Teaching Spatial Kinematics for Engineers

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Abstract — *Spatial kinematics is a challenging field because of its real world applications at serial and parallel manipulators. However, it is not easy to teach as on the one hand the students' spatial abilities need to be well developed. And on the other hand familiarity with calculus and vector algebra is substantial.*
After several years of experience I learned that one can expect from students to work with dual vectors. Only at first sight they look too abstract. But students soon estimate the advantages of this method:
(1) Though looking like common vectors, they allow to handle oriented lines. With dot product and cross product complicated 3D-geometric problems can be solved in a very brief and elegant way.
(2) Beside the correspondance between dual unit vectors and oriented lines, the general dual unit vectors symbolize screws, i.e., instantaneous motions. As each dual vector is a dual multiple of a dual unit vector, any instantaneous motion is a helical motion with well defined axis, angular velocity and translatory velocity.
(3) The spatial 'Three-Pole-Theorem' states, that the screw of the relative motion is the difference of the screws of the absolute motion and the guiding motion. Therefore the instantaneous motion of the endeffector equals the sum of the screws of the instanteous rotations between consecutive links.
This leads directly to the Jacobi-matrix of serial robots with the dual vectors of the revolute axes as columns. And it allows to solve, e.g., calibration problems at parallel robots, thus revealing that the 'purely abstract looking' dual vector algebra is a powerful tool for a plenty of real world applications.

Index Terms — *dual vector, forward kinematics, line coordinates, parallel manipulator, spatial kinematics, serial manipulator, screw, Jacobi matrix.*

DUAL UNIT VECTORS REPRESENTING ORIENTED LINES IN THE 3-SPACE

There is a tight connection between spatial kinematics and the geometry of lines in the Euclidean 3-space $\mathbb{E}^3$. Therefore we start with recalling the use of appropriate line coordinates (cf. [4], p. 155 ff)).

Any oriented line (*spear*) g with direction vector $\mathbf{g}$ and passing through point $\mathbf{a}$, i.e., $g = \mathbf{a} + \mathbb{R}\mathbf{g}$ can be uniquely represented by the pair of vectors $(\mathbf{g}, \hat{\mathbf{g}})$, the *direction vector* $\mathbf{g}$ and the *momentum vector* $\hat{\mathbf{g}}$, according to the definitions

$$\mathbf{g}\cdot\mathbf{g} = 1 \text{ and } \hat{\mathbf{g}} := \mathbf{a}\times\mathbf{g} \text{ which imply } \mathbf{g}\cdot\hat{\mathbf{g}} = 0 .$$

Conversely, any pair $(\mathbf{g}, \hat{\mathbf{g}})$ of vectors obeying $\mathbf{g}\cdot\mathbf{g} = 1$ and $\mathbf{g}\cdot\hat{\mathbf{g}} = 0$ defines a unique spear g because $\mathbf{a} := \mathbf{g}\times\hat{\mathbf{g}}$ is a point of this line, the pedal point of g with respect to the origin. It makes sense to replace the pair $(\mathbf{g}, \hat{\mathbf{g}})$ by the *dual vector*

$$\underline{\mathbf{g}} := \mathbf{g} + \varepsilon\hat{\mathbf{g}} , \tag{1}$$

where the *dual unit* ε obeys the rule $\varepsilon^2 = 0$. We extend the usual dot product of vectors to dual vectors and notice

$$\underline{\mathbf{g}}\cdot\underline{\mathbf{g}} = (\mathbf{g}+\varepsilon\hat{\mathbf{g}})\cdot(\mathbf{g}+\varepsilon\hat{\mathbf{g}}) = \mathbf{g}.\hat{\mathbf{g}} + 2\varepsilon\mathbf{g}.\hat{\mathbf{g}} = 1+\varepsilon 0 = 1 . \tag{2}$$

Hence we call $\underline{\mathbf{g}}$ a *dual unit vector*. In this sense the set of oriented lines in $\mathbb{E}^3$ can be seen as the dual extension of the unit sphere. The representation of oriented lines g in $\mathbb{E}^3$ by dual unit vectors $\underline{\mathbf{g}}$ brings about several advantages, and from now on we do not distinguish between oriented lines g and their representing dual vector $\underline{\mathbf{g}}$ as well as for points X and their coordinate vector $\mathbf{x}$.

THEOREM 1: *For two given oriented lines* $\underline{\mathbf{g}}, \underline{\mathbf{h}}$ *in* $\mathbb{E}^3$ *let* $\underline{\mathbf{n}}$ *denote the common normal endowed with an arbitrary orientation. If the helical motion along* $\underline{\mathbf{n}}$ *which transforms* $\underline{\mathbf{g}}$ *into* $\underline{\mathbf{h}}$ *(see Figure 1) has the angle* φ *of rotation and the length* $\hat{\varphi}$ *of translation and we combine them in the dual angle* $\underline{\varphi} = \varphi + \varepsilon\,\hat{\varphi}$, *then the following equations hold true:*

$$\underline{\mathbf{g}}\cdot\underline{\mathbf{h}} = \underline{\cos}\,\underline{\varphi} = \cos\varphi - \varepsilon\hat{\varphi}\sin\varphi \quad \textit{and} \quad \underline{\mathbf{g}}\times\underline{\mathbf{h}} = \underline{\sin}\,\underline{\varphi}\,\underline{\mathbf{n}} = \sin\varphi\mathbf{n} + \varepsilon(\sin\varphi\hat{\mathbf{n}} + \hat{\varphi}\cos\varphi\mathbf{n}) . \tag{3}$$

Here we use the *dual extension* of differentiable functions which is defined by

$$\underline{f}(\underline{x}) = \underline{f}(x+\varepsilon\hat{\varphi}) = f(x) + \varepsilon\hat{x}\,f'(x)\,.$$

This is the beginning of a Taylor series where due to $\varepsilon^2 = 0$ all higher powers are vanishing. This guarantees that identities like $\underline{\cos}^2\underline{x} + \underline{\sin}^2\underline{x} = 1$ are preserved under the dual extension as they are valid for the power series, too.

The notation ε originates from the fact that the dual unit can be seen as such a small number that its square is neclectable. Note that only dual numbers $\underline{x}$ with non-vanishing real part have an inverse $\underline{x}^{-1}$.

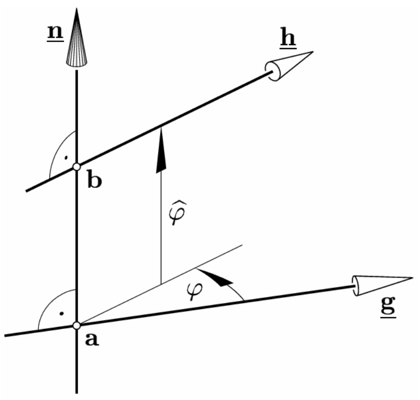

FIGURE 1: Dual angle $\underline{\varphi} = \varphi + \varepsilon\hat{\varphi}$ between $\underline{\mathbf{g}}$ and $\underline{\mathbf{h}}$.

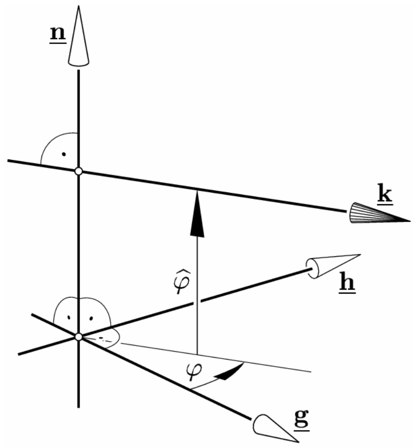

FIGURE 2: $\underline{\mathbf{k}} := \underline{\cos}\,\underline{\varphi}\,\underline{\mathbf{g}} + \underline{\sin}\,\underline{\varphi}\,\underline{\mathbf{h}}$

On the other hand we use in Theorem 1 the dual extension of the vector product according to

$$\underline{\mathbf{g}}\times\underline{\mathbf{h}} = (\mathbf{g}+\varepsilon\hat{\mathbf{g}})\times(\mathbf{h}+\varepsilon\hat{\mathbf{h}}) = \mathbf{g}\times\mathbf{h} + \varepsilon\,(\hat{\mathbf{g}}\times\mathbf{h} + \mathbf{g}\times\hat{\mathbf{h}})\,. \qquad (4)$$

Proof of Theorem 1: Suppose $\hat{\mathbf{g}} := \mathbf{a}\times\mathbf{g}$ and $\hat{\mathbf{h}} := \mathbf{b}\times\mathbf{h}$. Then the shortest distance between $\underline{\mathbf{g}}$ and $\underline{\mathbf{h}}$ reads

$$\hat{\varphi} = (\mathbf{b}-\mathbf{a})\cdot\frac{\mathbf{g}\times\mathbf{h}}{\sin\varphi} = \frac{1}{\sin\varphi}[\det(\mathbf{b},\mathbf{g},\mathbf{h}) - \det(\mathbf{a},\mathbf{g},\mathbf{h})] = \frac{1}{\sin\varphi}[-(\mathbf{b}\times\mathbf{h})\cdot\mathbf{g} - (\mathbf{a}\times\mathbf{g})\cdot\mathbf{h})] = \frac{-1}{\sin\varphi}(\hat{\mathbf{g}}\cdot\mathbf{h} + \mathbf{g}\cdot\hat{\mathbf{h}})\,.$$

If **a** and **b** are supposed to be the intersection points of $\underline{\mathbf{g}}$ and $\underline{\mathbf{h}}$ with the common normal $\underline{\mathbf{n}}$, then

$$\sin\varphi\,\hat{\mathbf{n}} = \mathbf{a}\times\sin\varphi\,\mathbf{n} = \mathbf{a}\times(\mathbf{g}\times\mathbf{h}) = (\mathbf{a}\cdot\mathbf{h})\,\mathbf{g} - (\mathbf{a}\cdot\mathbf{g})\,\mathbf{h} + [(\mathbf{a}-\mathbf{b})\cdot\mathbf{g}]\,\mathbf{h} = (\mathbf{a}\cdot\mathbf{h})\,\mathbf{g} - (\mathbf{b}\cdot\mathbf{g})\,\mathbf{h}\,.$$

The expression in brackets vanishes and could therefore be added without changing the value. On the other hand due to standard formulas from vector algebra we see

$$(\hat{\mathbf{g}}\times\mathbf{h}) + (\mathbf{g}\times\hat{\mathbf{h}}) = [(\mathbf{a}\times\mathbf{g})\times\mathbf{h}] + [\mathbf{g}\times(\mathbf{b}\times\mathbf{h})] = (\mathbf{a}\cdot\mathbf{h})\,\mathbf{g} - (\mathbf{g}\cdot\mathbf{h})\,\mathbf{a} + (\mathbf{g}\cdot\mathbf{h})\,\mathbf{b} - (\mathbf{g}\cdot\mathbf{b})\,\mathbf{h} =$$
$$= (\mathbf{a}\cdot\mathbf{h})\,\mathbf{g} - (\mathbf{b}\cdot\mathbf{g})\,\mathbf{h} + (\mathbf{g}\cdot\mathbf{h})(\mathbf{b}-\mathbf{a}) = \sin\varphi\,\hat{\mathbf{n}} + \hat{\varphi}\cos\varphi\,\mathbf{n}\,.$$ □

By dual extension it is possible to convert formulas from spherical geometry into formulas on spears. The following example should illustrate this so called *'principle of transference'* (German: Uebertragungsprinzip) which dates back to E. Study. A first example is given in

THEOREM 2: *Let* $\underline{\mathbf{g}}$ *and* $\underline{\mathbf{h}}$ *be two orthogonally intersecting spears with the common perpendicular* $\underline{\mathbf{n}}$. *Then*

$$\underline{\mathbf{k}} := \underline{\cos}\,\underline{\varphi}\,\underline{\mathbf{g}} + \underline{\sin}\,\underline{\varphi}\,\underline{\mathbf{h}} \qquad (5)$$

is the image of $\underline{\mathbf{g}}$ *under the helical motion along* $\underline{\mathbf{n}}$ *through the dual angle* $\underline{\varphi}$ (see Figure 2).

Proof: From the perpendicular intersection of $\underline{\mathbf{g}}$ and $\underline{\mathbf{h}}$ we conclude $\underline{\mathbf{g}}\cdot\underline{\mathbf{h}} = 0$ and $\underline{\mathbf{n}} := \underline{\mathbf{g}}\times\underline{\mathbf{h}}$. These equations imply $\underline{\mathbf{k}}\cdot\underline{\mathbf{k}} = \underline{\cos}^2\underline{\varphi}(\underline{\mathbf{g}}\cdot\underline{\mathbf{g}}) + \underline{\sin}^2\underline{\varphi}(\underline{\mathbf{h}}\cdot\underline{\mathbf{h}}) = 1$, and from Theorem 1 we conclude $\underline{\mathbf{g}}\times\underline{\mathbf{k}} := \underline{\sin}\,\underline{\varphi}(\underline{\mathbf{g}}\times\underline{\mathbf{h}}) = \underline{\sin}\,\underline{\varphi}\,\underline{\mathbf{n}}$ as stated. □

Another standard example is shown in [5]: The statement that in any spherical triangle with atmost one right angle the altitudes are concurrent leads by dual extension to a statement of J. Hjelmslev and F. Morley (1898): *At any skew hexagon with only right angles the common perpendiculars of opposite sides intersect a common line perpendicularily, provided atmost one pair of opposite sides is parallel.* Finally, in [6] the differential geometry of ruled surfaces is based on Frenet equations which are exactly the dual extensions of the Frenet equations for spherical curves. Here the curvature center of a spherical curve is replaced by the curvature axis (or Disteli axis) of the ruled surface.

In the sequel we need

THEOREM 3: *Any dual vector* $\underline{\mathbf{v}} = \mathbf{v} + \varepsilon\,\hat{\mathbf{v}}$ *is a dual multiple of a dual unit vector, i.e.,* $\underline{\mathbf{v}} = \underline{\lambda}\,\underline{\mathbf{g}}$ *with* $\underline{\mathbf{g}} \cdot \underline{\mathbf{g}} = 1$. *In the case* $\mathbf{v} \neq \mathbf{o}$ *the dual unit vector* $\underline{\mathbf{g}}$ *is uniquely determined up to its sign.*

Proof: We have to fulfill the equation $\mathbf{v} + \varepsilon\,\hat{\mathbf{v}} = (\lambda + \varepsilon\hat{\lambda})(\mathbf{g} + \varepsilon\,\hat{\mathbf{g}}) = \lambda\mathbf{g} + \varepsilon(\hat{\lambda}\mathbf{g} + \lambda\hat{\mathbf{g}})$. First we note that $\underline{\mathbf{v}} \cdot \underline{\mathbf{v}} = \underline{\lambda}^2\, \underline{\mathbf{g}} \cdot \underline{\mathbf{g}}$ implies $\mathbf{v} \cdot \mathbf{v} + 2\varepsilon(\mathbf{v} \cdot \hat{\mathbf{v}}) = \lambda^2 + 2\varepsilon\lambda\hat{\lambda}$. This gives $\mathbf{v} = \lambda\mathbf{g}$, $\hat{\mathbf{v}} = \hat{\lambda}\mathbf{g} + \lambda\hat{\mathbf{g}}$ and $\mathbf{v} \cdot \hat{\mathbf{v}} = \lambda\hat{\lambda}$. For $\lambda = \pm\|\,\mathbf{v}\,\| \neq 0$ we get

$$\mathbf{g} = \frac{1}{\lambda}\mathbf{v} \quad \text{and} \quad \hat{\mathbf{g}} = \frac{1}{\lambda}\left(\hat{\mathbf{v}} - \frac{\lambda\hat{\lambda}}{\lambda^2}\mathbf{v}\right) \tag{6}$$

as the solution. In the case $\lambda = 0$, i.e., $\underline{\mathbf{v}} = \varepsilon\,\hat{\mathbf{v}}$, we set $\hat{\lambda} = \|\,\hat{\mathbf{v}}\,\|$, $\hat{\lambda}\mathbf{g} = \hat{\mathbf{v}}$ and choose an arbitrary $\hat{\mathbf{g}}$ under $\hat{\lambda} \neq 0$, otherwise the unit vector $\underline{\mathbf{g}}$ can be chosen arbitrarily, too. □

DUAL VECTORS REPRESENTING SCREWS

In this section we demonstrate the use of dual vectors for describing instantaneous motions (for an introduction see also [1,2,3]): Let a rigid body representing the moving system Σ_1 perform a one-parameter motion against the frame, the fixed system Σ_0. We assume that cartesian coordinate frames are attached to each system Σ_i, $i = 0,1$, and we use the subscript *i* to indicate related coordinate vectors. Then the movement of Σ_1 against Σ_0 can analytically be described by the coordinate transformation which at each instant *t* gives the Σ_0-coordinate vector $\mathbf{x}_0$ of any point which with respect to the moving system Σ_1 has the coordinate vector $\mathbf{x}_1$. This coordinate transformation reads

$$\Sigma_1/\Sigma_0: \; \mathbf{x}_0 = \mathbf{u}_0(t) + \mathbf{A}(t) \cdot \mathbf{x}_1 \quad \text{with} \quad \mathbf{A}(t) \cdot \mathbf{A}(t)^T = \mathbf{I}_3 \text{ and } \det \mathbf{A}(t) = 1. \tag{7}$$

Here $\mathbf{I}_3$ denotes the unit matrix, $\mathbf{u}_0(t)$ is the Σ_0-coordinate vector of the origin of Σ_1, and $\mathbf{A}(t)$ is an orthogonal matrix, i.e., its transposed $\mathbf{A}(t)^T$ is at the same time its inverse $\mathbf{A}(t)^{-1}$. In order to figure out the distribution of velocity vectors ${}_X\mathbf{v}_{10}$ of points X attached to the moving system Σ_1 we differentiate and replace $\mathbf{x}_1$ by $\mathbf{x}_0$ due to (7). We thus obtain – after dropping the parameter *t* –

$${}_X\mathbf{v}_{10} = \dot{\mathbf{x}}_0 = \dot{\mathbf{u}}_0 + \dot{\mathbf{A}} \cdot \mathbf{x}_1 = (\dot{\mathbf{u}}_0 - \dot{\mathbf{A}} \cdot \mathbf{A}^T \cdot \mathbf{u}_0) + \dot{\mathbf{A}} \cdot \mathbf{A}^T \cdot \mathbf{x}_0 \tag{8}$$

because of $\dot{\mathbf{x}}_1 = \mathbf{o}$. The matrix $\dot{\mathbf{A}} \cdot \mathbf{A}^T$ is skew as differentiation of $\mathbf{A} \cdot \mathbf{A}^T = \mathbf{I}_3$ gives

$$\dot{\mathbf{A}} \cdot \mathbf{A}^T + \mathbf{A} \cdot \dot{\mathbf{A}}^T = \dot{\mathbf{A}} \cdot \mathbf{A}^T + (\dot{\mathbf{A}} \cdot \mathbf{A}^T)^T = \mathbf{O} = \text{zero matrix.}$$

There is a dual vector

$$\underline{\mathbf{q}}_{10} = \mathbf{q}_{10} + \varepsilon\hat{\mathbf{q}}_{10} \text{ such that } \dot{\mathbf{A}} \cdot \mathbf{A}^T \cdot \mathbf{x}_0 = \mathbf{q}_{10} \times \mathbf{x}_0 \text{ for all } \mathbf{x}_0 \in \mathbb{R}^3 \text{ and } \hat{\mathbf{q}}_{10} := \dot{\mathbf{u}}_0 - \dot{\mathbf{A}} \cdot \mathbf{A}^T \cdot \mathbf{u}_0. \tag{9}$$

We call this dual vector the *instantanous screw* as according to (8) it rules the distribution of those velocity vectors which the given motion instantaneously assigns to each point $X \in \Sigma_1$. We have

$${}_X\mathbf{v}_{10} = \hat{\mathbf{q}}_{10} + (\mathbf{q}_{10} \times \mathbf{x}_0). \tag{10}$$

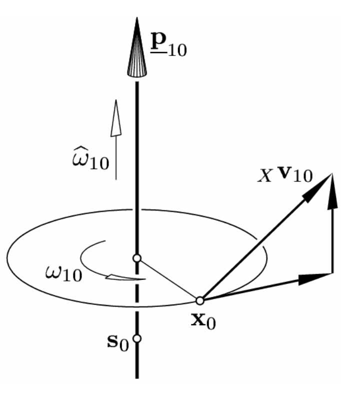

FIGURE 3: INSTANTANEOUS MOTION WITH SCREW $\underline{\mathbf{q}}_{10} = \underline{\omega}_{10}\,\underline{\mathbf{p}}_{10}$.

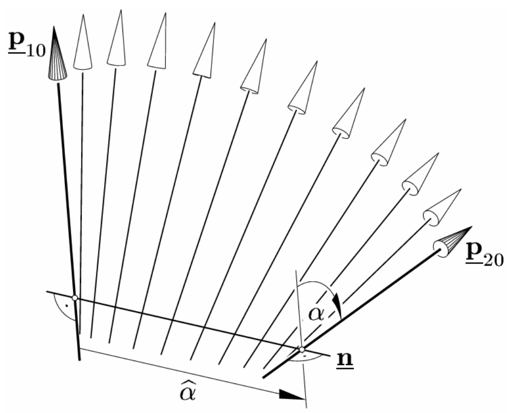

FIGURE 4: PORTION OF PLÜCKER'S CYLINDROID

Due to Theorem 3 the dual vector $\underline{\mathbf{q}}_{10}$ is a dual multiple of a dual unit vector, i.e.,

$$\underline{\mathbf{q}}_{10} = \underline{\omega}_{10}\,\underline{\mathbf{p}}_{10} \text{ or explicitely } \mathbf{q}_{10} + \varepsilon\hat{\mathbf{q}}_{10} = (\omega_{10} + \varepsilon\hat{\omega}_{10})(\mathbf{p}_{10} + \varepsilon\hat{\mathbf{p}}_{10}) = \omega_{10}\mathbf{p}_{10} + \varepsilon(\hat{\omega}_{10}\mathbf{p}_{10} + \omega_{10}\hat{\mathbf{p}}_{10}) \tag{11}$$

with $\underline{\mathbf{p}}_{10} \cdot \underline{\mathbf{p}}_{10} = 1$. In order to figure out the meaning of the dual scalar $\underline{\omega}_{10}$ and the oriented line $\underline{\mathbf{p}}_{10}$ we use a point $\mathbf{s}_0$ of this line and set $\hat{\mathbf{p}}_{10} = \mathbf{s}_0 \times \mathbf{p}_{10}$. Then according to (10) and (11) we get

$$ {}_X\mathbf{v}_{10} = \hat{\mathbf{q}}_{10} + (\mathbf{q}_{10} \times \mathbf{x}_0) = (\hat{\omega}_{10}\mathbf{p}_{10} + \omega_{10}\hat{\mathbf{p}}_{10}) + [\omega_{10}\mathbf{p}_{10} \times \mathbf{x}_0] = \hat{\omega}_{10}\mathbf{p}_{10} + \omega_{10}[\mathbf{p}_{10} \times (\mathbf{x}_0 - \mathbf{s}_0)]. \quad (12) $$

This reveals that ${}_X\mathbf{v}_{10}$ is the velocity vector of $\mathbf{x}_0$ under a helical motion about the instantaneous axis (ISA) $\underline{\mathbf{p}}_{10}$ with angular velocity ω_{10} and translatory velocity $\hat{\omega}_{10}$ (see Figure 3). We call $\underline{\omega}_{10}$ the *dual angular velocity* of the instantaneous motion. In this sense the dual unit vectors are at the same time the screws for instantaneous rotations with angular velocity 1.

Now it is clear why due to Theorem 2 the axis $\underline{\mathbf{p}}_{10}$ is uniquely determined only under $\omega_{10} \neq 0$: Otherwise the instantaneous motion is a translation and here only the direction of the axis is determined, but not the axis itself.

THEOREM 4: *When the instantaneous screw $\underline{\mathbf{q}}_{10}$ is expressed as a multiple $\underline{\mathbf{q}}_{10} = \underline{\omega}_{10}\underline{\mathbf{p}}_{10}$ of a dual unit vector, then $\underline{\mathbf{p}}_{10}$ is the instantaneous axis and $\underline{\omega}_{10}$ the dual angular velocity of the instantaneous helical motion.*

One has to be careful with dual vectors when coordinate transformations are applied as the two components of a dual vector behave different:

THEOREM 5: *If in Σ_0 the coordinates $\mathbf{x}_0$ of point X are replaced by $\mathbf{x}_0'$ according to $\mathbf{x}_0' = \mathbf{b}_0' + \mathbf{B} \cdot \mathbf{x}_0$ with orthogonal $\mathbf{B}$, then the components of any screw $\underline{\mathbf{q}}_{10} = \mathbf{q}_{10} + \varepsilon\hat{\mathbf{q}}_{10}$ (inclusive spears) have to be replaced by*

$$ \mathbf{q}_{10}' = \mathbf{B} \cdot \mathbf{q}_{10} \text{ and } \hat{\mathbf{q}}_{10}' = \mathbf{B} \cdot \hat{\mathbf{q}}_{10} + (\mathbf{b}_0' \times \mathbf{B} \cdot \mathbf{q}_{10}). \quad (13) $$

Proof: Because of the geometric meaning of the screw $\underline{\mathbf{q}}_{10} = \underline{\omega}_{10}\underline{\mathbf{p}}_{10}$ the coordinate transformation does affect the underlying spear, but not the dual angular velocity, i.e., $\underline{\mathbf{q}}_{10}' = \underline{\omega}_{10}\underline{\mathbf{p}}_{10}'$. And $\mathbf{p}_{10} + \varepsilon\hat{\mathbf{p}}_{10}$ is transformed into $\mathbf{p}_{10}' + \varepsilon\hat{\mathbf{p}}_{10}'$ by $\mathbf{p}_{10}' = \mathbf{B} \cdot \mathbf{p}_{10}$ and

$$ \hat{\mathbf{p}}_{10}' = \mathbf{s}_0' \times \mathbf{p}_{10}' = (\mathbf{b}_0' + \mathbf{B} \cdot \mathbf{s}_0) \times \mathbf{B} \cdot \mathbf{p}_{10} = (\mathbf{b}_0' \times \mathbf{B} \cdot \mathbf{p}_{10}) + \mathbf{B} \cdot (\mathbf{s}_0 \times \mathbf{p}_{10}) = (\mathbf{b}_0' \times \mathbf{B} \cdot \mathbf{p}_{10}) + \mathbf{B} \cdot \hat{\mathbf{p}}_{10}. \quad \square $$

The formulas for $\underline{\mathbf{q}}_{10}$ in (9) refer to coordinates in the fixed system Σ_0. However, for the sake of simplicity we avoid the more precise notation $\underline{\mathbf{q}}_{10|0}$. Eqs. (13) could be used to represent the screw also with respect to Σ_1, and then the coordinate vector should be denoted by $\underline{\mathbf{q}}_{10|1}$.

THEOREM 6: *For any instantaneous motion with the screw $\underline{\mathbf{q}}_{10}$ the 'path-normals', i.e., the lines $\underline{\mathbf{n}}$ perpendicular to ${}_X\mathbf{v}_{10}$ and passing through X, constitute a linear line complex as $\underline{\mathbf{n}} = \mathbf{n} + \varepsilon\hat{\mathbf{n}}$ obeys the linear homogeneous equation*

$$ \hat{\mathbf{q}}_{10} \cdot \mathbf{n} + \mathbf{q}_{10} \cdot \hat{\mathbf{n}} = 0 \quad (\Leftrightarrow \underline{\mathbf{q}}_{10} \cdot \underline{\mathbf{n}} \in \mathbb{R}). \quad (14) $$

This equation is independent from X. Hence any line $\pm\underline{\mathbf{n}}$, which is a path-normal at one of its points, is a path-normal at each point.

Proof: Due to (10) and $\hat{\mathbf{n}} = \mathbf{x}_0 \times \mathbf{n}$ we have

$$ 0 = {}_X\mathbf{v}_{10} \cdot \mathbf{n} = [\hat{\mathbf{q}}_{10} + (\mathbf{q}_{10} \times \mathbf{x}_0)] \cdot \mathbf{n} = \hat{\mathbf{q}}_{10} \cdot \mathbf{n} + \det(\mathbf{q}_{10}, \mathbf{x}_0, \mathbf{n}) = \hat{\mathbf{q}}_{10} \cdot \mathbf{n} + \mathbf{q}_{10} \cdot (\mathbf{x}_0 \times \mathbf{n}) = \hat{\mathbf{q}}_{10} \cdot \mathbf{n} + \mathbf{q}_{10} \cdot \hat{\mathbf{n}}. \quad \square $$

By (3) and (11) eq. (14) is equivalent to $\underline{\omega}_{10}\underline{\mathbf{p}}_{10} \cdot \underline{\mathbf{n}} = \underline{\omega}_{10}\cos\underline{\alpha} \in \mathbb{R}$, i.e., $(\omega_{10} + \varepsilon\hat{\omega}_{10})(\cos\alpha - \varepsilon\hat{\alpha}\sin\alpha) \in \mathbb{R}$ or $\hat{\omega}_{10}\cos\alpha - \omega_{10}\hat{\alpha}\sin\alpha = 0$. Hence the path normals $\underline{\mathbf{n}}$ of the instantaneous motion are characterized by

$$ \frac{\hat{\omega}_{10}}{\omega_{10}} = \hat{\alpha}\tan\alpha \quad (15) $$

with $\underline{\alpha}$ denoting the dual angle between the ISA $\underline{\mathbf{p}}_{10}$ and any orientation of $\underline{\mathbf{n}}$. The quotient on the left hand side is the pitch of the helical motion.

Finally, we need the spatial *Three-Pole-Theorem*

THEOREM 7: *If for three given systems $\Sigma_0, \Sigma_1, \Sigma_2$ the dual vectors $\underline{\mathbf{q}}_{10}$ and $\underline{\mathbf{q}}_{20}$ are the instantaneous screws of Σ_1/Σ_0 and Σ_2/Σ_0, respectively, then*

$$ \underline{\mathbf{q}}_{21} = \underline{\mathbf{q}}_{20} - \underline{\mathbf{q}}_{10}, \text{ i.e., } \underline{\omega}_{21}\underline{\mathbf{p}}_{21} = \underline{\omega}_{20}\underline{\mathbf{p}}_{20} - \underline{\omega}_{10}\underline{\mathbf{p}}_{10} \quad (16) $$

is the instantaneous screw of the relative motion Σ_2/Σ_1. The three corresponding linear line complexes are included in a pencil of line complexes.

Proof: According to (10) we have

$_X\mathbf{v}_{21} = {}_X\mathbf{v}_{20} - {}_X\mathbf{v}_{10} = [\hat{\mathbf{q}}_{20} + (\mathbf{q}_{20}\times\mathbf{x}_0)] - [\hat{\mathbf{q}}_{10} + (\mathbf{q}_{10}\times\mathbf{x}_0)] = (\hat{\mathbf{q}}_{20} - \hat{\mathbf{q}}_{10}) + [(\mathbf{q}_{20} - \mathbf{q}_{10})\times\mathbf{x}_0]$ for each point X . □

Let a line $\underline{\mathbf{n}}$ intersect the ISAs $\underline{\mathbf{p}}_{10}$ of Σ_1/Σ_0 and $\underline{\mathbf{p}}_{20}$ of Σ_2/Σ_0 orthogonally, i.e., $\underline{\mathbf{p}}_{10}\cdot\underline{\mathbf{n}} = \underline{\mathbf{p}}_{20}\cdot\underline{\mathbf{n}} = 0$. Then we obtain $\underline{\omega}_{21}\underline{\mathbf{p}}_{21}\cdot\underline{\mathbf{n}} = (\underline{\omega}_{20}\underline{\mathbf{p}}_{20}\cdot\underline{\mathbf{n}} - \underline{\omega}_{10}\underline{\mathbf{p}}_{10}\cdot\underline{\mathbf{n}}) = 0$ which means that the line $\underline{\mathbf{n}}$ does intersect the axis $\underline{\mathbf{p}}_{21}$ of Σ_2/Σ_1 orthogonally, too, provided $\omega_{21} \neq 0$.

Let two skew axes $\underline{\mathbf{p}}_{10}$ and $\underline{\mathbf{p}}_{20}$ be given, i.e., $\underline{\mathbf{p}}_{10}\cdot\underline{\mathbf{p}}_{20} \notin \mathbf{R}$. When for the corresponding dual velocities we replace $\underline{\omega}_{20}$ by $c\,\underline{\omega}_{20}$ with a constant c varying in $\mathbf{R}$, then the axes $\underline{\mathbf{p}}_{21}$ of the relative motions Σ_2/Σ_1 constitute a *cylindroid* or *Plücker conoid* (see [4], p. 181). Figure 4 gives an impression of the cylindroid by showing some generators 'between' $\underline{\mathbf{p}}_{10}$ and $\underline{\mathbf{p}}_{20}$.

Now the principle of transference can be used to convert theorems from spherical kinematics into those of spatial kinematics. One example is the dual extension of the Euler-Savary-formula (see [6]). It dates back to M. Disteli (1914) and deals with the curvature axes of ruled surfaces which are traced by lines under a spatial motion. Other examples concerning overconstrained spatial mechanisms derived from spherical ones are presented in [5].

APPLICATIONS

EXAMPLE 1: *Infinitesimal forward and inverse kinematics of 6R robots:*
A serial robot is an open kinematic chain of links $\Sigma_0, \Sigma_1, ..., \Sigma_6$. Any two consecutive links are connected by a revolute joint with the axis $\underline{\mathbf{p}}_{10}, \underline{\mathbf{p}}_{21}, ..., \underline{\mathbf{p}}_{65}$, respectively.

Given: Any posture of a 6R robot with instantaneous angular velocites $\omega_{10}, \omega_{21}, ..., \omega_{65} \in \mathbf{R}$ of the relative motions about the axes. All axes are given in the same coordinate system.
Required: What is the instantaneous motion of the endeffector Σ_6 ?

This instantaneous motion is defined by the screw $\underline{\mathbf{q}}_{60}$. From the Three-Pole-Theorem (Theorem 7) we conclude that $\underline{\mathbf{q}}_{60} = \omega_{10}\underline{\mathbf{p}}_{10} + \omega_{21}\underline{\mathbf{p}}_{21} + ... + \omega_{65}\underline{\mathbf{p}}_{65}$. This can also be written in matrix form: We combine the coordinates $\underline{\mathbf{p}}_{10}, ..., \underline{\mathbf{p}}_{65}$ of the axes as columns in a 6×6-matrix J. Then the resulting screw reads

$$\underline{\mathbf{q}}_{60} = \sum_{i=1..6} \omega_{i\,i-1}\mathbf{p}_{i\,i-1} = J\cdot\Omega$$

with Ω as the column vector of given angular velocities. J is called *Jacobi matrix*. In the regular case ($\det J \neq 0$) we can also solve the inverse problem: For given $\underline{\mathbf{q}}_{60}$ we get the corresponding angular velocities $\omega_{10}, ..., \omega_{65}$ by solving a linear system of equations. If there is a rank deficiency of J, then the instantaneous degree of freedom of Σ_6/Σ_0 is less than 6. Just in this case the columns of J are linearly dependent. This is equivalent to the statement that the six axes are included in a linear line complex.

EXAMPLE 2: *Calibration of Stewart-Gough-Platforms:*
Given: Any posture of a Stewart-Gough-Platform, i.e., a parallel manipulator where the platform Σ_1 is connected with the frame Σ_0 by six telescopic legs. The anchor points in Σ_0 are denoted by $\mathbf{a}_1, ..., \mathbf{a}_6$, those in the platform Σ_1 by $\mathbf{b}_1, ..., \mathbf{b}_6$. We assume that for all these points the instantaneous coordinates are given in the same coordinate system.
Required: Suppose that by precise measurements a mislocation of the platform against the frame has been detected. How to figure out which leg is mainly responsible for this deviation.

There is a (small) helical motion which transports the actual posture into the target posture. After solving this registration problem we get the axis $\underline{\mathbf{p}}_{10}$, the angle φ_{10} of rotation and the length $\hat{\varphi}_{10}$ of translation. Let us assume that this movement is performed within – say – 1 second. This defines a screw $\underline{\mathbf{q}}_{10}$, and the instantaneous helical motion of Σ_1 assigns to each of its anchor points $\mathbf{b}_1, ..., \mathbf{b}_6$ a velocity vector according to (10). The component of this vector in direction of the leg $\mathbf{a}_i\mathbf{b}_i$ gives the corresponding variation in length which has to be carried out within 1 second. So, the leg with the largest velocity component should be most responsible for the mislocation. Of course, the reliability on this result needs to be fostered by iterated measurements in different postures.

How to compute these variations? Let d_i denote the distance $\|\mathbf{b}_i - \mathbf{a}_i\|$. Then the carrier line of the leg oriented in the direction $\mathbf{a}_i\mathbf{b}_i$ has the coordinate components

$$\mathbf{l}_i = \frac{\mathbf{b}_i - \mathbf{a}_i}{d_i} \text{ and } \hat{\mathbf{l}}_i = \mathbf{b}_i \times \mathbf{l}_i .$$

From $d_i^2 = (\mathbf{b}_i - \mathbf{a}_i)^2$ we obtain bei differentiation $d_i \dot{d}_i = (\mathbf{b}_i - \mathbf{a}_i) \cdot \dot{\mathbf{b}}_i$, hence

$$\dot{d}_i = \mathbf{l}_i \cdot \dot{\mathbf{b}}_i = \mathbf{l}_i \cdot [\hat{\mathbf{q}}_{10} + (\mathbf{q}_{10} \times \mathbf{b}_i)] = \mathbf{l}_i \cdot \hat{\mathbf{q}}_{10} + \det(\mathbf{l}_i, \mathbf{q}_{10}, \mathbf{b}_i) = \mathbf{l}_i \cdot \hat{\mathbf{q}}_{10} + (\mathbf{b}_i \times \mathbf{l}_i) \cdot \mathbf{q}_{10} = \mathbf{l}_i \cdot \hat{\mathbf{q}}_{10} + \hat{\mathbf{l}}_i \cdot \mathbf{q}_{10}.$$

We form a 6×6 -matrix $\overline{J}$ with rows consisting of the coordinates of $(\hat{\mathbf{l}}_i, \mathbf{l}_i)$ written in this order. Then we get

$$\dot{D} = \overline{J} \cdot \underline{\mathbf{q}}_{10}$$

with $\dot{D}$ denoting the column of $\dot{d}_i$. So we get the variation of leg lengths by multiplying this Jacobi matrix $\overline{J}$ with the screw. In singular postures, which are characterized by $\det \overline{J} = 0$, there are infinitesimal self motions of the platform while the lengths of all telescopic legs remain fixed. Just in singular postures the rows in $\overline{J}$ are linearly dependent and therefore the six lines $\mathbf{a}_i\mathbf{b}_i$ included in a linear line complex.

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