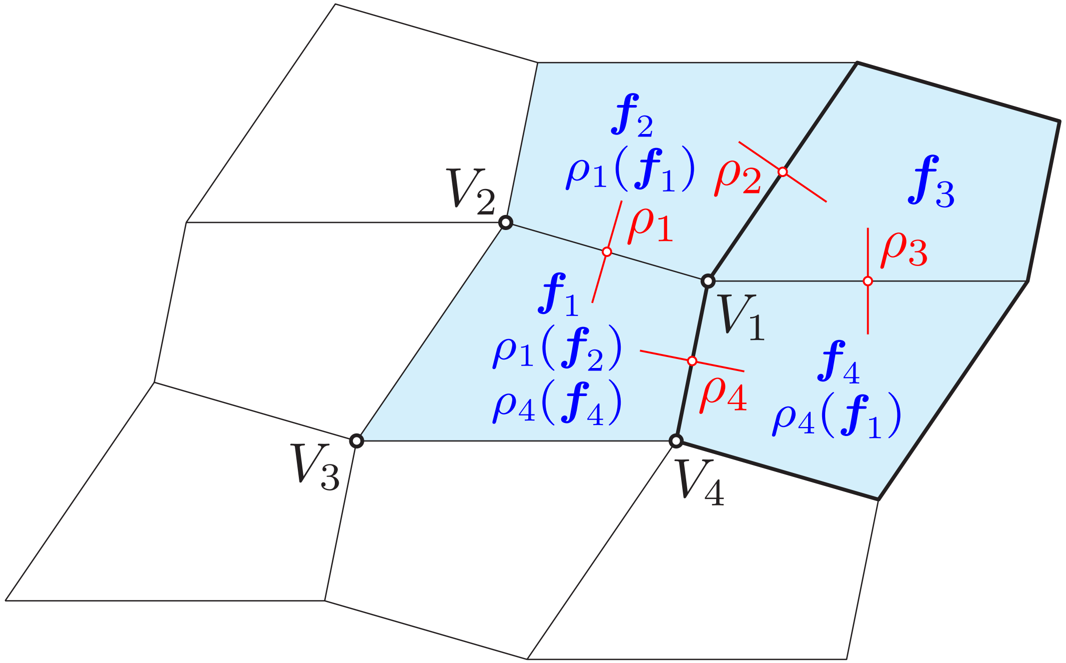

We apply the four $180°$-rotations (= half-rotations) consecutively to the quadrangle $\boldsymbol{f}_1$. Then it is mapped via $\boldsymbol{f}_2$, $\boldsymbol{f}_3$, $\boldsymbol{f}_4$ onto itself, hence

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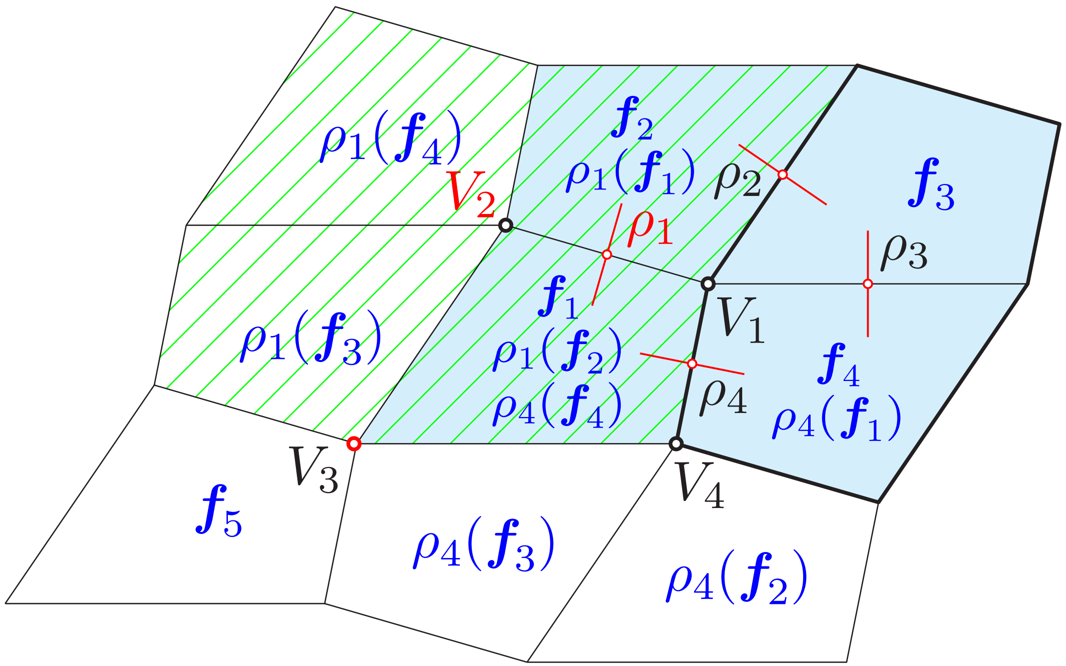
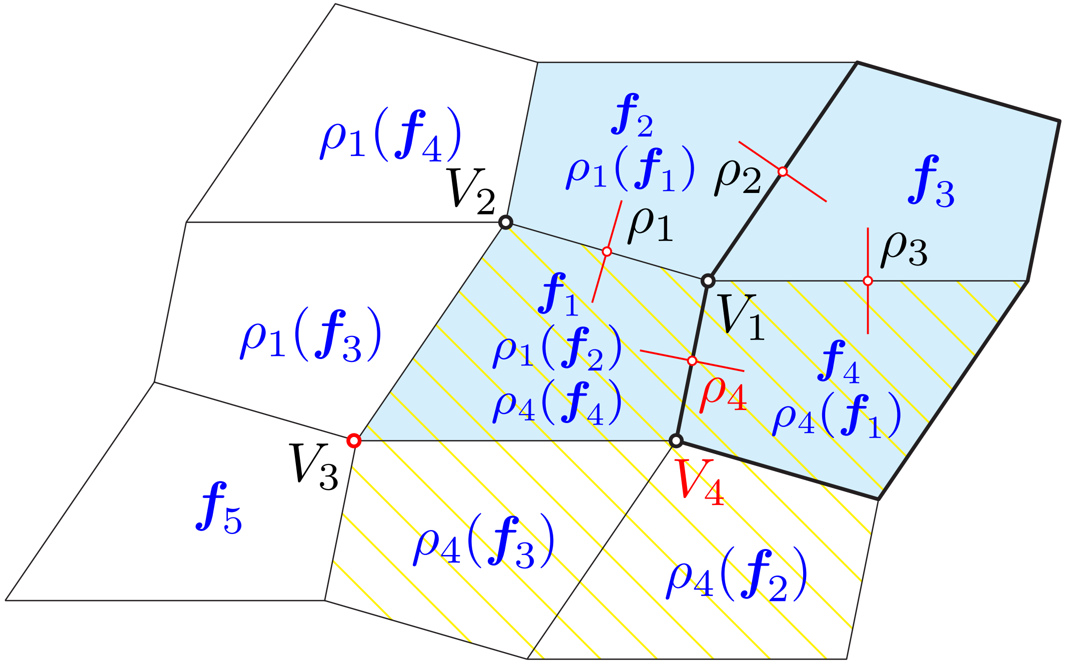


ρ_1 maps the pyramide with vertex V_1 onto the pyramide with vertex V_2.

ρ_4 maps the pyramide with vertex V_1 onto the pyramide with vertex V_4.

There are two possibilities to continue the flection onto the fourth pyramide with vertex V_3 .

2. Two examples of flexible quad meshes



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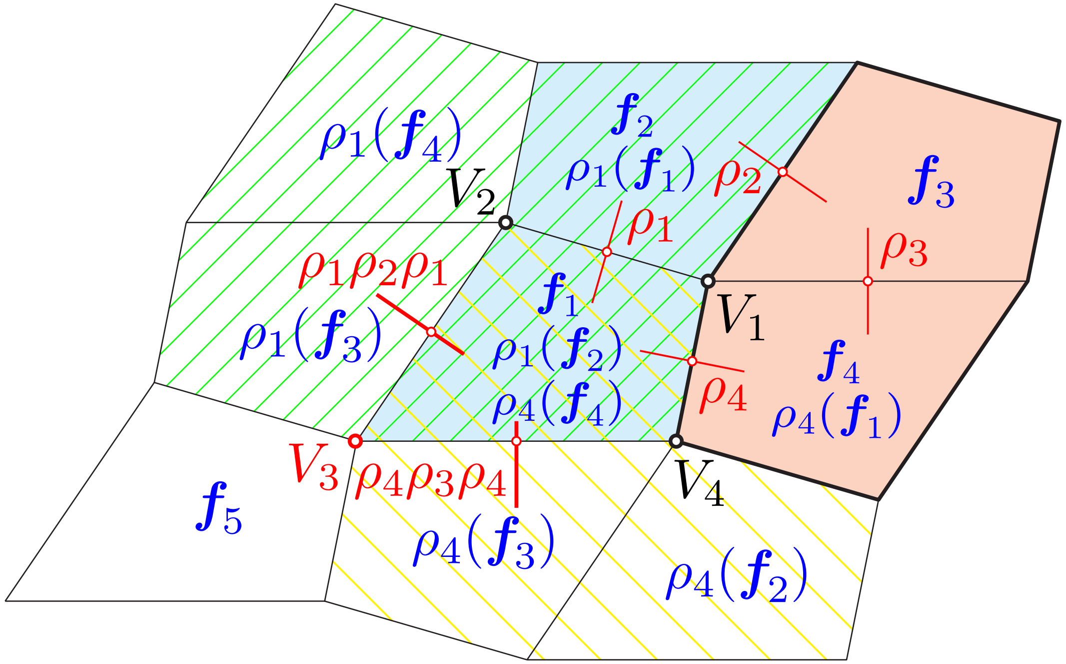


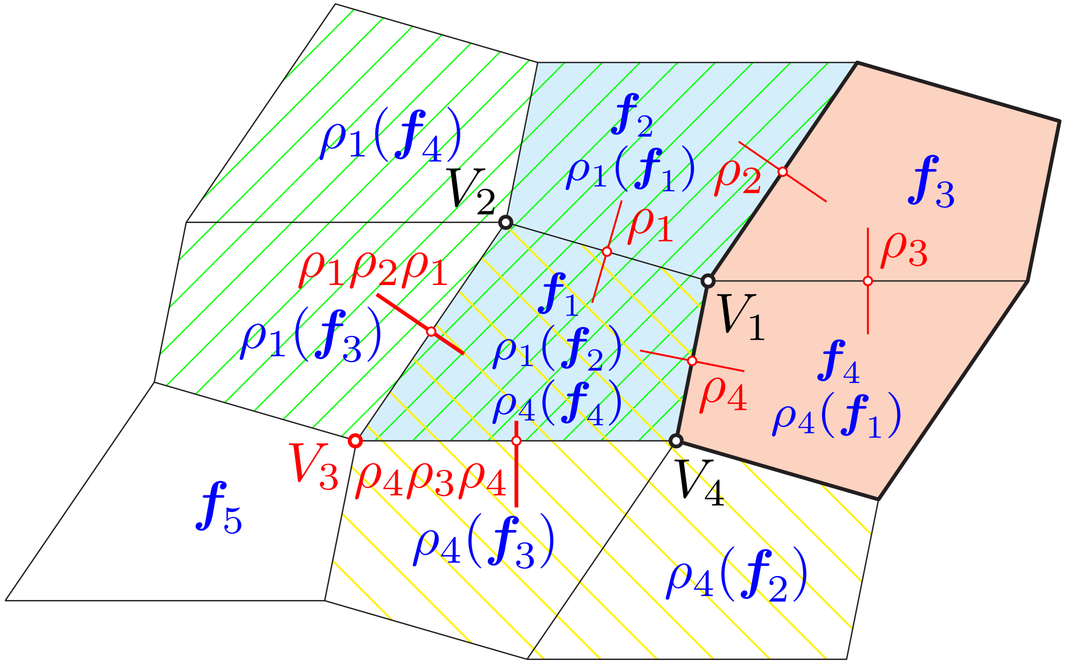
$\rho_1 \circ \rho_2 \circ \rho_1$ maps the pyramide with vertex V_2 onto the pyramide with vertex V_3.

$\rho_4 \circ \rho_3 \circ \rho_4$ maps the pyramide with vertex V_4 onto the pyramide with vertex V_3.

$\boldsymbol{f}_5$ is the image of $\boldsymbol{f}_2 = \rho_1(\boldsymbol{f}_1)$ under $\rho_1 \circ \rho_2 \circ \rho_1$, and image of $\boldsymbol{f}_4 = \rho_4(\boldsymbol{f}_1)$ under $\rho_4 \circ \rho_3 \circ \rho_4$, as $\boldsymbol{f}_5 = \rho_1 \circ \rho_2(\boldsymbol{f}_1) = \rho_4 \circ \rho_3(\boldsymbol{f}_1)$.

The complete pose arises from the “hexagon” $f_3 \cup f_4$ under iterations of the helical motions $\rho_1 \circ \rho_2$ and $\rho_4 \circ \rho_1$.

2. Two examples of flexible quad meshes



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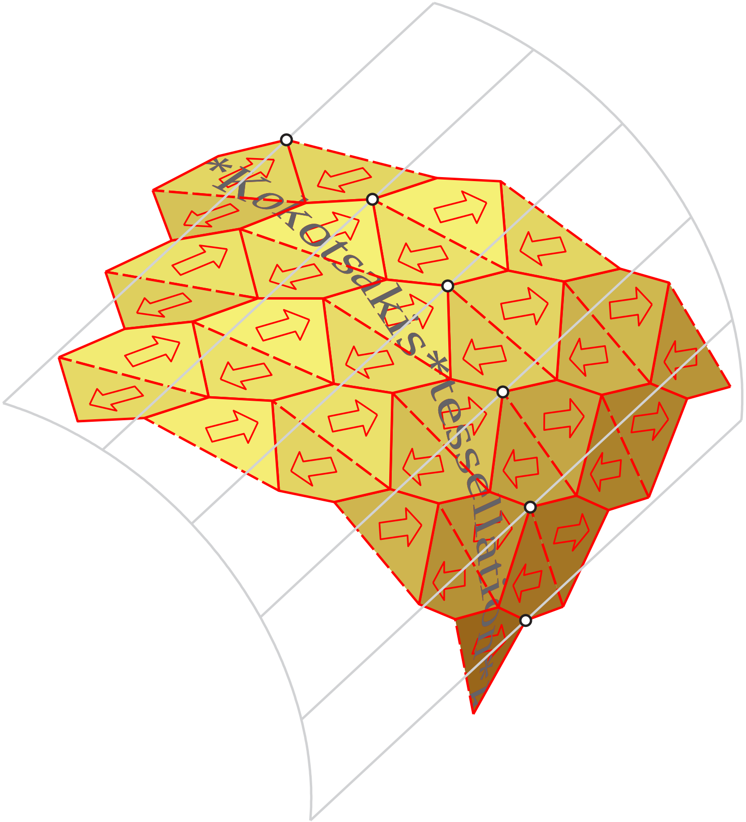
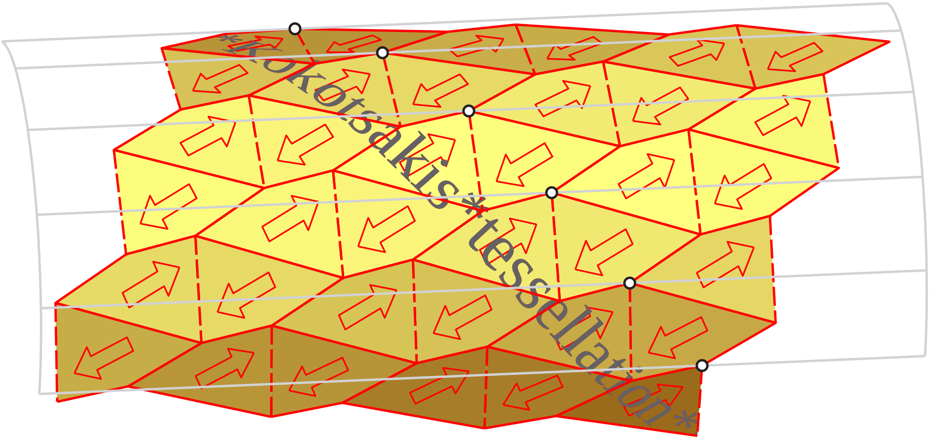
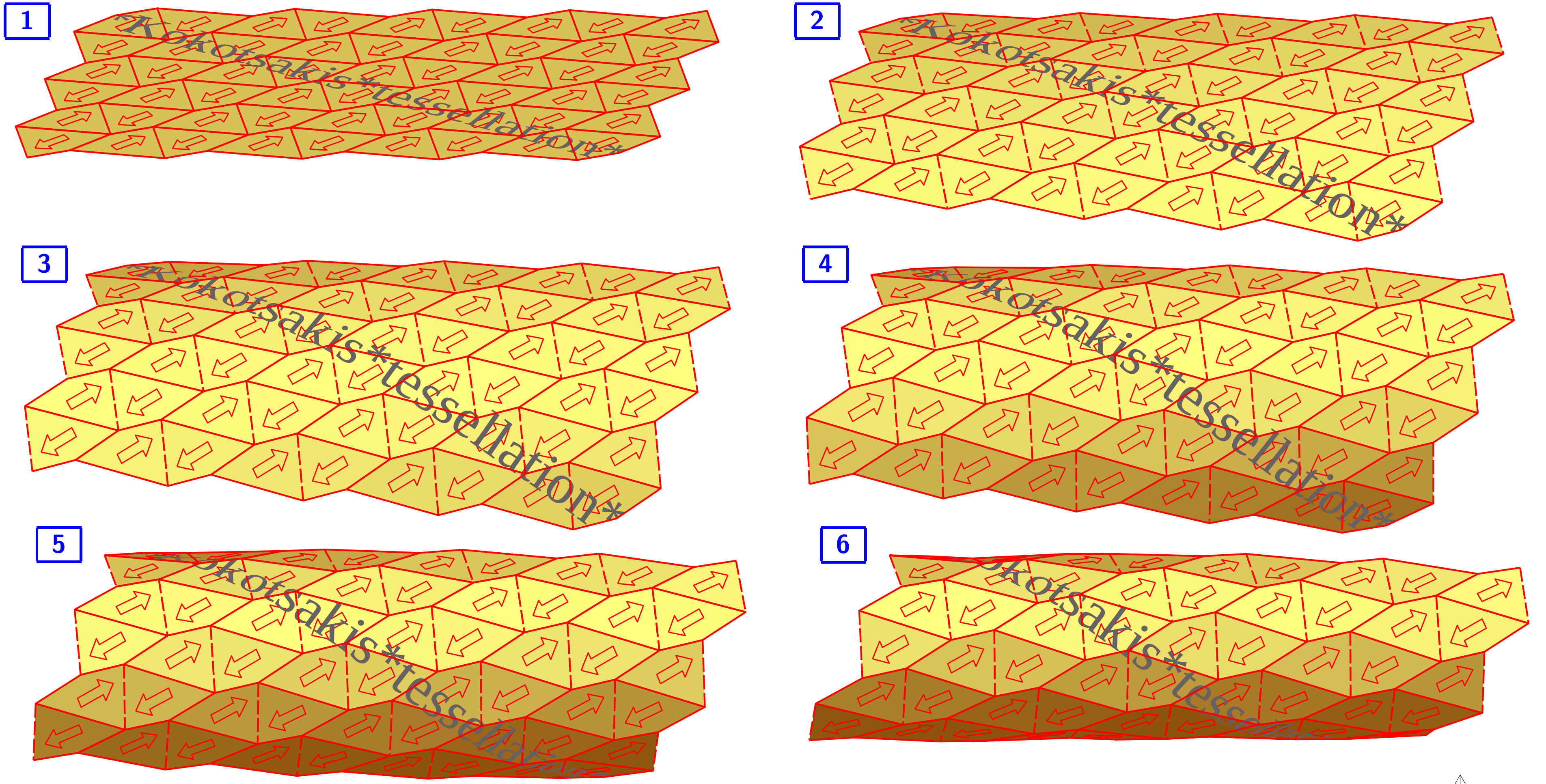




For each pose all vertices are located on a right circular cylinder.

2. Two examples of flexible quad meshes



2. Two examples of flexible quad meshes

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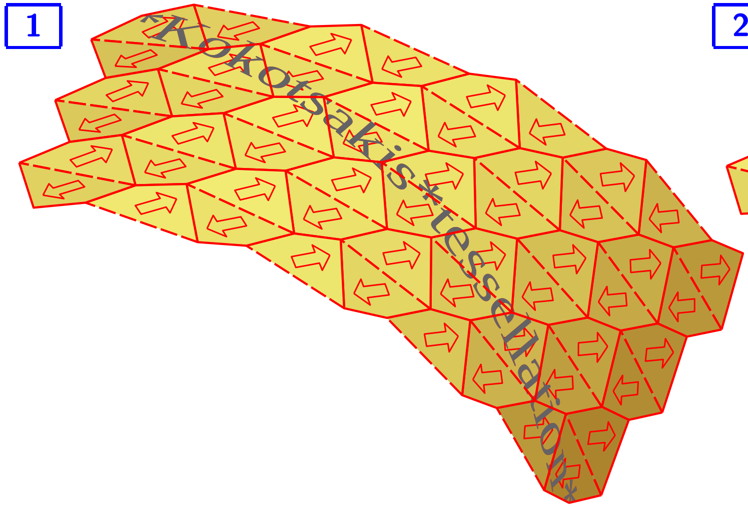

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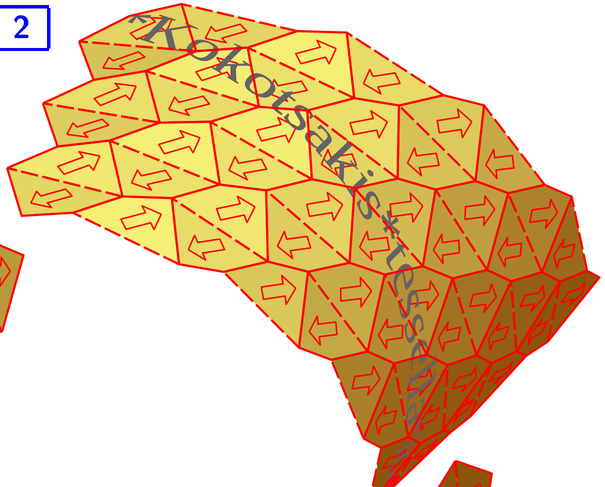

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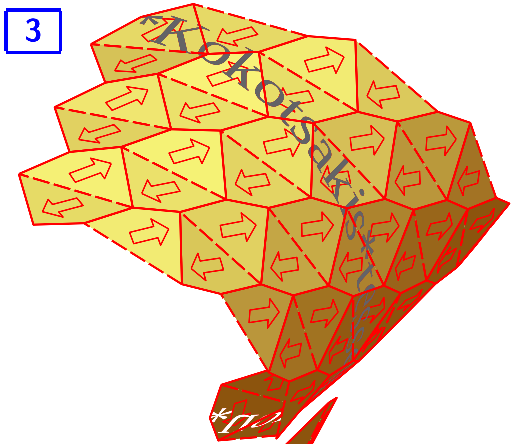

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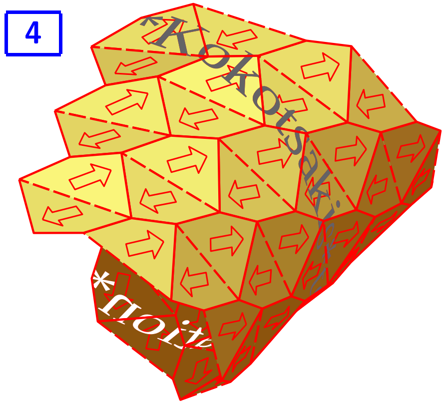

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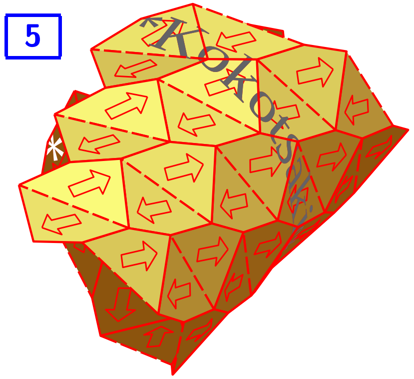

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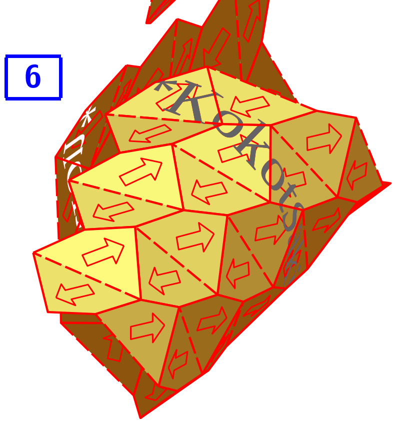

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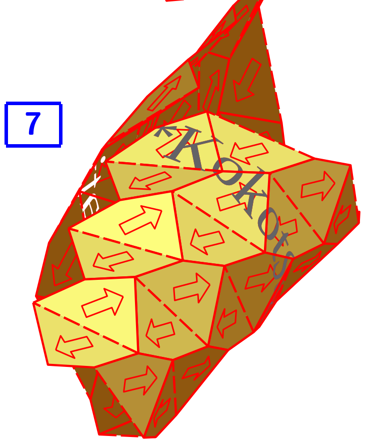
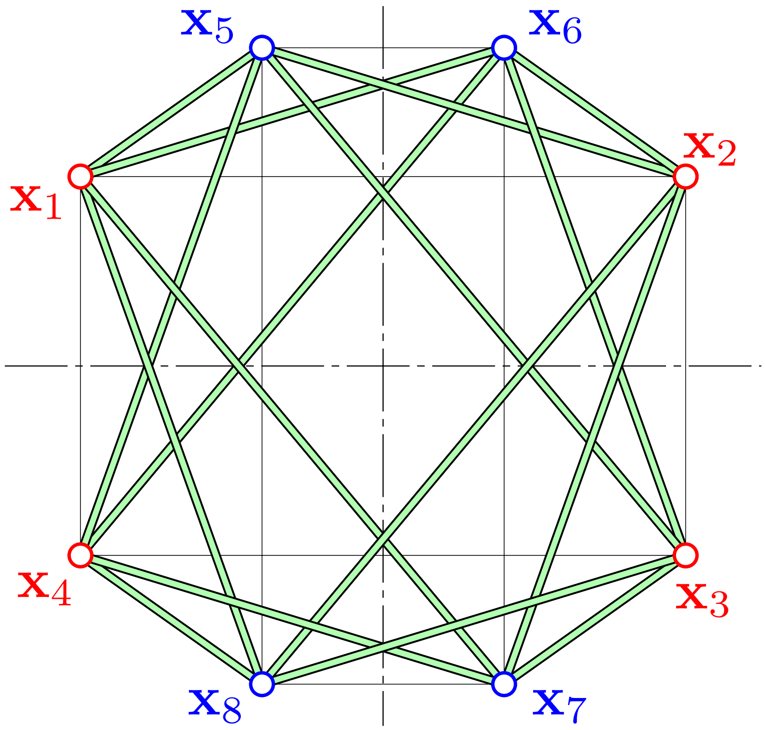


3. What means flexible?

Definition: A *framework* F in $\mathbb{R}^d$ consists of a set V of *vertices* $\{\mathbf{x}_1, \dots, \mathbf{x}_v\}$ and a set of *edges* $\{(i,j) \mid i<j,\ 1 \le i,j \le v\}$.
For any edge $\mathbf{x}_i\mathbf{x}_j$ of F the length is denoted by $l_{ij} := \|\mathbf{x}_j - \mathbf{x}_i\|$.



F is called *continuously flexible* if there is a continuous family F_t of frameworks for $0 \le t \le 1$ with $F_0 = F$ and $\|\mathbf{x}_j(t) - \mathbf{x}_i(t)\| = l_{ij}$, provided there are at least two vertices which do not keep their distance constant.

The family F_t is called a *flexion* of F .

3. What means flexible?

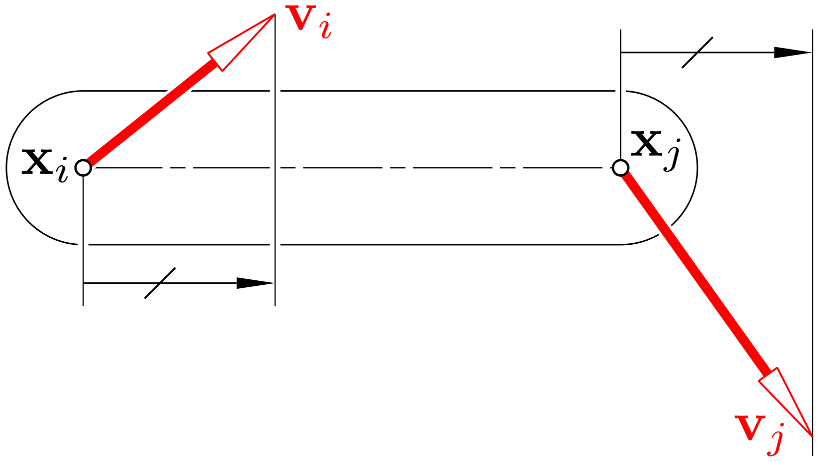
Suppose, the lengths of edges are only infinitesimally constant.

Definition:

A framework is called infinitesimally flexible $\iff$

to each vertex $\mathbf{x}_i$ a velocity vector $\mathbf{v}_i$ can be assigned such that

- for any edge $\mathbf{x}_i\mathbf{x}_j$ the projection theorem holds, and
- the assignment is nontrivial, i.e., the velocity vectors do not originate from a motion of the framework as a rigid body.



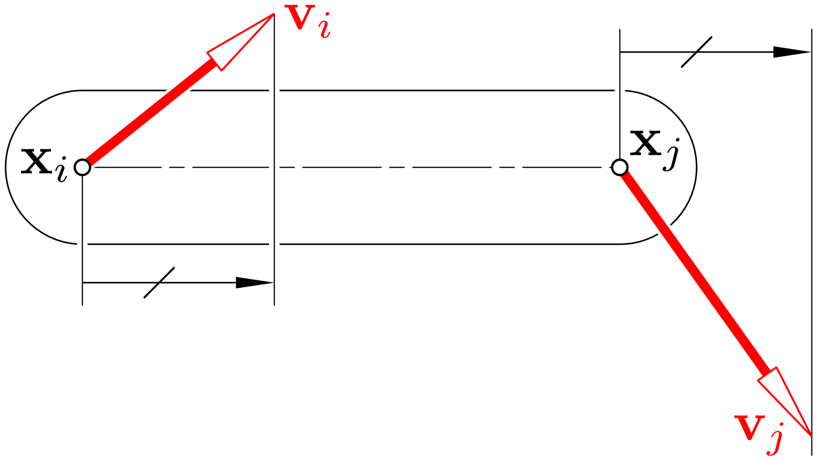
Projection theorem

3. What means flexible?

$$(\mathbf{x}_i - \mathbf{x}_j) \cdot (\mathbf{x}_i - \mathbf{x}) = \text{const.} \implies$$

$$(\mathbf{x}_i - \mathbf{x}_j) \cdot (\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j) = 0 \iff$$

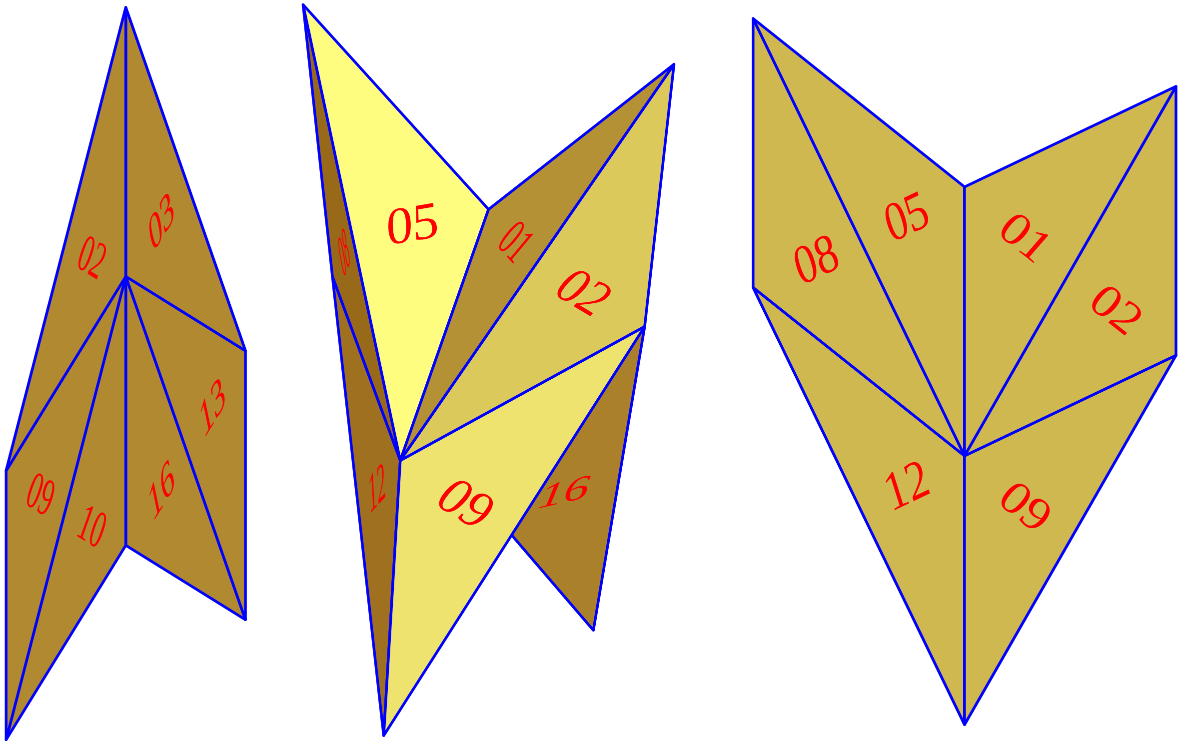
$$(\mathbf{x}_i - \mathbf{x}_j) \cdot \underbrace{\dot{\mathbf{x}}_i}_{\mathbf{v}_i} = (\mathbf{x}_i - \mathbf{x}_j) \cdot \underbrace{\dot{\mathbf{x}}_j}_{\mathbf{v}_j}$$



Projection theorem

A a real-world model of an infinitesimally flexible framework is really slightly flexible due to bendings of the faces and clearances at the vertices and edges.

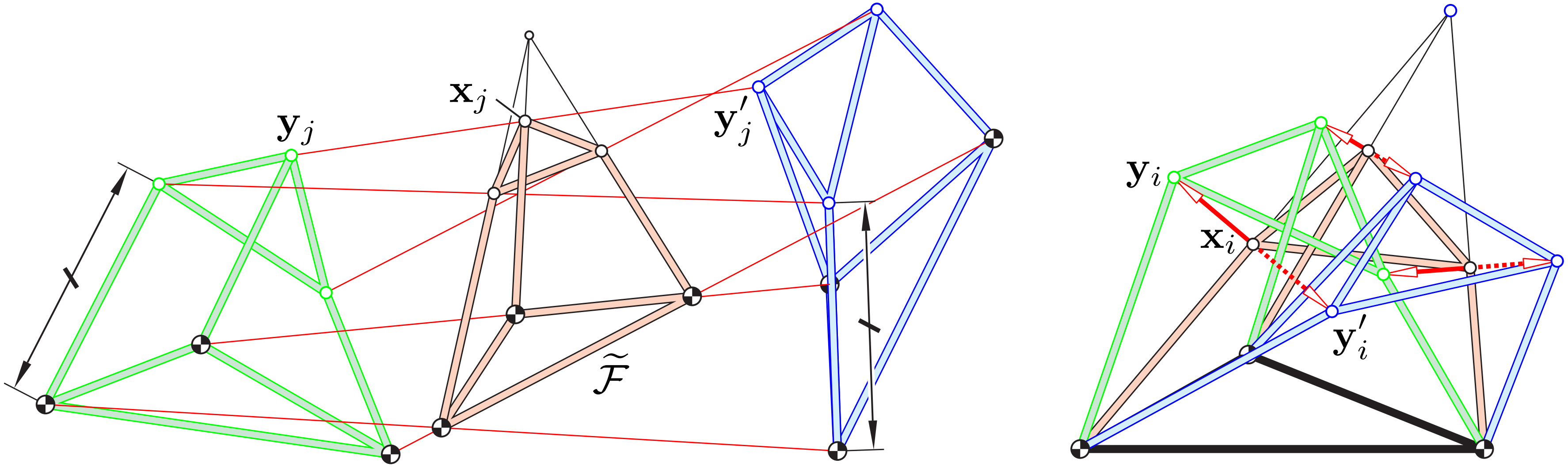
3. What means flexible?



This polyhedron called *“Vierhorn”* is locally rigid, but can snap between its spatial shape and two flat realizations in the planes of symmetry (W. Wunderlich, C. Schwabe).

At the science exposition *“Phänomena”* 1984 in Zürich this polyhedron was exposed and falsely stated that it is continuously flexible.

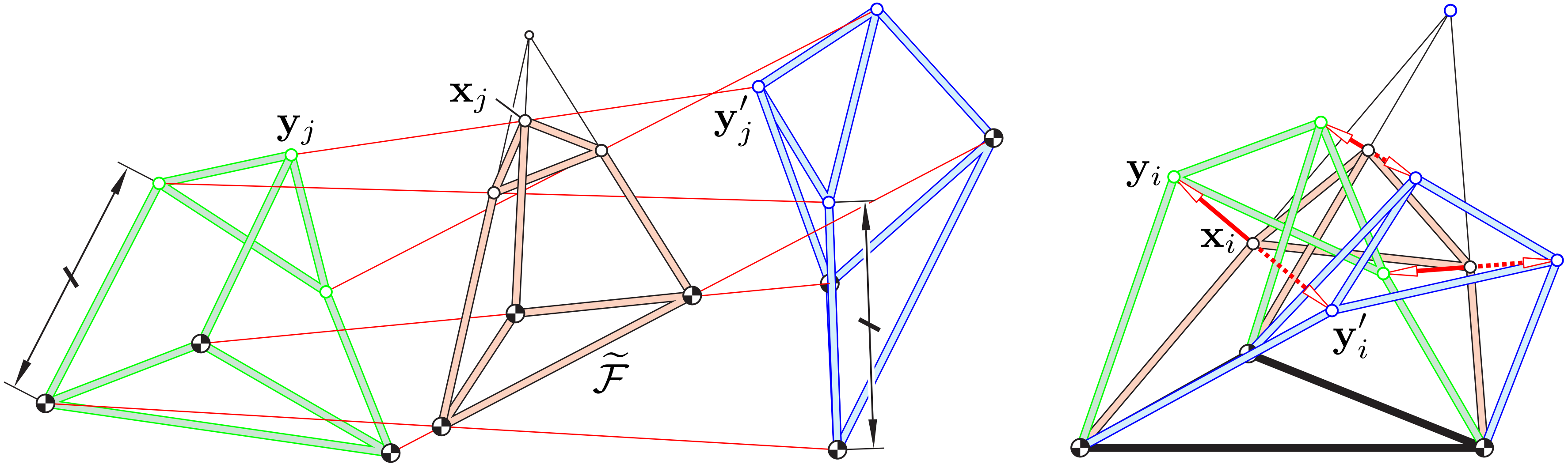
3. What means flexible?



Theorem (W. Whiteley (1990)), *Principle of "**averaging**":*

Let $\mathbf{y}_1, \ldots, \mathbf{y}_v$ *and* $\mathbf{y}'_1, \ldots, \mathbf{y}'_v$ *be vertices of two incongruent realizations of a framework* $\mathcal{F}$*. Then the midpoints* $\mathbf{x}_i = \frac{1}{2}(\mathbf{y}_i + \mathbf{y}'_i)$ *constitute an infinitesimally flexible framework* $\widetilde{\mathcal{F}}$ *of the same combinatorial structure with velocity vectors* $\mathbf{x}_{i,1} = \frac{1}{2}(\mathbf{y}_i - \mathbf{y}'_i)$*, and vice versa.*

3. What means flexible?



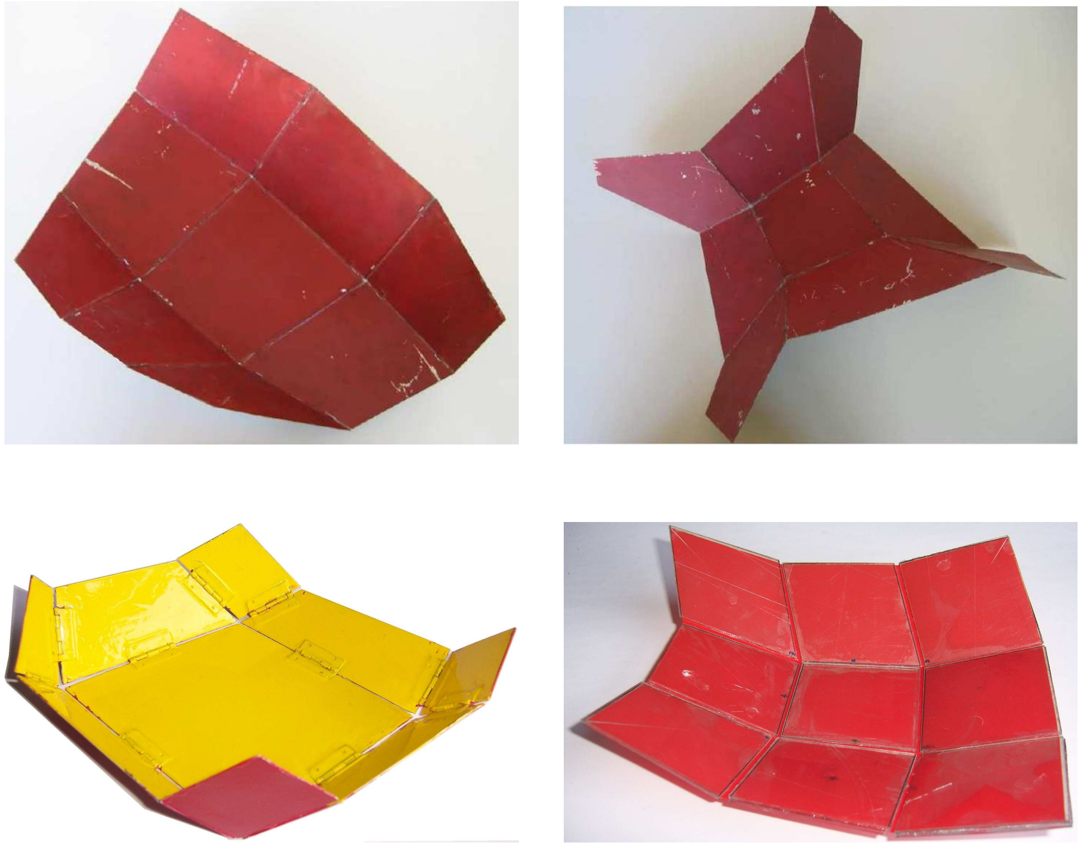
Proof: The condition $(\mathbf{y}_i - \mathbf{y}_j)^2 - (\mathbf{y}'_i - \mathbf{y}'_j)^2 = 0$ can be rewritten as

$$(\mathbf{y}_i - \mathbf{y}_j + \mathbf{y}'_i - \mathbf{y}'_j) \cdot (\mathbf{y}_i - \mathbf{y}_j - \mathbf{y}'_i + \mathbf{y}'_j) = 0$$

$$\Big(\underbrace{(\mathbf{y}_i + \mathbf{y}'_i)}_{2\mathbf{x}_i} - \underbrace{(\mathbf{y}_j + \mathbf{y}'_j)}_{2\mathbf{x}_j}\Big) \cdot \Big(\underbrace{(\mathbf{y}_i - \mathbf{y}'_i)}_{2\,\mathbf{v}_i} - \underbrace{(\mathbf{y}_j - \mathbf{y}'_j)}_{2\,\mathbf{v}_j}\Big) = 0 \;\ldots\; \text{Projection Thm.}$$

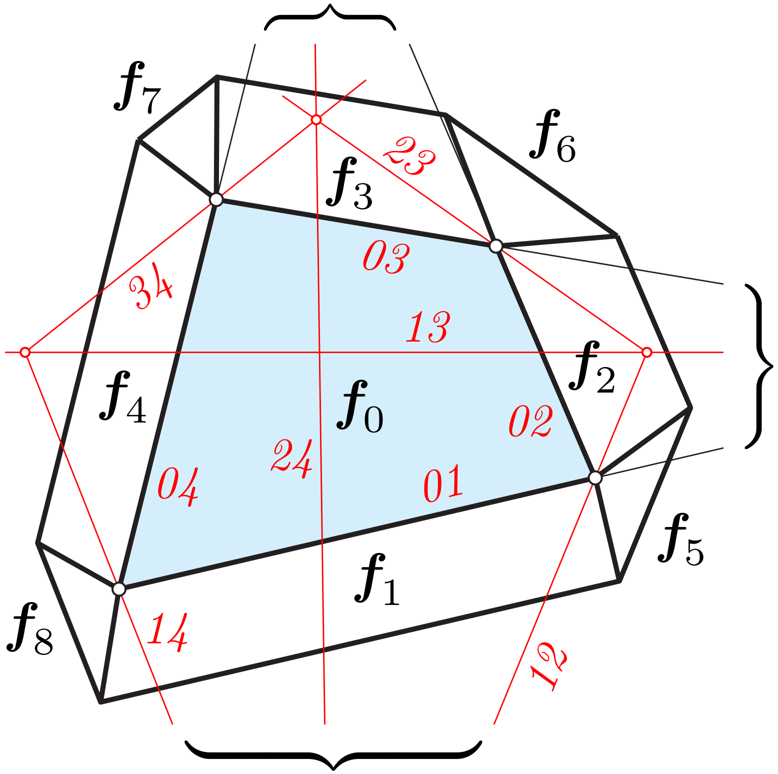
3. What means flexible?

Let us return to flexible Kokotsakis meshes.



courtesy Nadja Posselt, Uwe Hanke, TU Dresden

3. What means flexible?



Theorem: (A. Kokotsakis (1932))

A Kokotsakis mesh is *infinitesimally flexible* $\iff$ *the points of intersection between the traces of* (f_1, f_3), (f_5, f_6) *and* (f_7, f_8) *are collinear.*

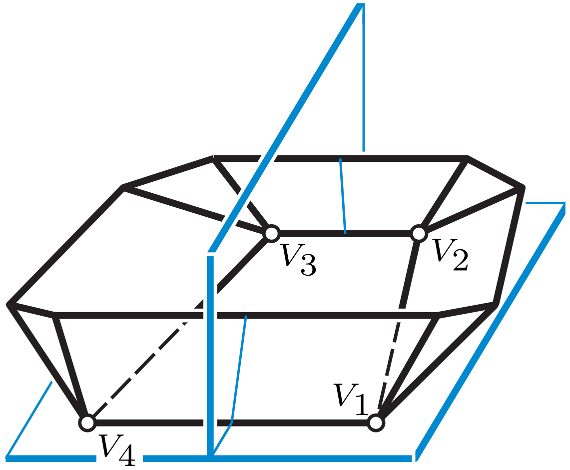
This is equivalent to the collinearity of the intersection points (f_2, f_4), (f_6, f_7) *and* (f_8, f_5).

The principle of “averaging” gives rise to snapping Kokotsakis meshes.

4. Flexible Kokotsakis meshes

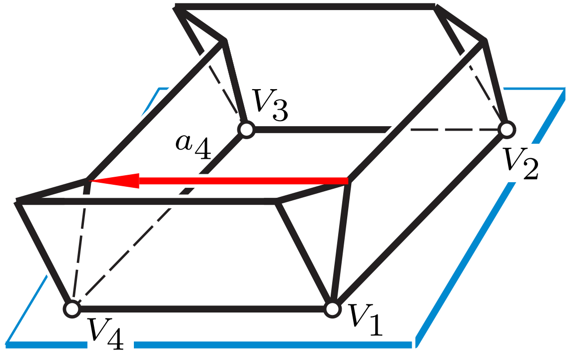
I. Planar-symmetric type (KOKOTSAKIS 1932):

The reflection in the plane of symmetry of V_1 and V_4 maps each horizontal fold onto itself while the two vertical folds are exchanged.

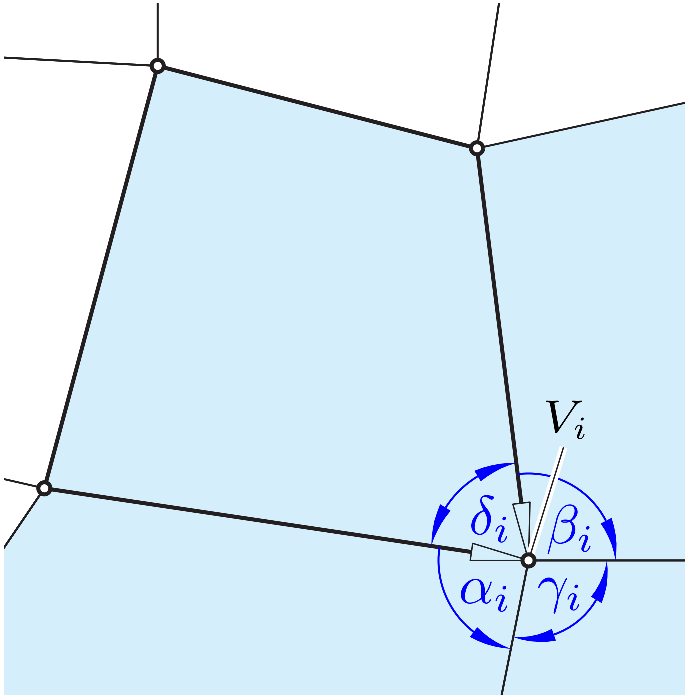


II. Translational type:

There is a translation $V_1 \mapsto V_4$ and $V_2 \mapsto V_3$ mapping the three faces on the right hand side onto the triple on the left hand side.



4. Flexible Kokotsakis meshes



Miura-ori is a special case of

Theorem: [KOKOTSAKIS 1932]
A Kokotsakis mesh is flexible when at each vertex V_i opposite angles are either equal or supplementary, i.e.,

$$\alpha_i = \beta_i, \quad \gamma_i = \delta_i \text{ or}$$
$$\alpha_i = \pi - \beta_i, \quad \gamma_i = \pi - \delta_i\,.$$

A discrete conjugate net where all vertices are of this type is called Voss surface:

- Its folds are geodesics,
- it is continuously flexible.

4. Flexible Kokotsakis meshes

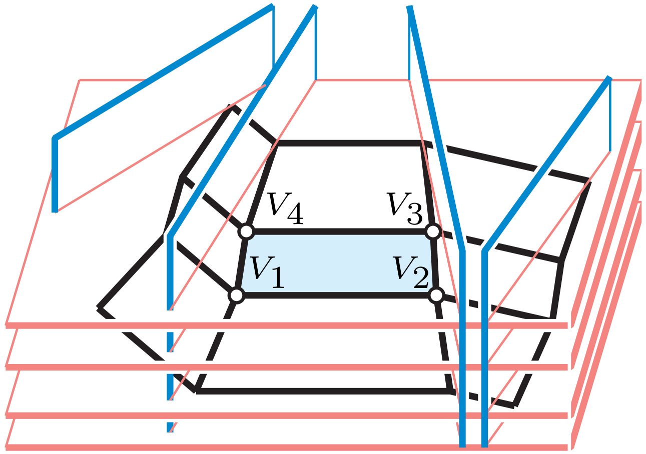
III. Generalized isogonal type:

A. Kokotsakis (1932): At each vertex opposite angles are congruent (Voss surface).

G. Nawratil (2010): At at least two of the four pyramides opposite angles are congruent.

IV. Orthogonal type (Graf, Sauer 1931):

Here the horizontal folds are located in parallel (say: horizontal) planes, the vertical folds in vertical planes (T-flat).

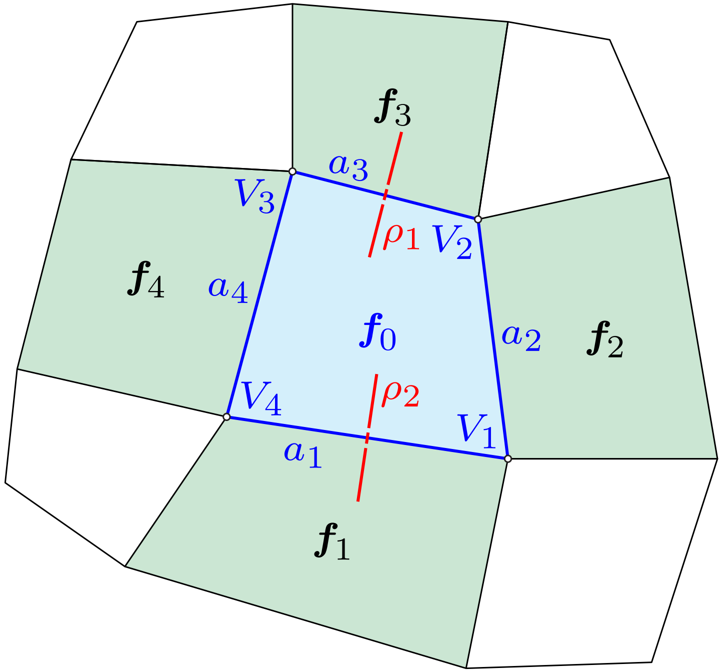


4. Flexible Kokotsakis meshes

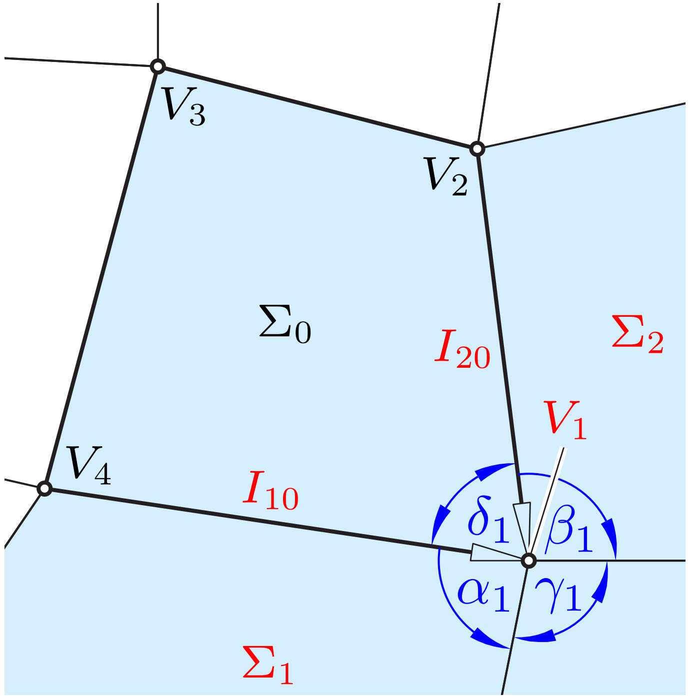
V. Line-symmetric type (H.S. 2009):

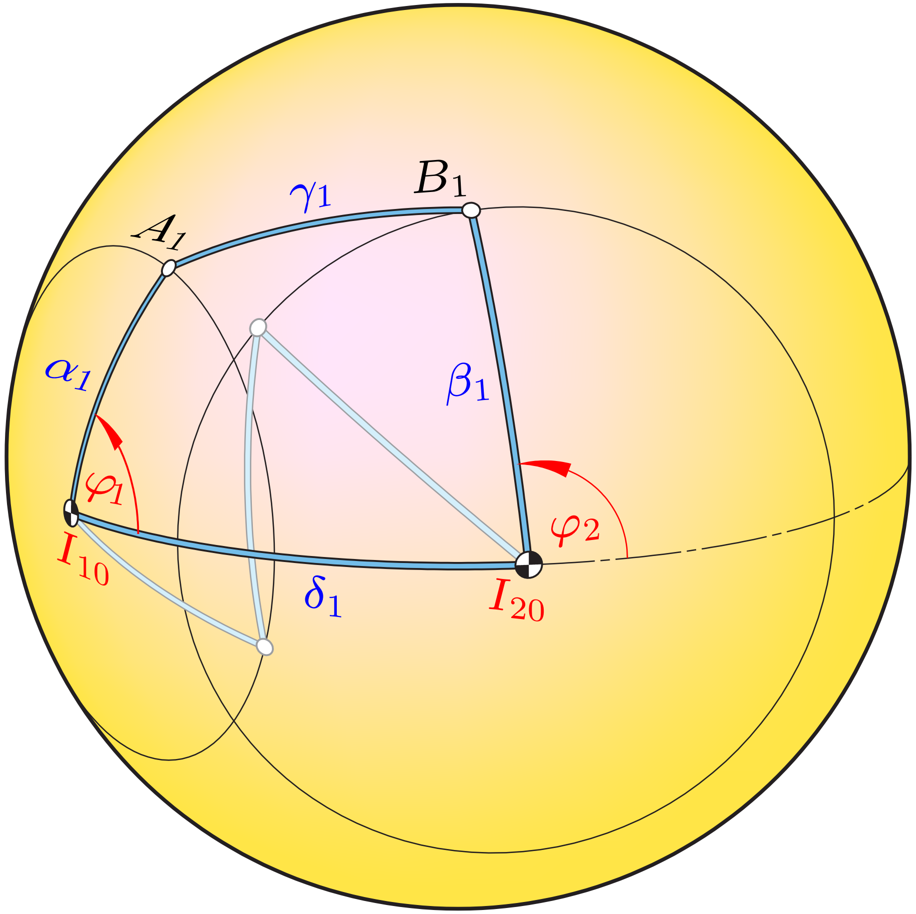
A line-reflection maps the pyramide at V_1 onto that of V_4; another one exchanges the pyramides at V_2 and V_3.

This includes Kokotsakis' example of a flexible tessellation.



4. Flexible Kokotsakis meshes

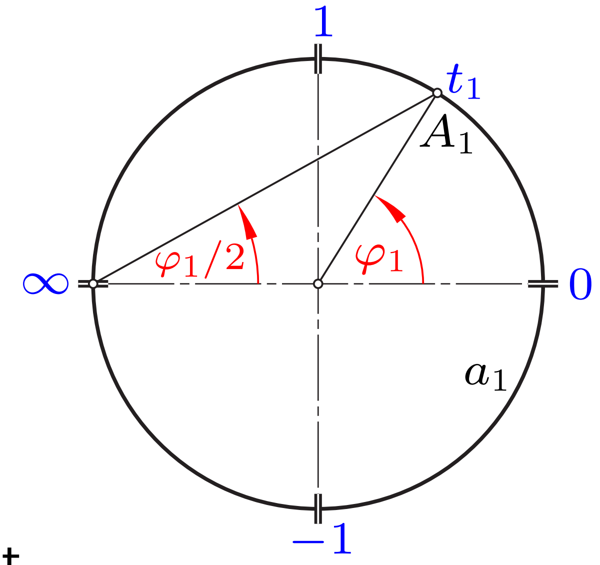


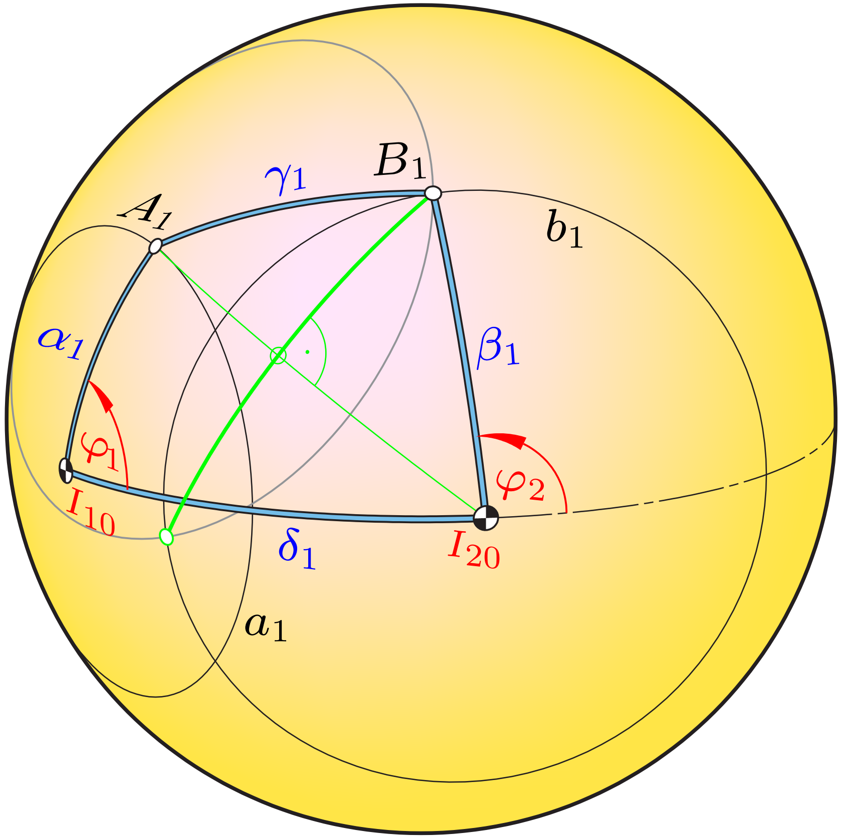


Four-bar motion Σ_2/Σ_1 and its spherical image

$$0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < 180°$$

4. Flexible Kokotsakis meshes





We set

$$t_1 := \tan\frac{\varphi_1}{2}, \quad t_2 := \tan\frac{\varphi_2}{2}\,.$$

t_1, t_2 are projective coordinates on the path circles a_1, b_1 of A_1 and B_1, resp., and obtain

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0 \quad \text{with} \quad c_{ik} = f(\alpha_1, \ldots, \delta_1)$$

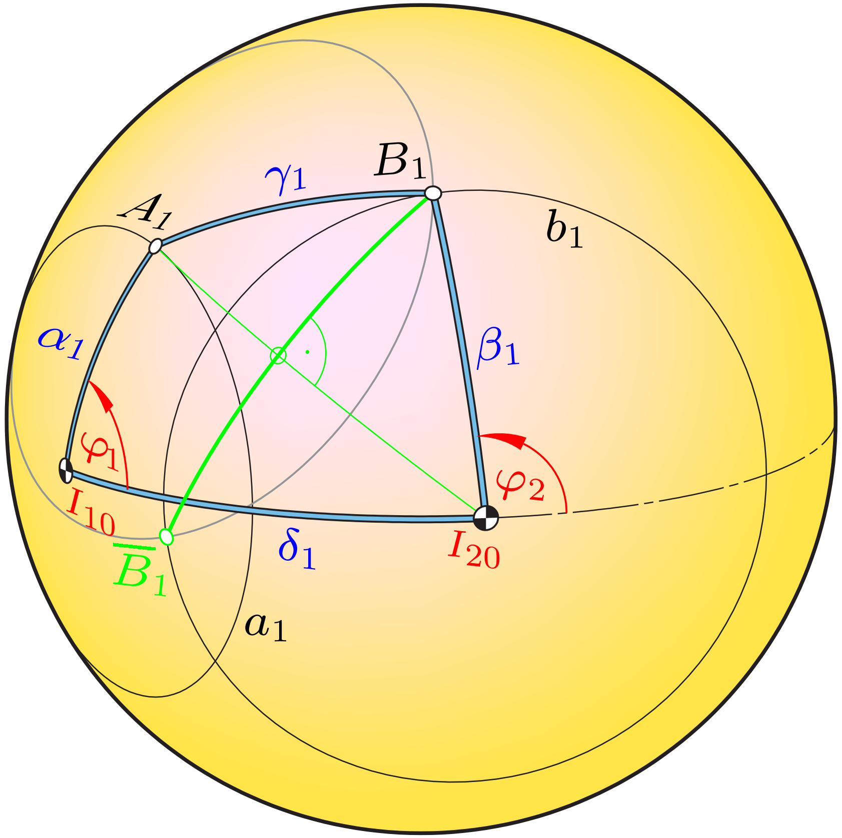
4. Flexible Kokotsakis meshes

The transmission $\varphi_1 \mapsto \varphi_2$ by the four-bar linkage defines a 2-2-correspondence between the circles a_1 and b_1:

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$

. . . like in the plane.

$$\left(t_1 := \tan\frac{\varphi_1}{2}, \quad t_2 := \tan\frac{\varphi_2}{2}\right)$$



4. Flexible Kokotsakis meshes

The coefficients in the biquadratic equation

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$

are:

$$\begin{aligned}
c_{22} &= \sin\tfrac{\alpha_1-\beta_1+\gamma_1+\delta_1}{2}\ \sin\tfrac{\alpha_1-\beta_1-\gamma_1+\delta_1}{2}\\
c_{20} &= \sin\tfrac{\alpha_1+\beta_1+\gamma_1+\delta_1}{2}\ \sin\tfrac{\alpha_1+\beta_1-\gamma_1+\delta_1}{2}\\
c_{11} &= -2\,\sin\alpha_1\,\sin\beta_1 \neq 0\\
c_{02} &= \sin\tfrac{\alpha_1+\beta_1+\gamma_1-\delta_1}{2}\ \sin\tfrac{\alpha_1+\beta_1-\gamma_1-\delta_1}{2}\\
c_{00} &= \sin\tfrac{\alpha_1-\beta_1+\gamma_1-\delta_1}{2}\ \sin\tfrac{\alpha_1-\beta_1-\gamma_1-\delta_1}{2}
\end{aligned}$$

4. Flexible Kokotsakis meshes

The 2-2-correspondence

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$

depends only on the ratio of the coefficients.

Theorem:

For any spherical four-bar linkage the coefficients c_{ik} *are algebraically dependent:* c_{11} *is a root of a 6th-degree polynomial with coefficients depending on* $c_{00}, c_{02}, c_{20}, c_{22}$.

Conversely, in the complex extension any choice of coefficients in the biquadratic equation above defines the spherical four-bar linkage uniquely — up to replacement of vertices by their antipodes. However, the vertices need not be real.

4. Flexible Kokotsakis meshes

The 2-2-correspondence

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$

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4. Flexible Kokotsakis meshes

Particular cases of the 2-2-correspondence:

1) The 2-2-correspondence between a_1 and b_1 splits into two projectivities $\Longleftrightarrow$ the quadrangle is a spherical isogram, i.e., $\beta_1 = \alpha_1$ and $\delta_1 = \gamma_1$ $(c_{00} = c_{22} = 0)$.

In this case (. . . isogonal type)

$$t \;\mapsto\; t_2 = \frac{\sin\alpha_1 \pm \sin\gamma_1}{\sin(\alpha_1 - \gamma_1)}\, t_1 \quad \text{for} \quad \alpha_1 \neq \gamma_1,\, \pi - \gamma_1$$

combines two linear functions.

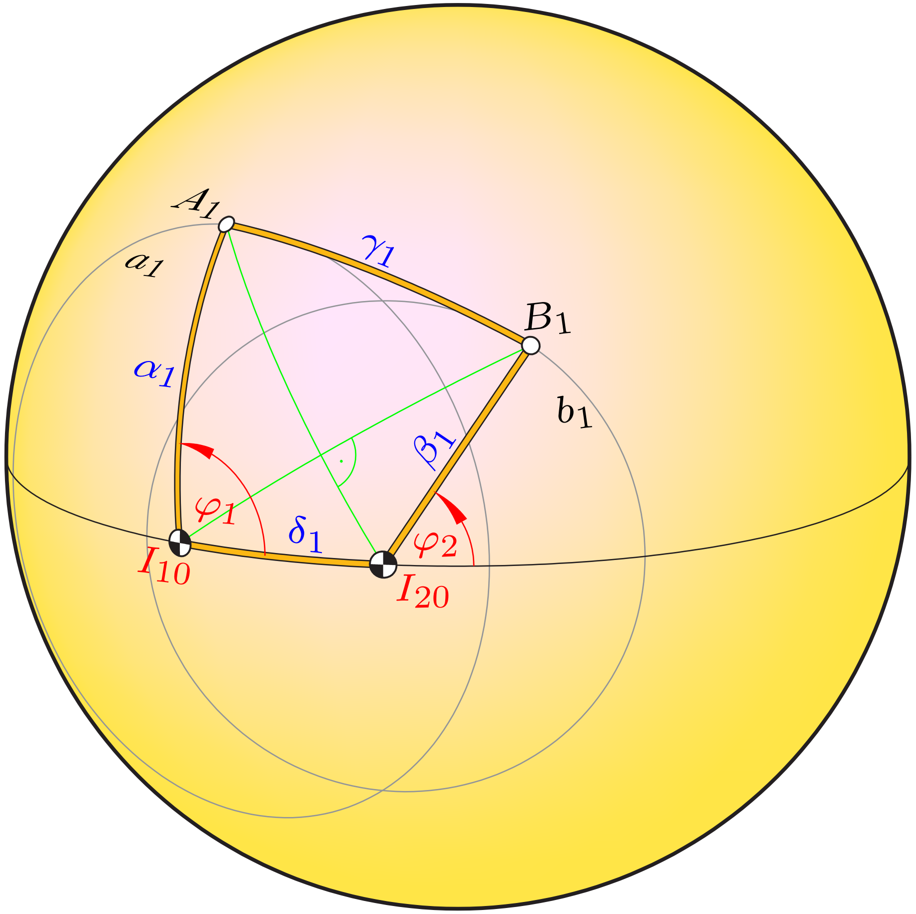
4. Flexible Kokotsakis meshes

2) Under the condition

$$\cos\alpha_1\ \cos\beta_1 = \cos\gamma_1\ \cos\delta_1$$

(equivalent to $\det(c_{ik}) = 0$) each quadrangle has orthogonal diagonals (. . . orthogonal type).

The 2-2-correspondence maps pairs of points on a_1 aligned with I_{20} onto pairs of points on b_2 located on the orthogonal line through I_{10}.



4. Flexible Kokotsakis meshes

3) Deltoid case:

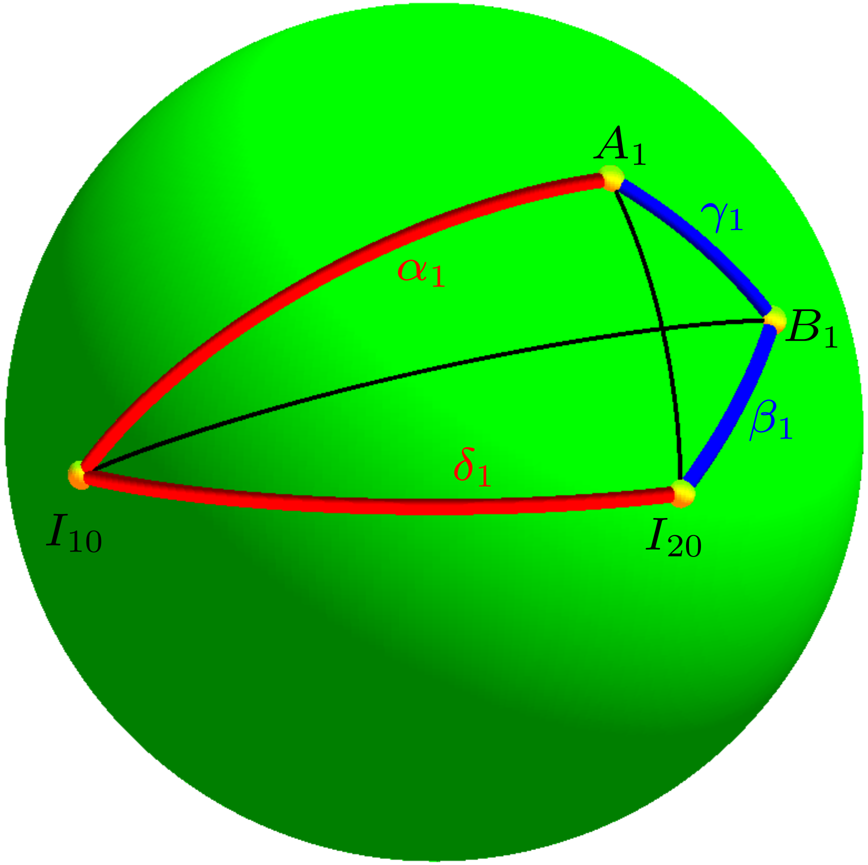
$$\alpha_1 = \delta_1 \implies c_{00} = c_{02} = 0\,.$$

The 2-2-correspondence splits:

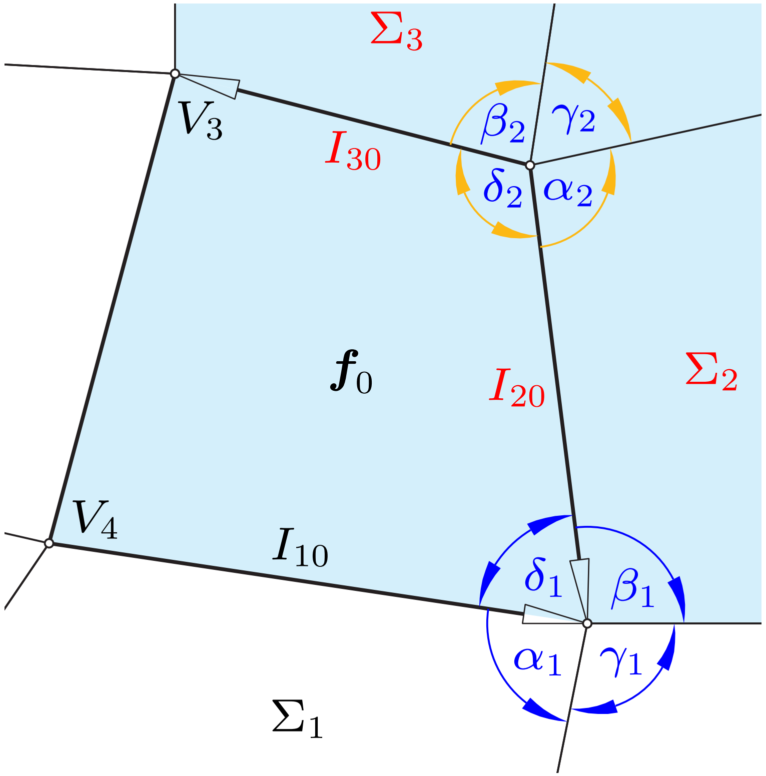
$$t_1\left(c_{22}t_1t_2^2 + c_{20}t_1 + c_{11}t_2\right) = 0;$$

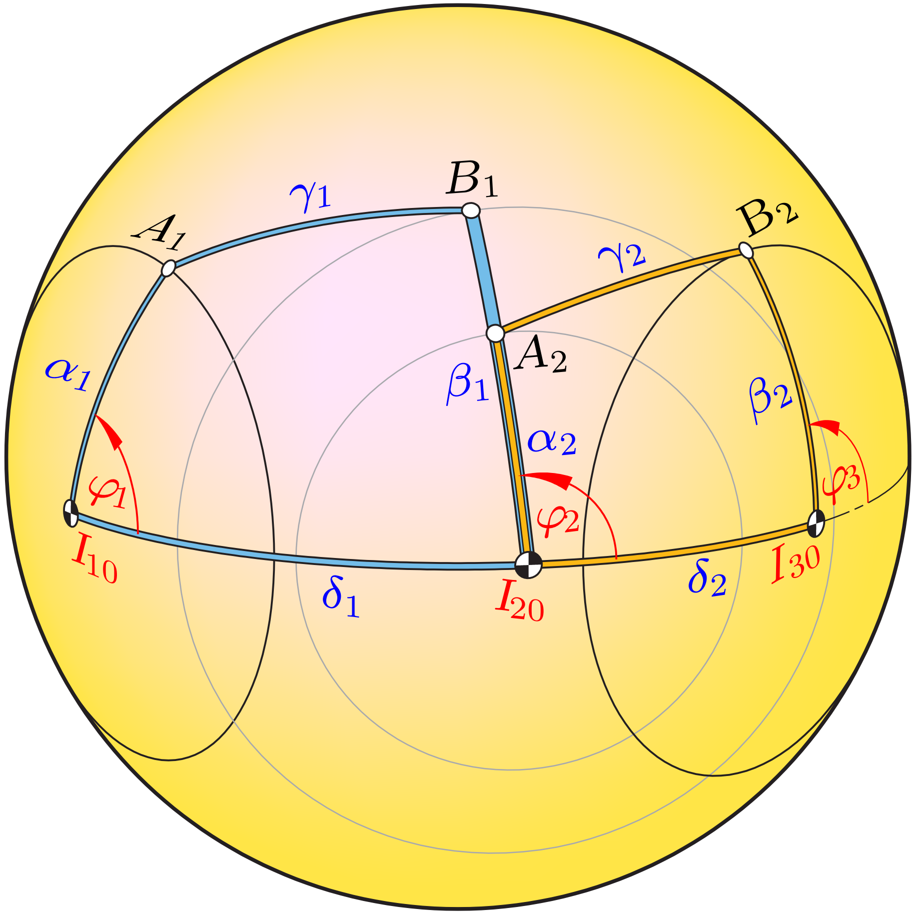
a) $t_1 = 0$ corresponds to all $t_2 \in \mathbb{R}$,

b) 1-2-correspondence.



4. Flexible Kokotsakis meshes





Composition of two four-bars Σ_2/Σ_1 and Σ_3/Σ_1 and their spherical images

4. Flexible Kokotsakis meshes

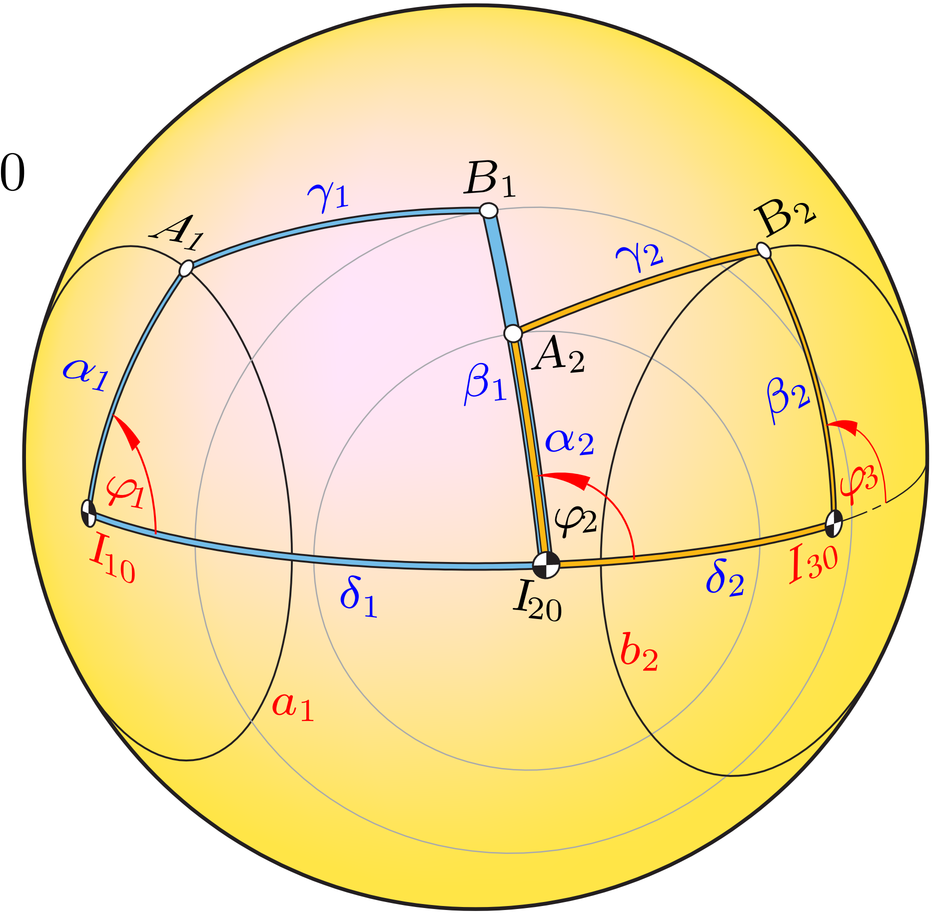
$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$
$$d_{22}t_3^2t_2^2 + d_{20}t_3^2 + d_{02}t_2^2 + d_{11}t_3t_2 + d_{00} = 0$$

The four-bar transmissions are equivalent to these two bilinear equations.

We eliminate t_2 by computing the resultant with respect to t_2. Thus we obtain a biquartic equation in

$$t_1 = \tan\frac{\varphi_1}{2} \text{ and } t_3 = \tan\frac{\varphi_3}{2}\,,$$

i.e., a **4-4-correspondence** between $A_1 \in a_1$ and $B_2 \in b_2$.



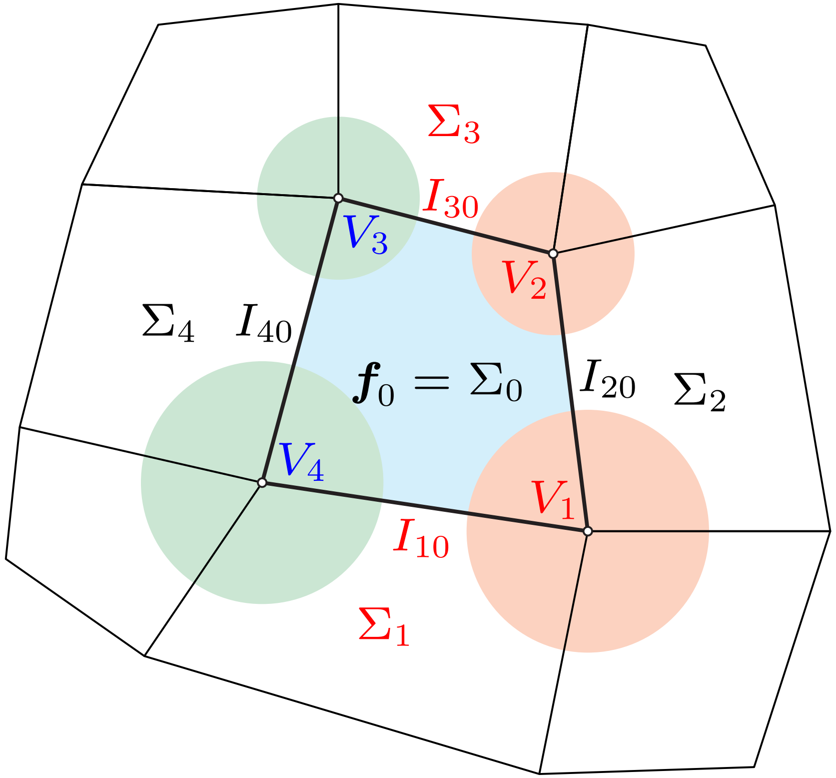
4. Flexible Kokotsakis meshes

Continuous flexibility of a Kokotsakis mesh for $n = 4$ means:
The 4-4-correspondence or – in the reducible case – one of its components can be decomposed in two different ways.

Particular cases:
III: In the isogonal case $(n \geq 4)$

$$\alpha_1 = \beta_1,\ \gamma_1 = \delta_1,\ \alpha_2 = \beta_2,\ \gamma_2 = \delta_2$$

the composition of two linear functions $t_1 \mapsto t_2$ and $t_2 \mapsto t_3$ is again linear (Kokotsakis, Graf, Sauer).



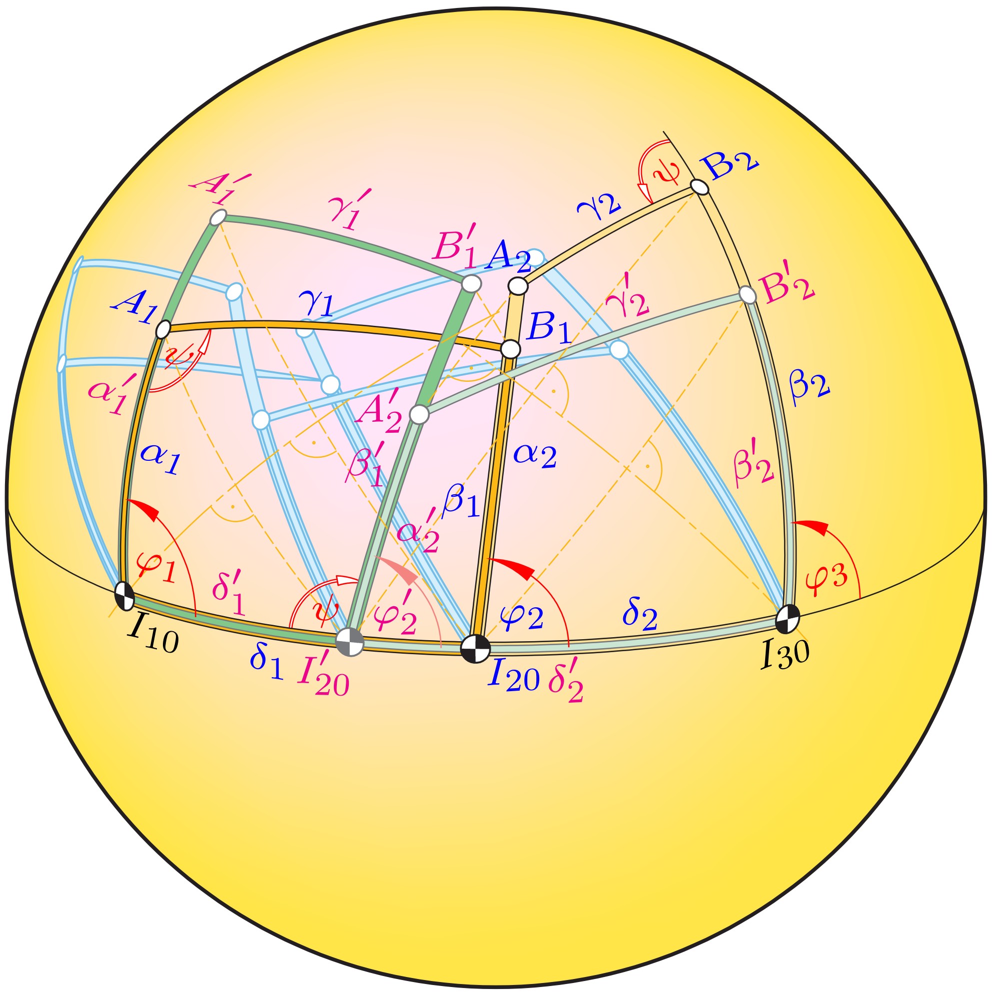
4. Flexible Kokotsakis meshes

V: Under the conditions

$$\alpha_1 + \beta_2 = \delta_1 + \delta_2$$

$$\mathrm{s}\alpha_1\mathrm{s}\gamma_1 : \mathrm{s}\beta_2\mathrm{s}\gamma_2 = \mathrm{s}\beta_1\mathrm{s}\delta_1 : \mathrm{s}\alpha_2\mathrm{s}\delta_2 = (\mathrm{c}\beta_1\mathrm{c}\delta_1 - \mathrm{c}\alpha_1\mathrm{c}\gamma_1) : (\mathrm{c}\beta_2\mathrm{c}\gamma_2 - \mathrm{c}\alpha_2\mathrm{c}\delta_2)$$

the 4-4-correspondance between t_1 and t_3 can be decomposed in two ways in the product of two 2-2-correspondences.



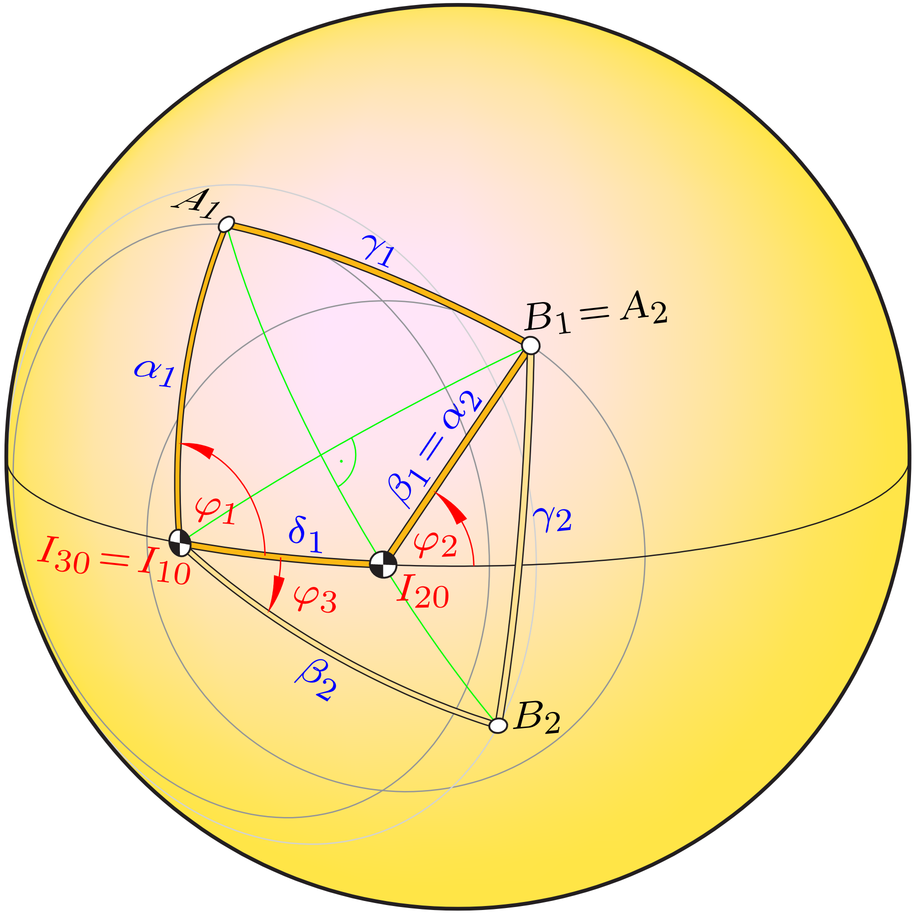
4. Flexible Kokotsakis meshes

IV: T-flat: Under the conditions

$$\cos\alpha_1\ \cos\beta_1 = \cos\gamma_1\ \cos\delta_1\,,\quad \alpha_2 = \beta_1\,,$$
$$\cos\alpha_2\ \cos\beta_2 = \cos\gamma_2\ \cos\delta_2\,,\quad \delta_2 = -\delta_1\,,$$

both four-bars share the orthogonal diagonals.

Due to GRAF and SAUER (1931) there is a second decomposition of the same kind; all four-bars share one diagonal (spherical DIXON mechanism).



4. Flexible Kokotsakis meshes

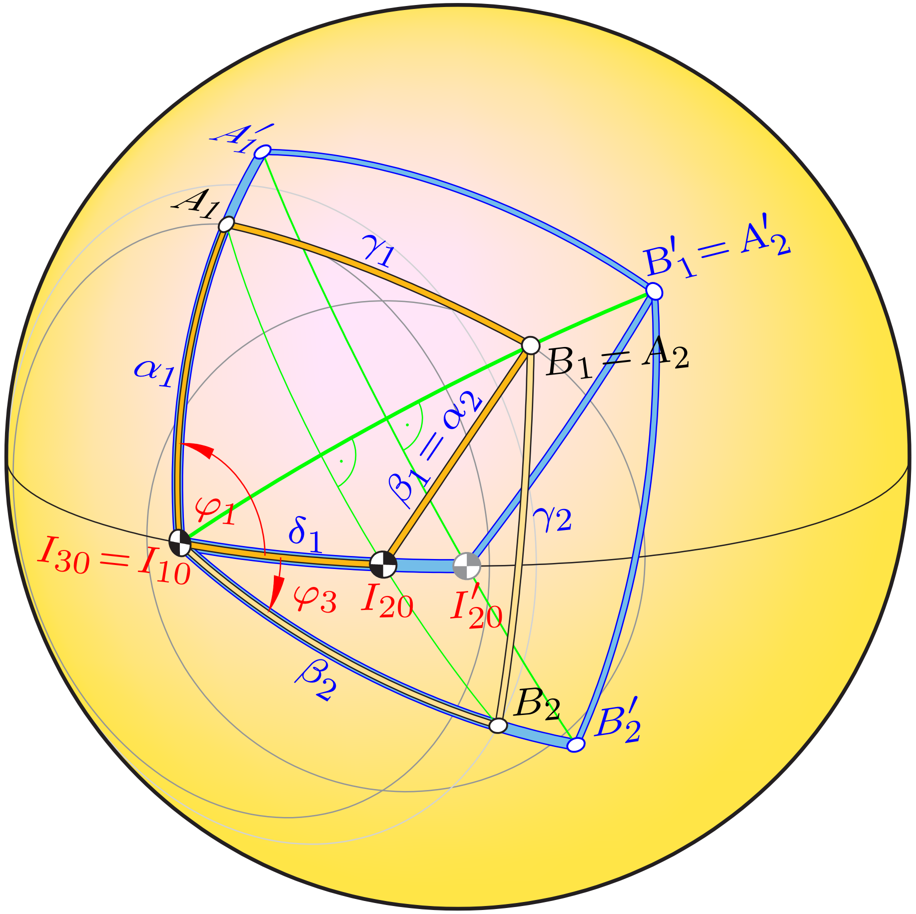
The 4-4-correspondence is the square of a 2-2-correspondence

$$c_{21}t_1^2t_3 + c_{12}t_1t_3^2 + c_{10}t_1 + c_{01}t_3 = 0$$

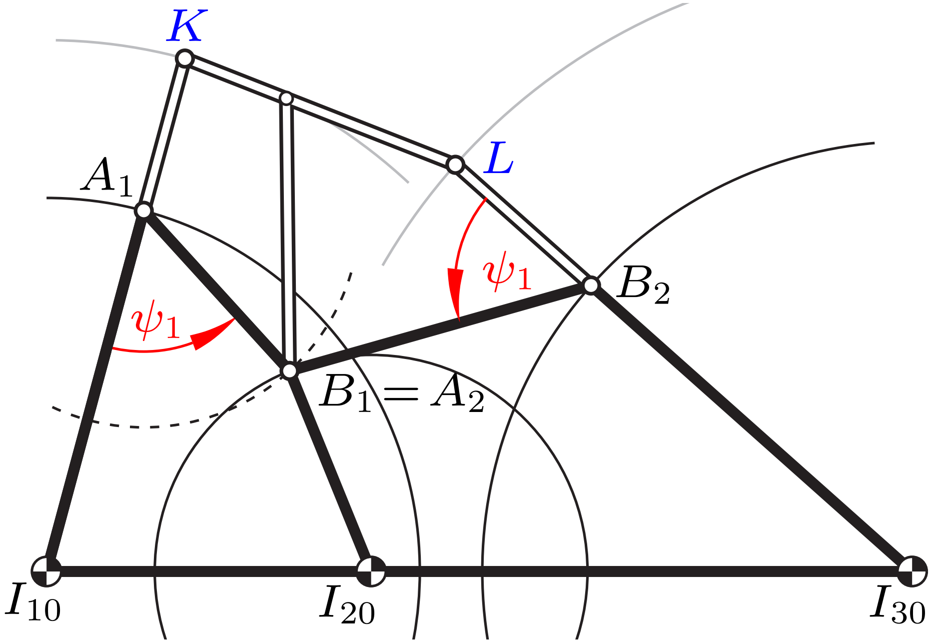
with coefficients depending on $\tan\alpha_1$, $\tan\delta_1$, $\tan\beta_2$, only.

In all known non-trivial examples (III, IV, V) the 4-4-correspondence between t_1 and t_3 is **reducible**.

There is a new example of a reducible composition:

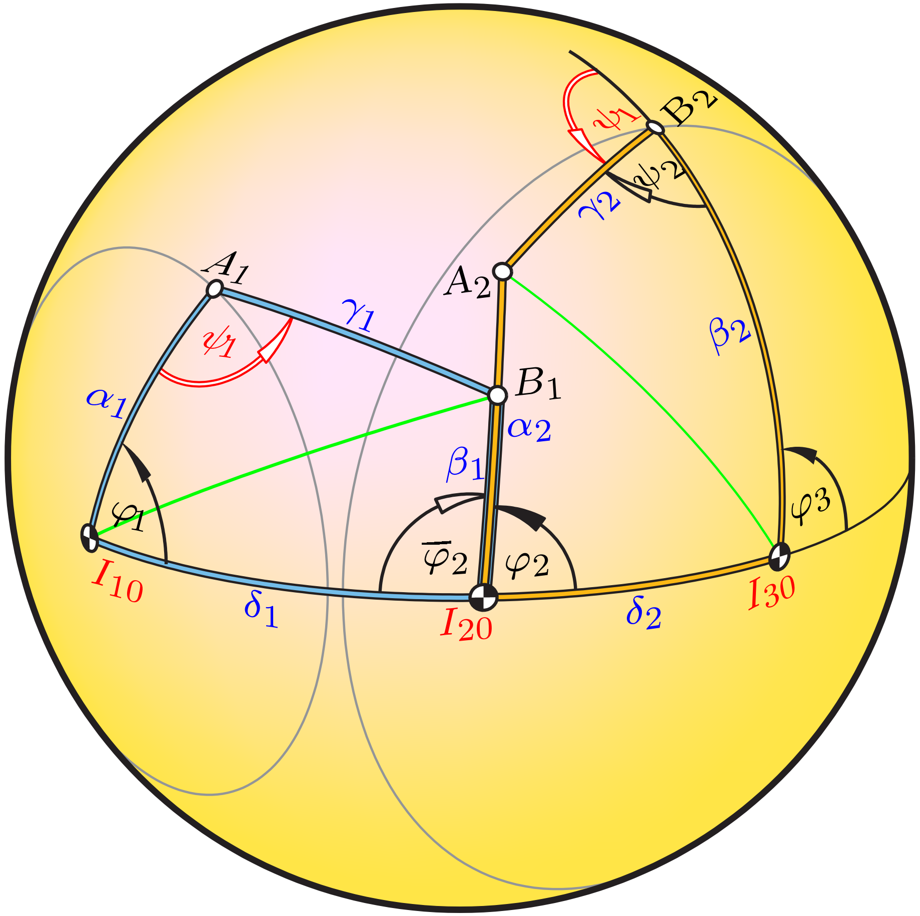


5. Flexibility vs. reducibility of meshes



BURMESTER's focal mechanism

Right hand figure: Reducible spherical composition obeying DIXON's angle condition for ψ_1



5. Flexibility vs. reducibility of meshes

For the composition of two spherical four-bars Dixon's angle condition $\sphericalangle\, I_{10}A_1B_1 = \pm \sphericalangle\, \overline{I}_{30}B_2A_2$ is equivalent to the statement that the discriminants of both 2-2-correspondences with respect to t_2

$$D_1 = (c_{11}t_2)^2 - 4(c_{22}t_2^2 + c_{20})(c_{02}t_2^2 + c_{00}) \quad \text{and}$$
$$D_2 = (d_{11}t_2)^2 - 4(d_{22}t_2^2 + d_{02})(d_{20}t_2^2 + d_{00})$$

are proportional.

Then the **4-4-correspondence is reducible**.

5. Flexibility vs. reducibility of meshes

Theorem: (G. Nawratil, 2011)

There are 4 non-trivial cases where the 4-4-correspondence is reducible:

1. Isogonal case: *One of the spherical quadrangles is isogonal.*

2. Dixon case: *The two spherical four-bars obey* Dixon*'s angle condition.*

3. Orthogonal case: *Both spherical quadrangles are orthogonal and share one diagonal (T-type).*

4: Deltoid case: *One of the quadrangles is a deltoid.*

5. Flexibility vs. reducibility of meshes

Conjecture:

Apart from the trivial translatory type I and planar-symmetrical type II there is no continuously flexible Kokotsakis-mesh with irreducible 4-4-correspondence.

Pro-arguments: The (complete) 4-4-correspondence defines its components, i.e., the 10 coefficients $c_{00}, \ldots, d_{22}$ uniquely — up to a common factor.

Once the conjecture is proved, the only candidates for flexible Kokotsakis-meshes are the four cases mentioned before. This should enable to classify of all flexible Kokotsakis-meshes.

5. Flexibility vs. reducibility of meshes

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