Gears and belt drives for non-uniform transmission

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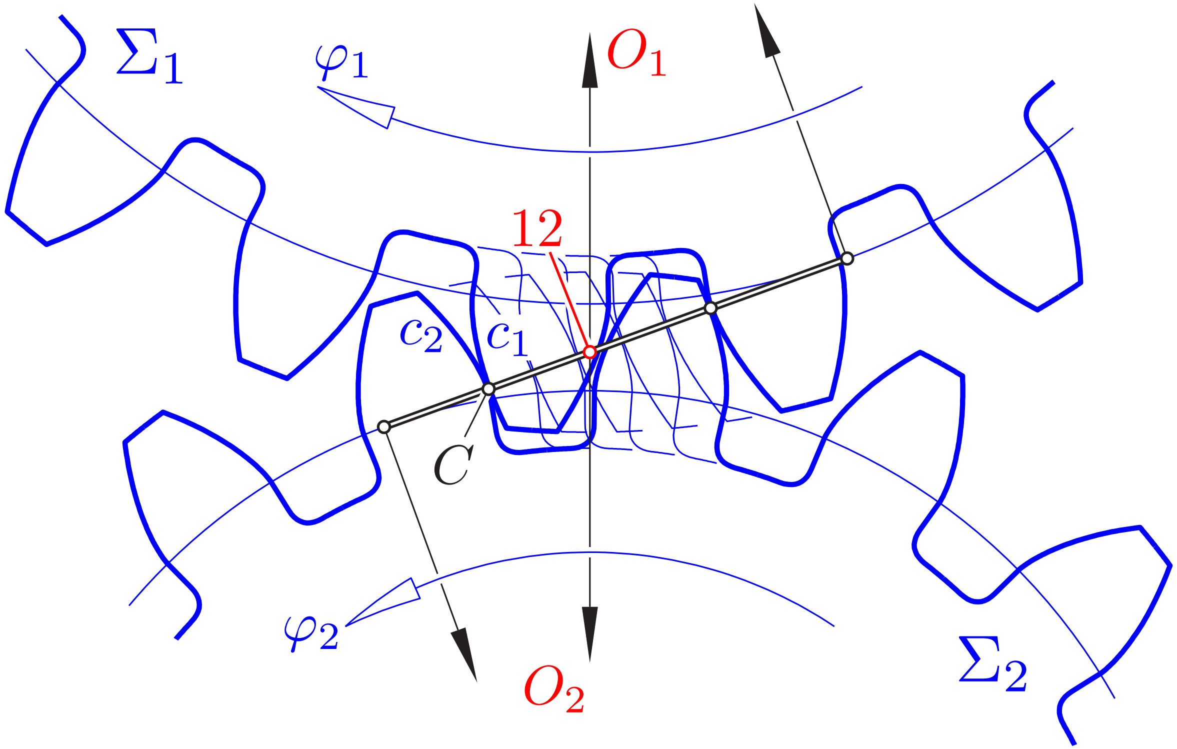
Table of contents

1. Revisiting S. Finsterwalder's principle of gearing

The driving wheel Σ_1 rotates about O_1 through φ_1, the out-put wheel Σ_2 rotates about O_2 through φ_2.

Then the relative pole 12 divides the segment O_1O_2 in the ratio of instantaneous angular velocities, i.e.,



$$\overline{O_1\,12} : \overline{O_2\,12} = \dot{\varphi}_2 : \dot{\varphi}_1 = \omega_2 : \omega_1.$$

Non-uniform transmission

REQUIRED: Gears that transmit rotary motion according to some

Transmission function: $\varphi_2 = f(\varphi_1)$ for $0 \leq \varphi_1 \leq 2\pi$.

Function f is assumed to be strictly monotonic, quite often differentiable, and

$$f(\varphi_1 + 2\pi) = f(\varphi_1) + 2\pi/n \quad \text{for} \quad n \in \mathbb{Z} \setminus \{0\}.$$

(n full input rotations $\leftrightarrow$ 1 output rotation). n is called *global transmission ratio*.

Non-uniform transmission

The transmission function $f(\varphi_1)$ defines the associated *perturbation function*

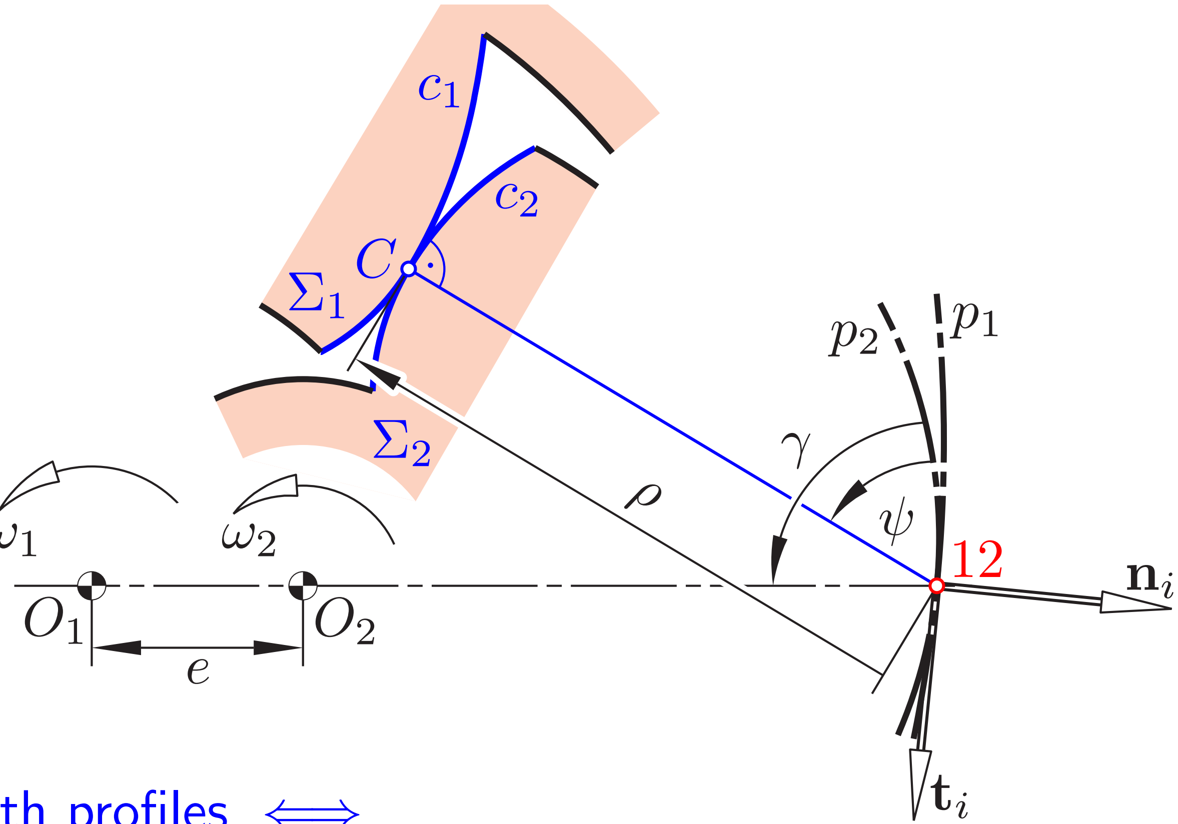
$$g(\varphi_1) := nf(\varphi_1) - \varphi_1 \quad \text{or} \quad f(\varphi_1) = \frac{1}{n}[\varphi_1 + g(\varphi_1)].$$

Because of

$$g(\varphi_1 + 2\pi) = nf(\varphi_1 + 2\pi) - \varphi_1 - 2\pi = nf(\varphi_1) - \varphi_1 = g(\varphi_1)$$

function $g(\varphi_1)$ is periodic and can be set up as a Fourier series.

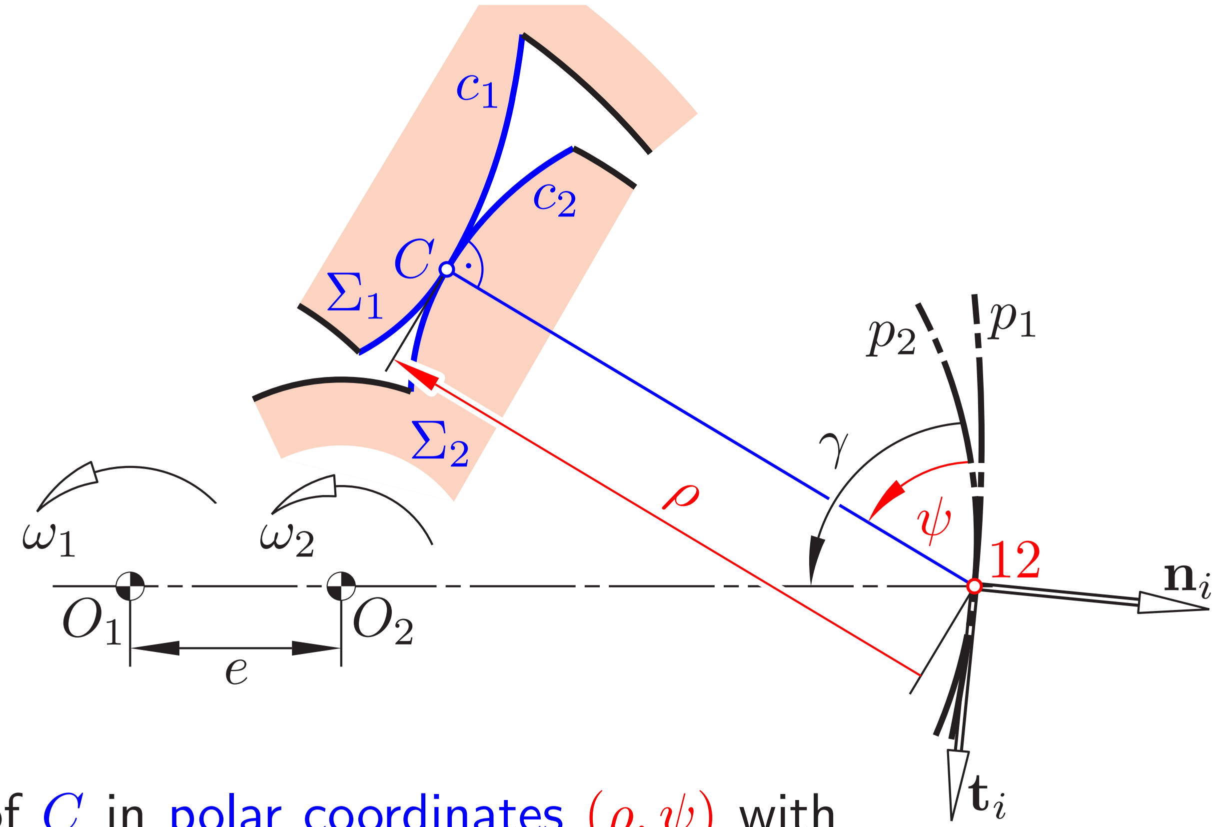
Law of gearing



c_1 and c_2 are conjugate tooth profiles $\Longleftrightarrow$

the common normal line at the point C of contact passes always through the relative pole 12.

Law of gearing



We express the position of C in polar coordinates (ρ, ψ) with respect to the relative pole 12 and the pole tangent.

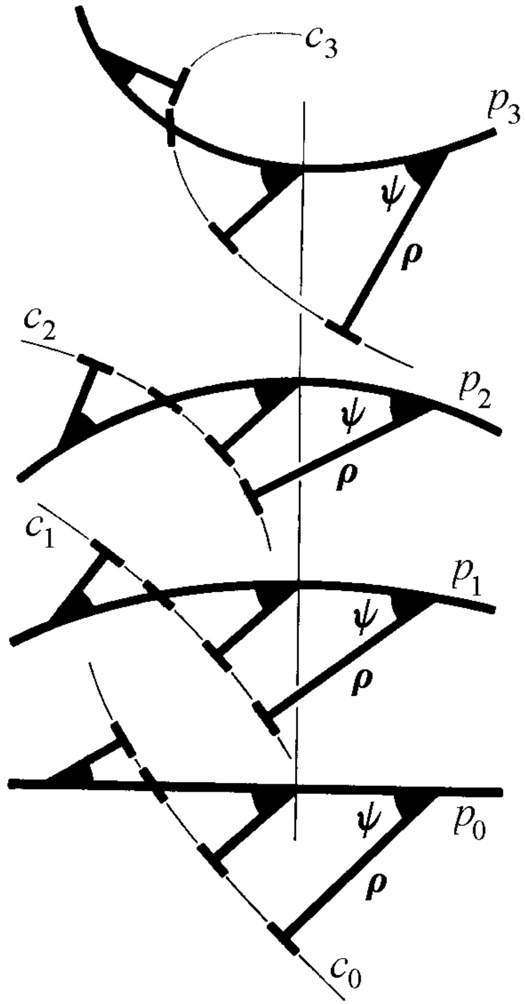
We may suppose $0 \leq \psi < \pi$ for $\rho \in \mathbb{R}$.

S. Finsterwalder's principle of gearing

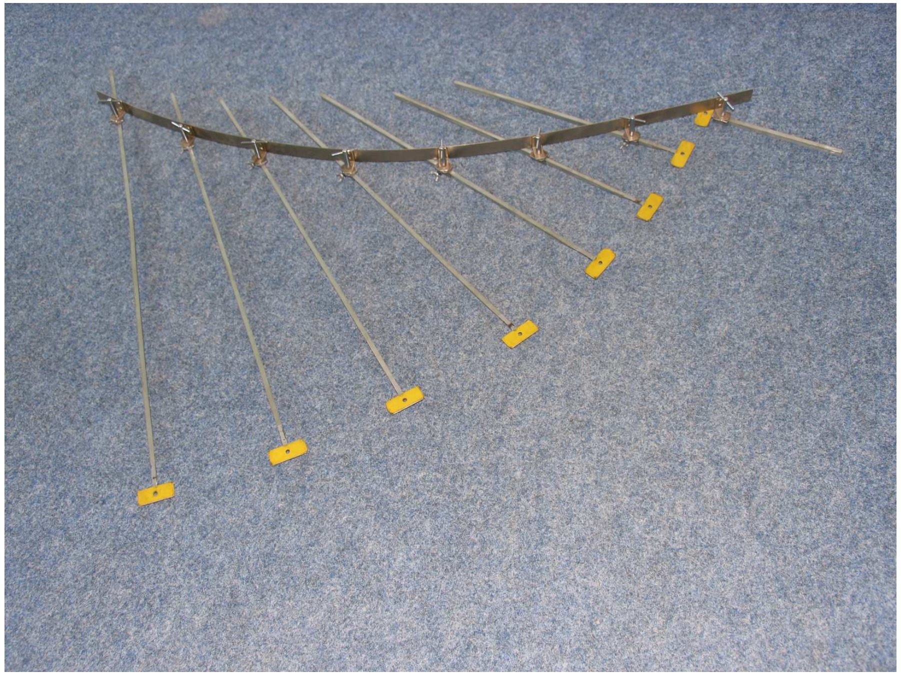
As the two polodes p_1, p_2 are in contact at 12, the coordinates (ρ, ψ) of $C \in c_i$ are the same with respect to p_1 and p_2.

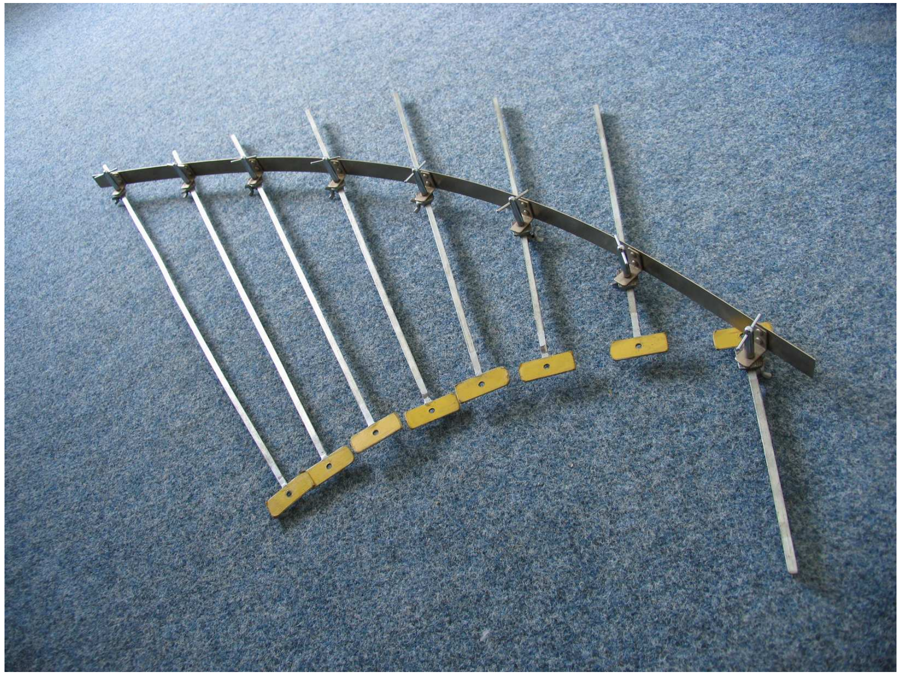
We imagine the polode p_1 as a flexible metal band and replace c_1 by a discrete set of line elements, each attached to the polode p_1 by fixing the angle ψ and the distance ρ.

For any flex p_2 of p_1 the curve c_2 formed by the attached line elements is conjugate to c_1 if the relative motion Σ_2/Σ_1 is defined by p_2 rolling along p_1.



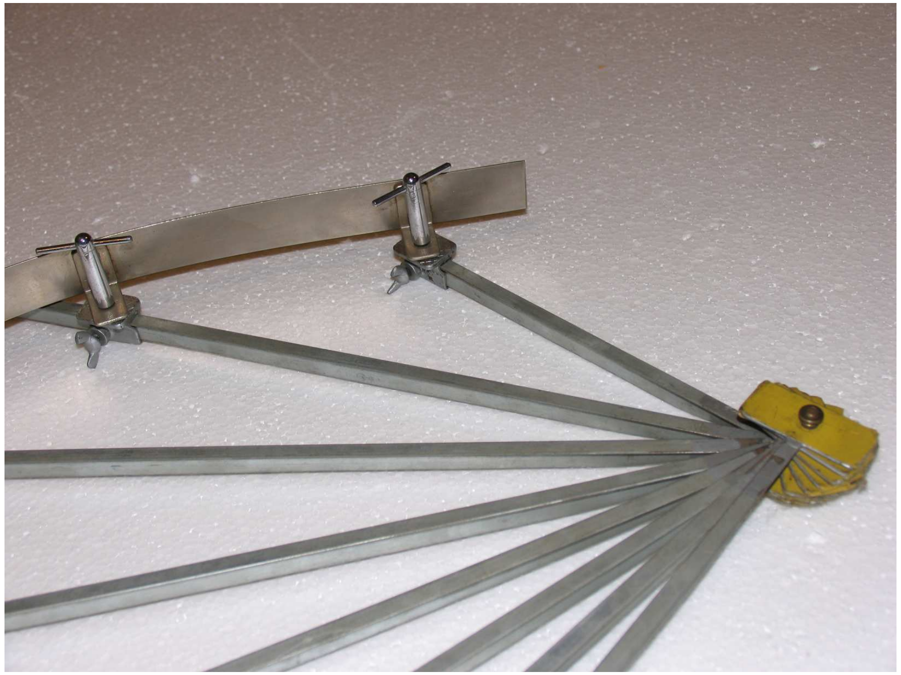
Original model from Munich

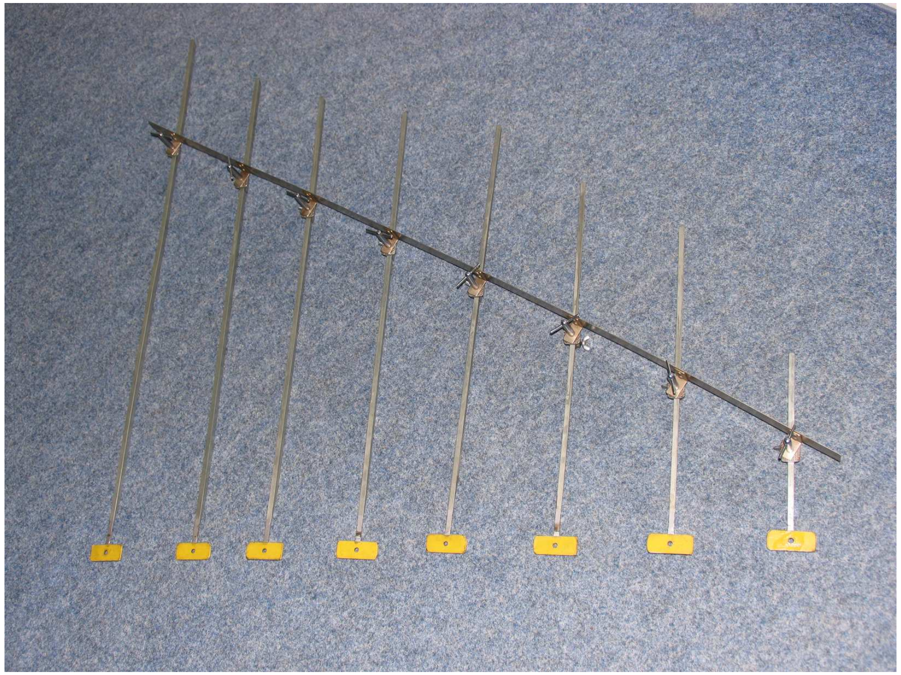




S. FINSTERWALDER, 1862-1951, professor for geometry
Technische Universität München

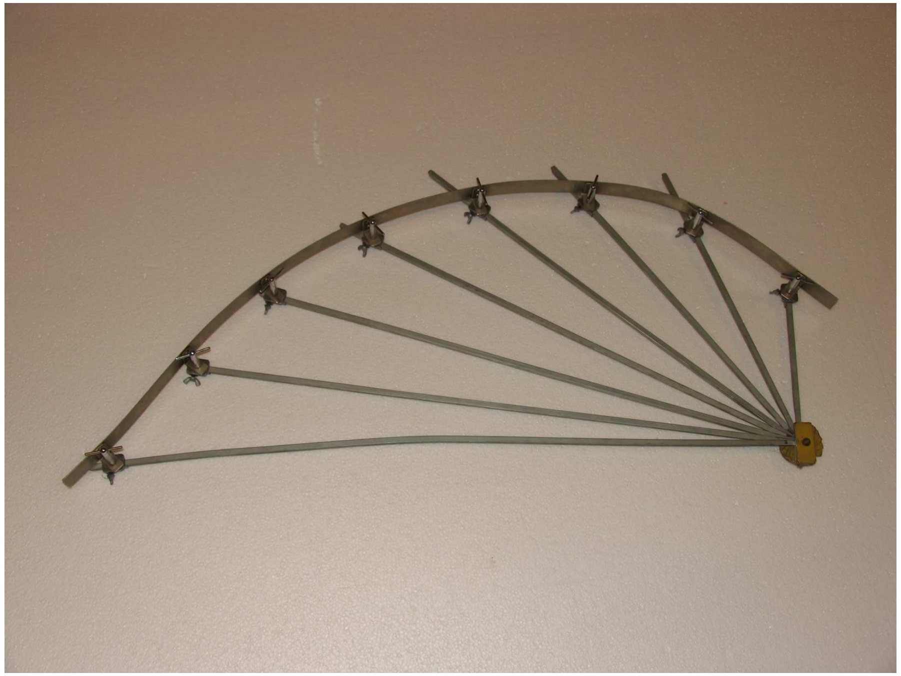
Original model from Munich





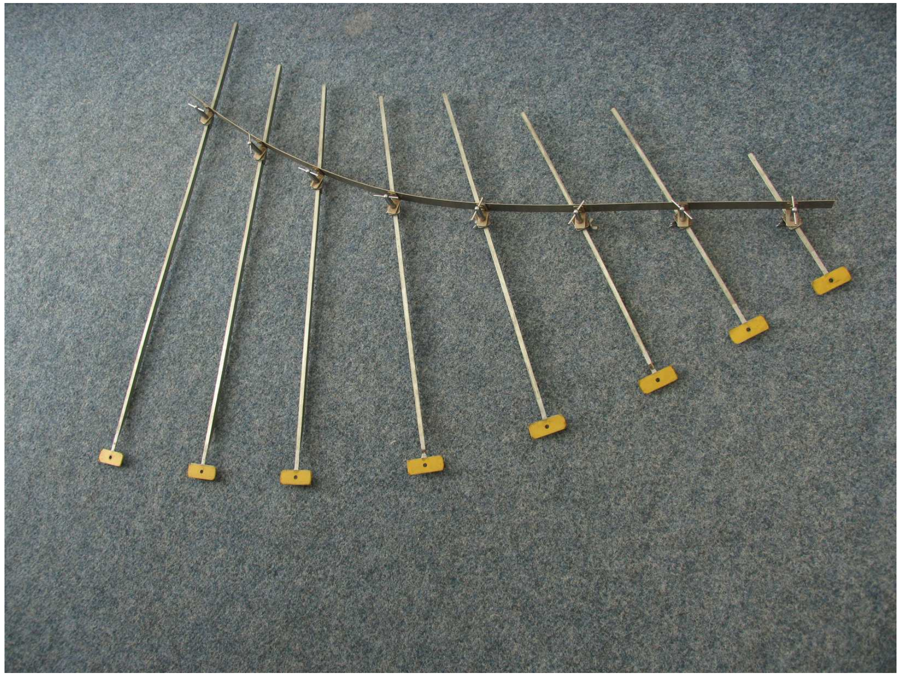
A straight line at the rack defines general involute gearing.
Photos: courtesy H. Hartl (TU Munich)

Original model from Munich



A particular initial profile is a point

Original model from Munich



Bending the metal band transforms the point into particular conjugate profiles

Algorithm for computing conjugate tooth profiles

1. Compute the polodes p_1 and p_2 for the relative motion Σ_2/Σ_1.

2. Rectify p_1, i.e., stretch it into the straight line p_0. Freely choose the corresponding rack tooth profile c_0 as long as it is nowhere orthogonal to p_0 and compute the polar coordinates (ρ, ψ) with respect to p_0.

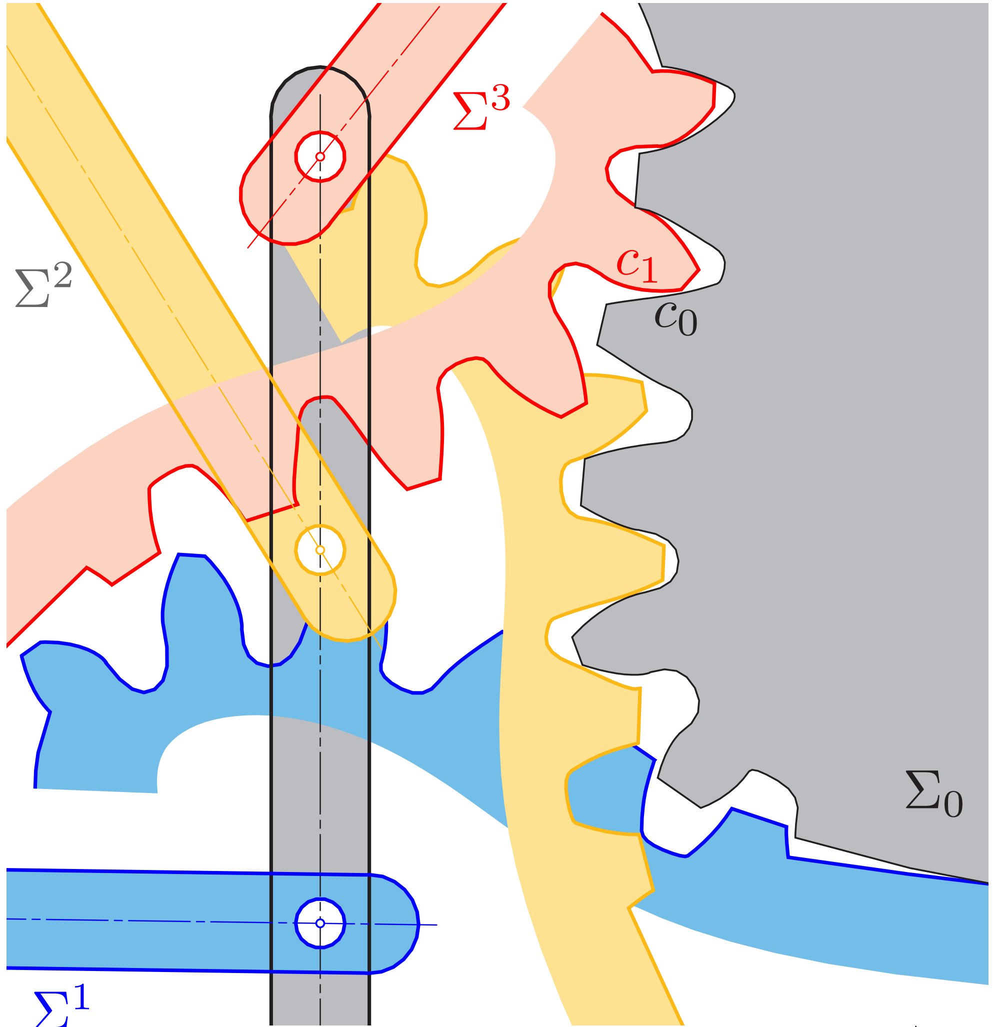
3. Bend p_0 back into p_1 and p_2 and use (ρ, ψ) with respect to the polodes to compute c_1 and c_2.

4. The applicable segments of c_1 and c_2 are arrived at by inspecting their relative movement in view of the contact ratio under avoidance of local and global undercuts.

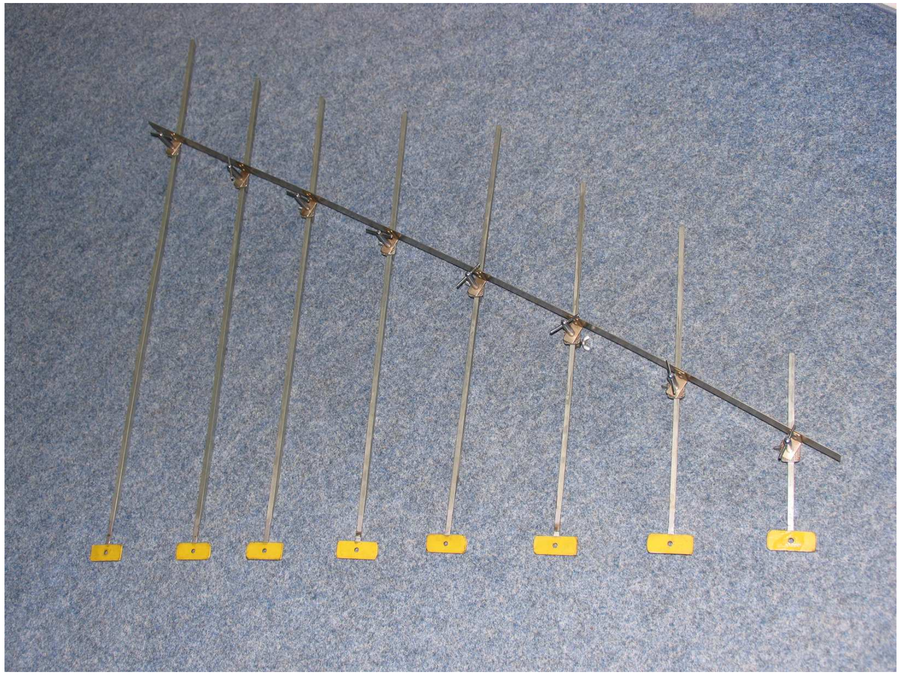
A real world example

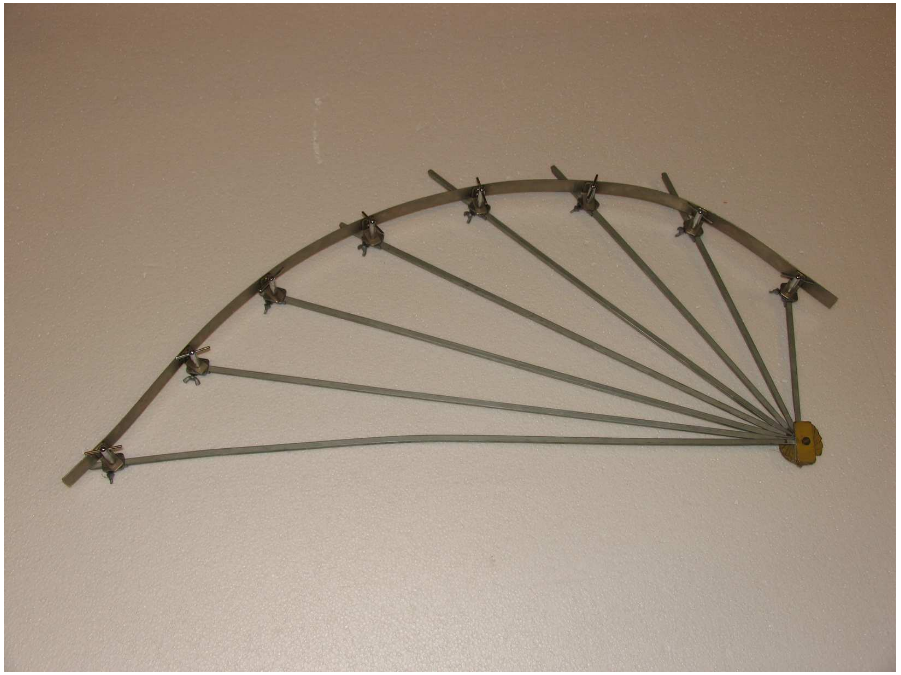
Mechanism for emptying garbage containers

(3 Positions Σ^1, Σ^2, Σ^3)



Equivalence to the principle of Camus





FINSTERWALDER's method is the discretized version
of general cycloid gearing (principle of CAMUS)

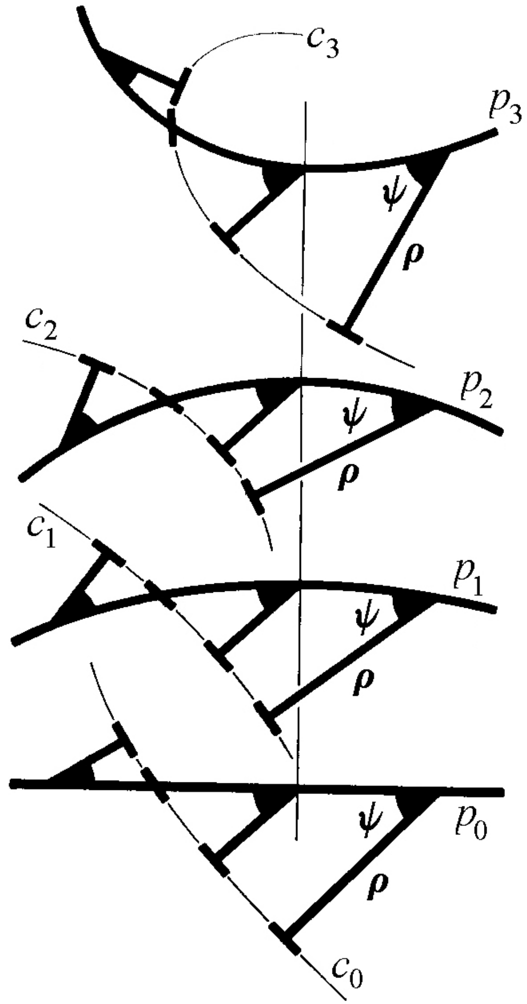
Equivalence to the principle of Camus

Choose p_0 as auxiliary curve with c_0 attached. Then c_i, $i = 1, 2$, can be seen as envelope of c_0 while p_0 is rolling along p_i.

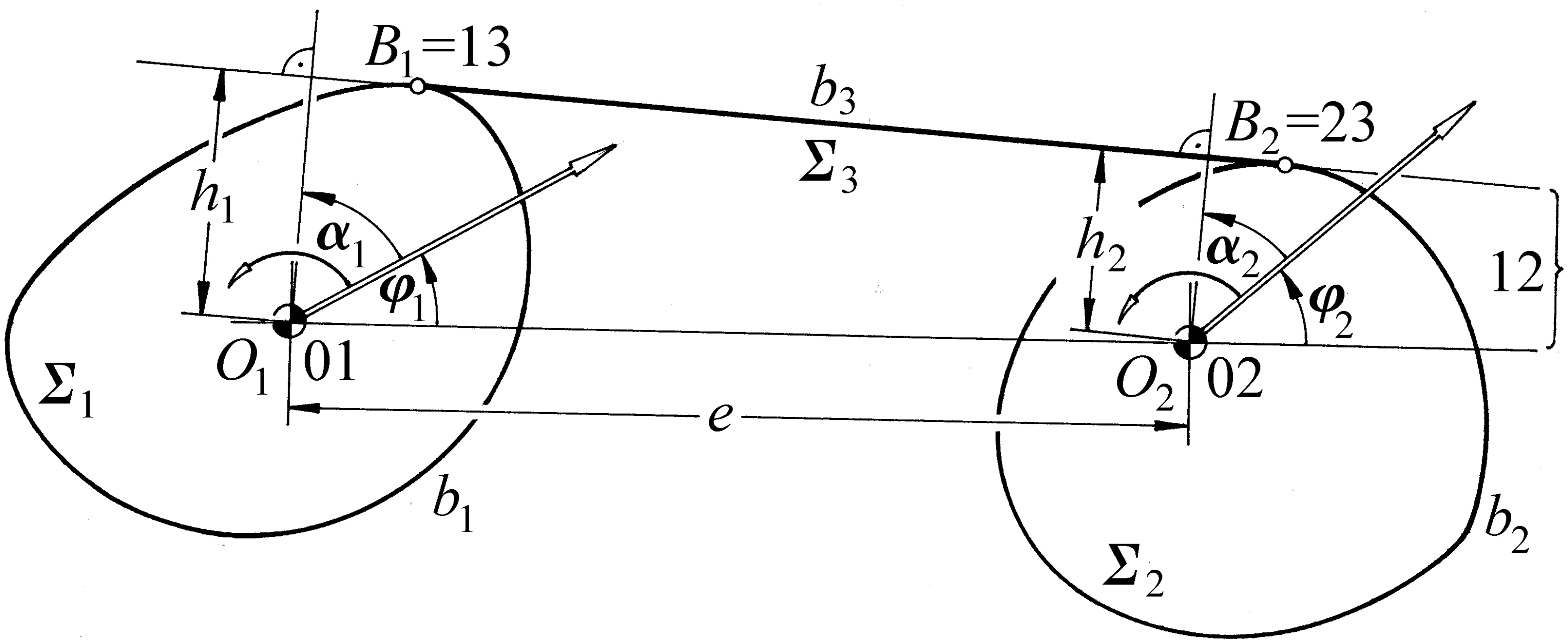
This is exactly the principle of CAMUS in the general form.

Choose for any flex p_0 the profile c_0 as a point P_0. Then c_i is the path of P_0 while p_0 is rolling along p_i.

This is the more popular principle of CAMUS and basis for cycloid gearing.



2. Non-uniform belt drives



Belt drives offer another option to transmit rotary motion by a given transmission function $\varphi_2 = f(\varphi_1)$.

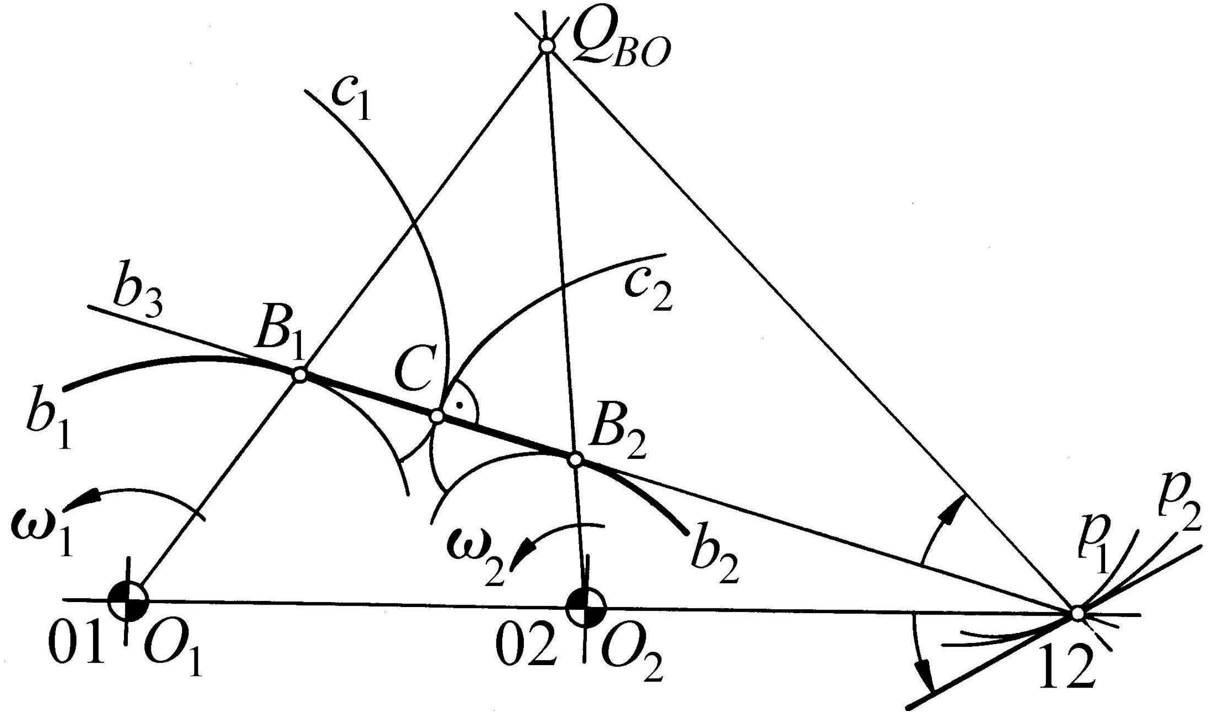
We call corresponding pully profiles b_1, b_2 *conjugate*.

Relation between belt drives and gears

The *upper belt span* between the contact points B_1 and B_2 defines a new system Σ_3.

As line b_3 rolls on b_1 and b_2, points B_1 and B_2 are the relative poles 13 and 23, respectively.

This implies

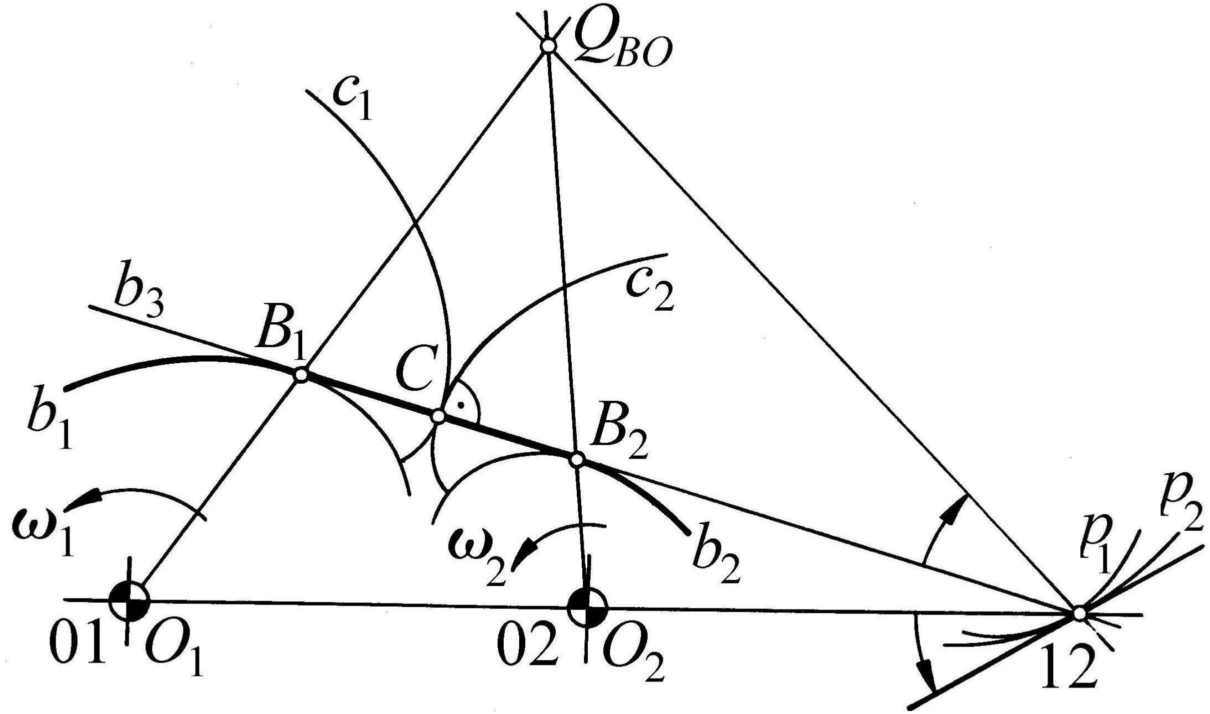


Theorem 1: At each instant the upper belt span must be aligned with the relative pole 12.

Relation between belt drives and gears

Under the relative motion Σ_3/Σ_1 an arbitrary point C attached to line b_3 traces an involute c_1 of b_1.

The path of C under Σ_3/Σ_2 is an involute c_2 of b_2, which contacts c_1 at C.



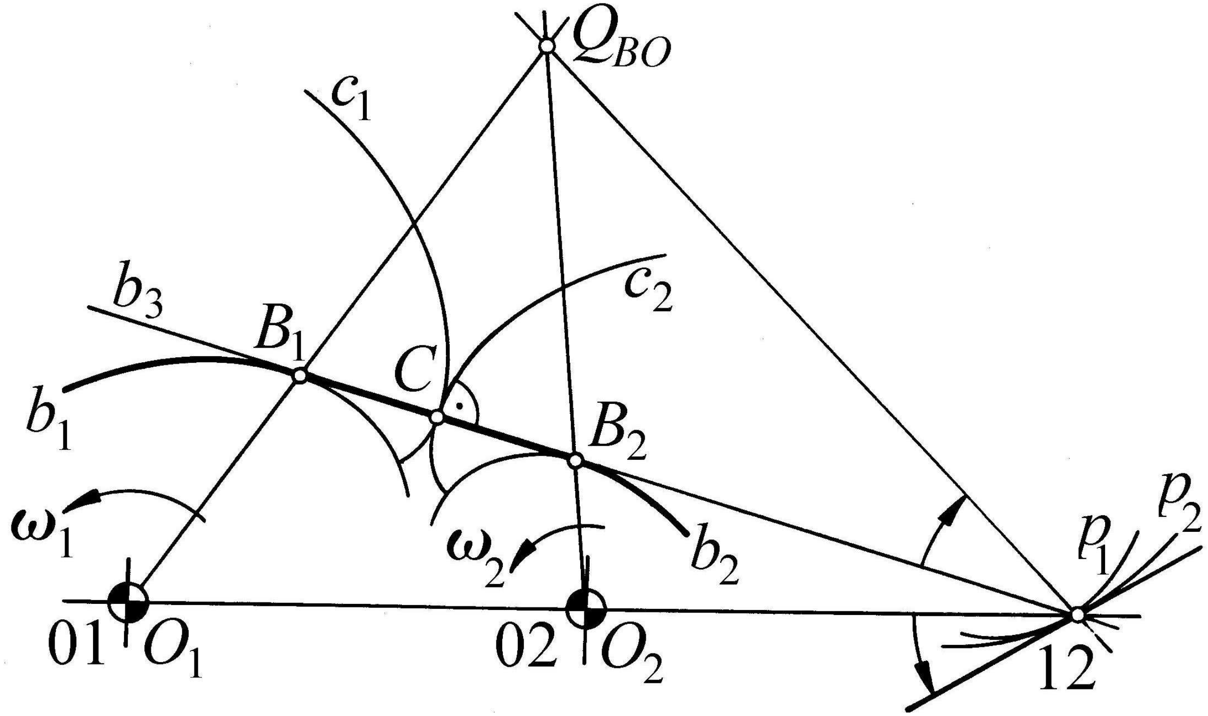
Theorem 2: b_1 and b_2 are conjugate pulley profiles $\Longleftrightarrow$ b_1, b_2 are evolutes of conjugate tooth profiles c_1, c_2.

Relation between belt drives and gears

B_1 and B_2 are curvature centers of c_1, c_2, resp., at point C.

c_1 envelopes c_2 under the relative motion Σ_1/Σ_2.

$\Longrightarrow$ B_2 results from B_1 under BOBILLIER's construction.



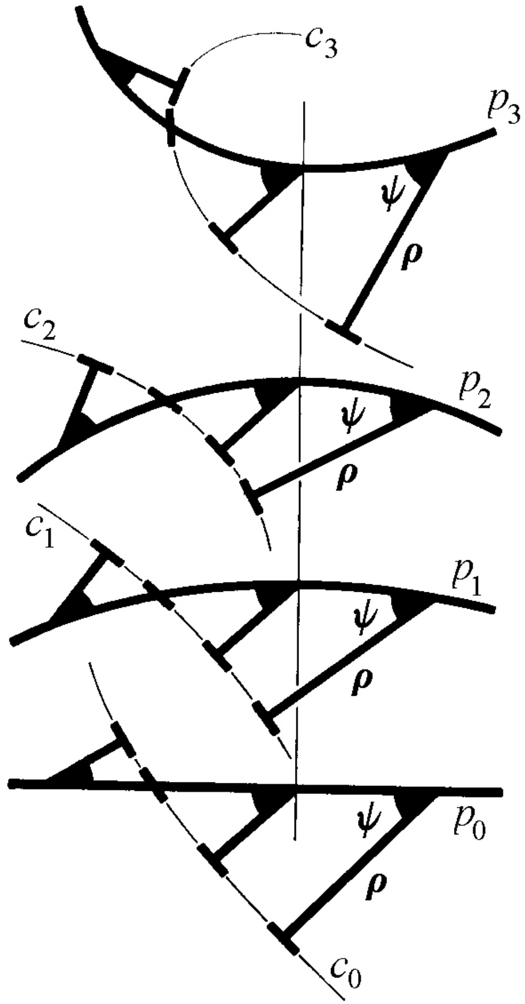
Theorem 3: The endpoints B_1 and B_2 are corresponding under the curvature transformation of the relative motion Σ_1/Σ_2.

Relation between belt drives and gears

Pulley profiles b_1, b_2 must be closed curves — contrary to the tooth profiles c_1, c_2.

Recalling FINSTERWALDER's method, we focus on the envelopes of the attached normal lines making angle ψ with the polodes. These envelopes b_1, b_2 are conjugate pulley profiles.

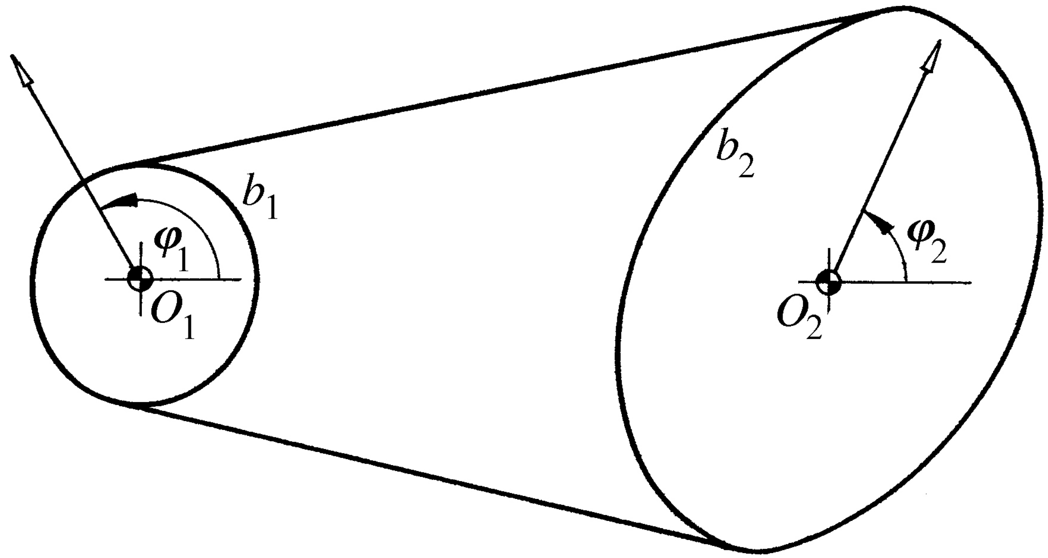
Theorem 4: For any transmission function $\varphi_2 = f(\varphi_1)$ and any driving pulley b_1 there is a unique conjugate profile b_2. However, b_2 needs not be convex.



3. Strict non-uniform belt drives

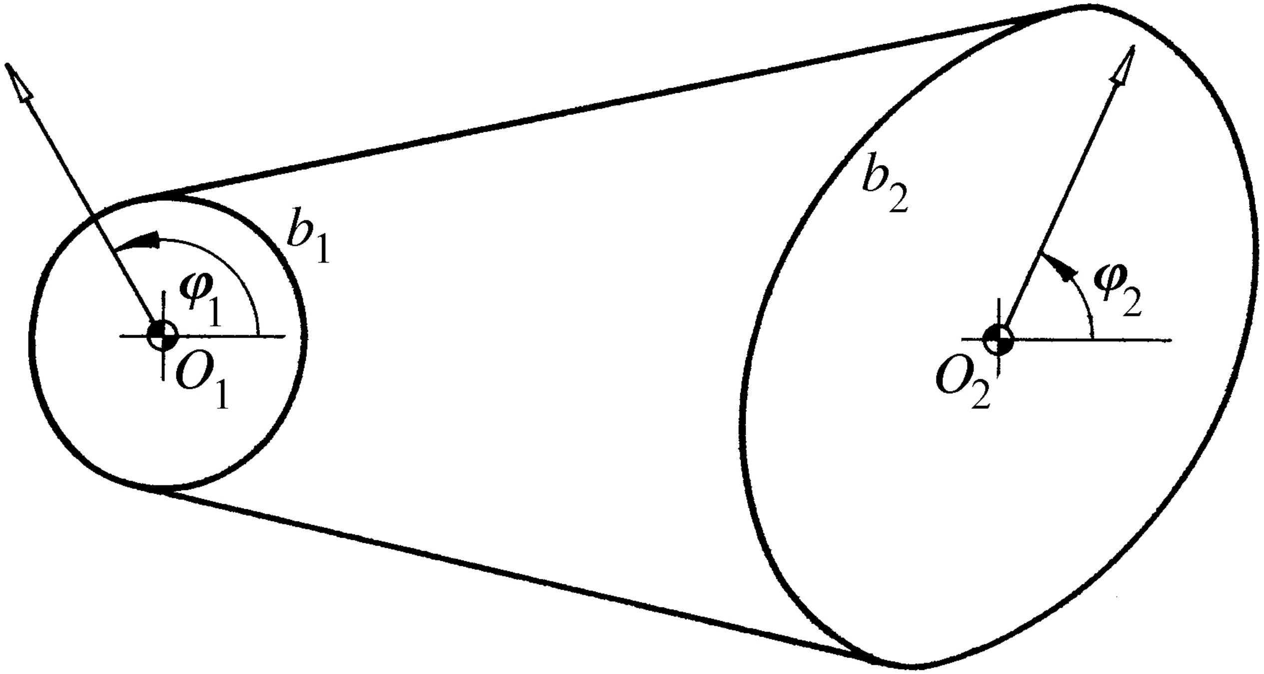
A belt drive with belt slack zero, i.e., when length of the surrounding belt with taut spans remains constant, is called *strict*.

In this case the lower belt span rolls on b_1 and b_2, too.



Theorem 5 [J. Hoschek 1982]:
Conjugate pulley profiles b_1, b_2 operate without needing a tightener $\Longleftrightarrow$ at each instant both the upper and the lower belt span are aligned with the relative pole 12.

3. Strict non-uniform belt drives



Strict belt drive for the transmission function

$$f(\varphi_1) = 0.5\,[\varphi_1 + 0.2\,\sin\varphi_1 + 0.16\sin 2\varphi_1 + 0.008\cos 2\varphi_1].$$

The length of the belt varies within approx. 0.002%.

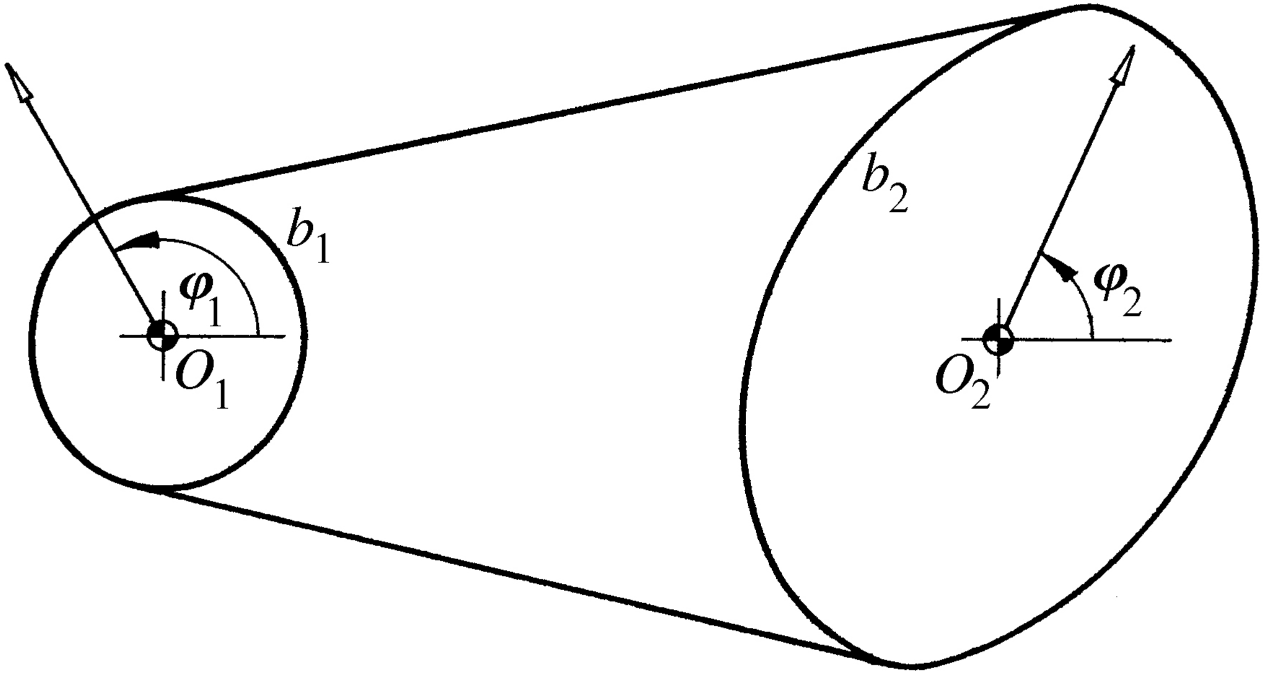
Exact strict belt drives

The following nontrivial — but more academic — example of a non-uniformly transmitting strict belt drive is known:

W. Wunderlich and P. Zenow discovered 1975 that for the line-symmetric motion with ellipses as polodes p_1, p_2 two congruent *ellipses* b_1, b_2 *confocal with* p_1 *and* p_2, resp., are pulley profiles for a strict belt drive and with global gear ratio $-1:1$.

Remark: In the uniform $1:1$-case any convex disk together with a translated copy constitute a strict belt drive.

Algorithm for computing strict belt drives



Theorem 5 implies an algorithm for computing strict belt drives — a slight modification of J. Hoschek's method (1982):

Algorithm for computing strict belt drives

1. In an arbitrary initial position ($\varphi_1 = \varphi_1^{(0)}$) we specify an upper belt span passing through 12. We attach this line to Σ_1.

2. We rotate Σ_1 and simultanously Σ_2 until this line becomes a lower belt span ($\varphi_1 = \varphi_1^{(1)}$). This must be tangent to the conjugate profile b_2, too, and we attach it to Σ_2.

3. We rotate Σ_2 and simultaneously Σ_1 until this line again covers an upper belt span ($\varphi_1 = \varphi_1^{(2)}$). In general, this gives a new tangent line of b_1.

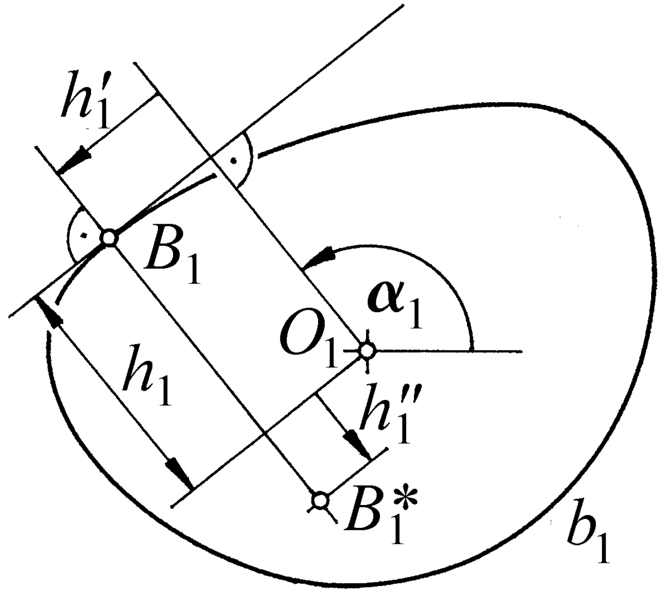
Iteration gives a finite set of lines tangent to b_1.

Algorithm for computing strict belt drives

In contrast to HOSCHEK's method using Bézier curves, we represent the tangent lines by their support function $h_1 = h_1(\alpha_1)$.

By a least square method the best approximating Fourier series (of given order) is computed.

The same is done for tangent lines of b_2.



On the existence of non-uniform belt drives

This algorithm works well in all examples with global transmission ratio $n \neq 1$. The computed lines obviously envelope a unique curve b_1.

This observation together with some arguments give rise to

Conjecture 1: For a given non-uniform transmission function $\varphi_2 = f(\varphi_1)$ with global transmission ratio $n \neq 1$ there is a one-parametric set of pairs of conjugate pulley profiles b_1, b_2 for a strict belt drive.

However, these profiles are convex only if the given transmission lies sufficiently close to the uniform case with the same global ratio n.

A rigorous mathematical proof is open but there is some supporting evidence:

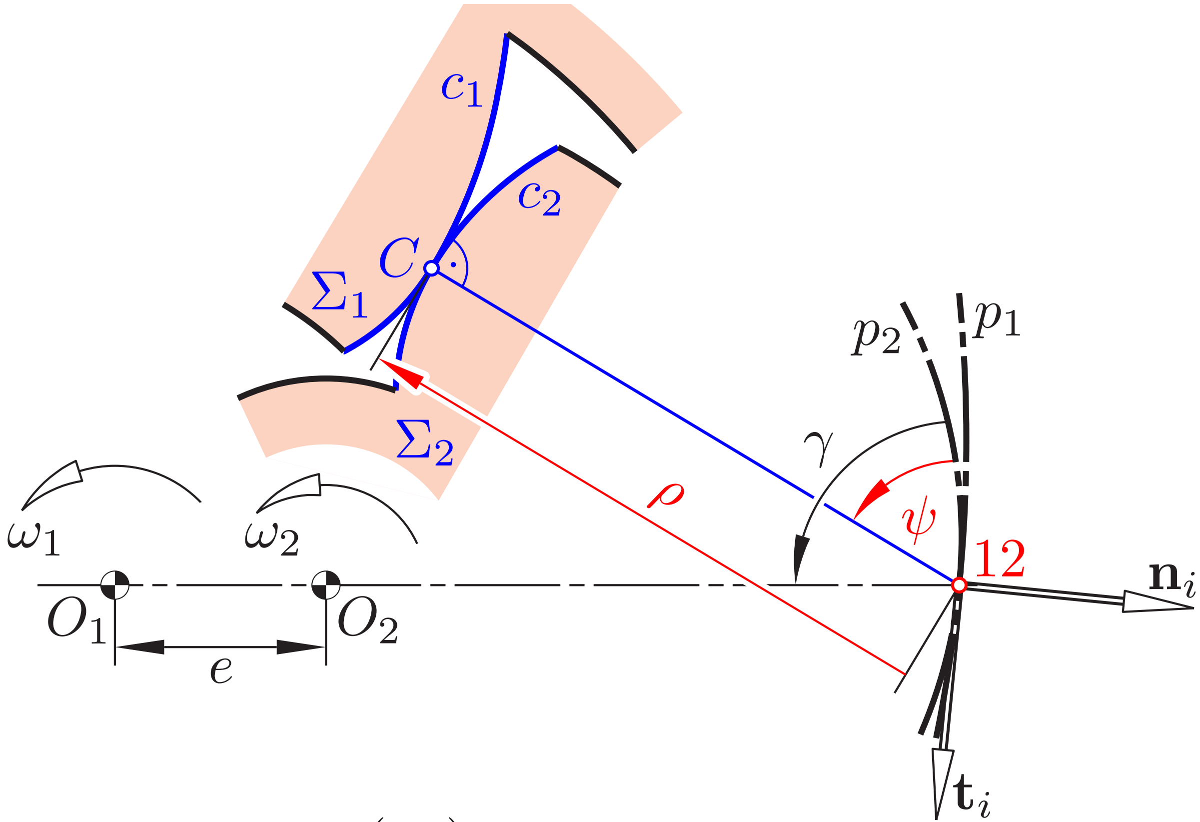
On the existence of non-uniform belt drives

For convex relative polodes and analytic $\varphi_2 = f(\varphi_1)$ the mapping

$$\psi(\varphi_1^{(0)}) \mapsto \psi(\varphi_1^{(2)})$$

of lines is analytic, too.

Lines tangent to p_1 are fixed. Passing through the center ($\psi = \gamma$) is preserved. And the support function obeys



$$\frac{h_1(\varphi_1^{(2)})}{h_1(\varphi_1^{(0)})} = \frac{\Omega(\varphi_1^{(2)})}{\Omega_1(\varphi_1^{(1)})} \quad \text{for} \quad \Omega(\varphi_1) = \frac{\omega_2(\varphi_1)}{\omega_1(\varphi_1)} \quad \text{and} \quad h_1 = r_1 \cos(\gamma - \psi).$$

On the existence of non-uniform belt drives

The excluded case $n = 1$ shows a strange behavior that was observed – but not reported – by J. Hoschek:

The lines obtained by the above algorithm do not envelope any curve.

It can be proved that in this case a starting line passing through O_1 after iteration does not rotate fully about O_1 but approaches a limiting position. By continuity, this seems to contradict the required convexity of b_1 and leads to

Conjecture 2: There is no strict belt drive for non-uniform transmission with global transmission ratio $n = 1$.

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