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Generating Solids by Sweeping Polyhedra

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Abstract. Let a one-parametric motion β and the boundary representation of a polyhedron $\mathbf{P}$ be given. Our goal is to determine the solid $\mathbf{S}_0$ swept by $\mathbf{P}$ under β: The complete boundary $\partial\mathbf{S}_0$ of $\mathbf{S}_0$ contains a subset of the enveloping surface Φ_0 of the moving polyhedron's boundary $\partial\mathbf{P}$ together with portions of the boundaries of the initial and the final positions of $\mathbf{P}$. For each intermediate position of $\mathbf{P}$ the curve of contact $c_{\partial\mathbf{P}}$ between $\partial\mathbf{P}$ and Φ_0 is called the characteristic curve $c_{\partial\mathbf{P}}$ of the surface $\partial\mathbf{P}$. However, in general only a subset of $c_{\partial\mathbf{P}}$ gives the characteristic curve $c_{\mathbf{P}}$ of the solid $\mathbf{P}$ which is defined as the curve of contact between $\partial\mathbf{P}$ and $\partial\mathbf{S}_0$.

After a short introduction into instantaneous spatial kinematics, these two characteristic curves $c_{\partial\mathbf{P}}$ and $c_{\mathbf{P}}$ are characterized locally. Then some global problems are discussed that arise when the boundary representation of a polyhedral approximation of $\mathbf{S}_0$ is derived automatically. The crucial point here is the determination of self-intersections at the envelope Φ_0. For the global point of view the motion β is restricted to the case of a helical motion with fixed axis and parameter.

Key Words: Kinematics, enveloping surfaces, constructive solid geometry.
MSC 1994: 53A17

1. Introduction

The generation of surfaces in the Euclidean 3-space $\mathbb{E}^3$ by sweeping curves is well known. Our task is different: We study the generation of a new solid $\mathbf{S}_0$ by sweeping a given solid $\mathbf{P}$ which is supposed to be a polyhedron with given boundary representation. Our main aim is to compute the boundary representation of a polyhedral approximation $\mathbf{P}_0$ of the swept solid $\mathbf{S}_0$ for a given one-parameter motion $\beta(t)$, $a \le t \le b$, of $\mathbf{P}$. Finally, a computerized algorithm is developed in order to solve our problem automatically.

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There is of course a trivial way of deriving $\mathbf{P}_0$ by discretizing the motion: Subdivide the given definition interval by $t_0 = a < t_1 < \cdots < t_n = b$. Then the union over all positions

$$\mathbf{P}_0 := \bigcup_{i=0}^{n} \beta(t_i)\mathbf{P}$$

gives a rough approximation of the swept volume $\mathbf{S}_0$. But instead of smooth bounding surfaces the boundary looks rather crumpled (compare Fig. 1[1] and Fig. 2). One can of course improve this approximation by applying a smoothing operation, and it will turn out in chapter 4 that, in some periods of the motion, this is the only way to handle the problem. Nevertheless we discuss an approach where results from the geometry of envelopes are involved.

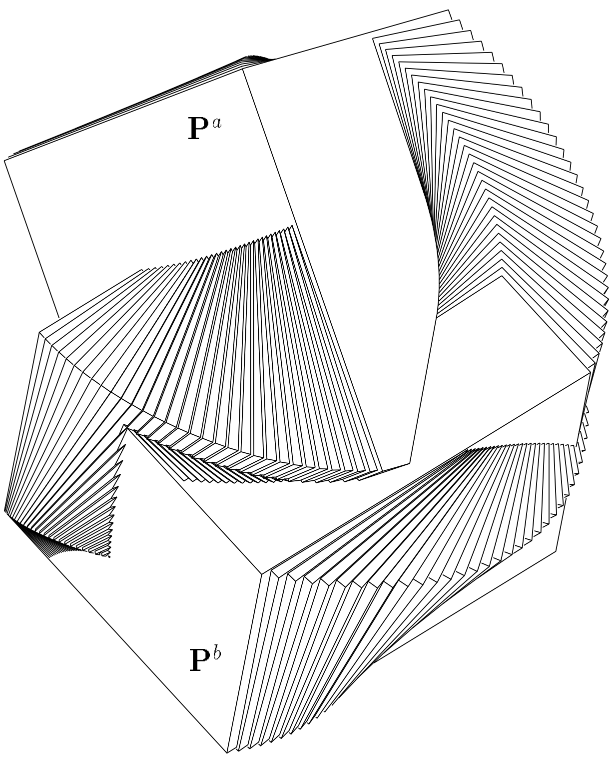

Figure 1: Rough approximation of $\mathbf{S}_0$

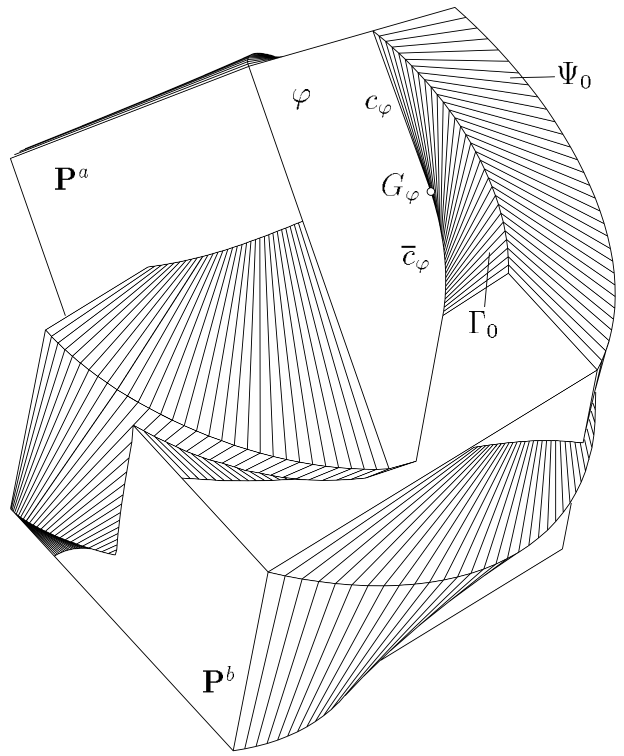

Figure 2: Exact swept solid $\mathbf{S}_0$

The boundary $\partial\mathbf{S}_0$ of the solid $\mathbf{S}_0$ is a subset of the enveloping surface Φ_0 of the piecewise smooth boundary $\partial\mathbf{P}$ under the motion β. Each intermediate position of $\partial\mathbf{P}$ is in contact with the enveloping Φ_0 along the *characteristic curve* $c_{\partial\mathbf{P}}$. At the beginning we present the theory of envelopes and the related formulas from spatial kinematics.

However, it turns out that this local theory is not sufficient, even when only a small portion of the motion is regarded. Usually, there are self-intersections of Φ_0, already from the initial position on. It is the crucial point of our problem to figure out and to exclude those parts of Φ_0 that are located in the interior of $\mathbf{S}_0$. Therefore it is necessary to distinguish between the characteristic curve $c_{\partial\mathbf{P}}$ of $\partial\mathbf{P}$ and the *characteristic curve* $c_\mathbf{P}$ of the solid $\mathbf{P}$ which is the curve of contact between $\partial\mathbf{P}$ and the boundary $\partial\mathbf{S}_0$ of the swept solid. Note that $c_{\partial\mathbf{P}}$ is based on the surface model of $\mathbf{P}$ while $c_\mathbf{P}$ is related to the solid model.

[1] This figure has been obtained by the 3d-software package CAD-3D which had been developed at the Institute for Geometry in Vienna (cf. [6]). This software is based on the boundary representations of polyhedra. As an alternative also decompositions techniques for solids could be used.

For the global point of view we will restrict β to a pure helical motion with fixed axis and fixed helical parameter. In this case the characteristic $c_{\partial\mathbf{P}}$ is fixed with respect to the moving $\mathbf{P}$. We have to distinguish whether the total screwing angle is finite of our interval of definition reads $-\infty < t < \infty$. Only in the latter case is the characteristic $c_{\mathbf{P}}$ parameter-invariant too, like $c_{\partial\mathbf{P}}$. Note that for a small helical parameter, for each position of $\mathbf{P}$, also the following and the previous turns of $\mathbf{S}_0$ have to be tested in view of intersections.

One application of our theory is the following: Find the shape of an object which is manufactured from the solid $\mathbf{R}_0$ by milling, provided $\mathbf{P}$ is seen as a polyhedral approximation of the milling cutter. We solve this problem by determining the difference $\mathbf{R}_0\backslash\mathbf{S}_0$, where $\mathbf{S}_0$ is swept by $\mathbf{P}$ under the relative motion $\beta(t)$ of the milling cutter against the given solid $\mathbf{R}_0$ (cf. [1]).

2. The surface characteristic $c_{\partial\mathbf{P}}$

The characteristic curve $c_{\partial\mathbf{P}}$ of any intermediate position $\beta(t_0)\mathbf{P}$, $a < t_0 < b$, of the moving polyhedron is defined as the curve of contact between the boundary $\partial\mathbf{P}$ and the enveloping surface Φ_0[2]. This curve $c_{\partial\mathbf{P}}$ consists of closed polygons on $\partial\mathbf{P}$ (note Fig. 5, left). The sides of $c_{\partial\mathbf{P}}$ are located either on edges e or in the faces f of $\mathbf{P}$ at the characteristic line c_φ of the plane $\varphi = span(f)$. The geometric characterization of these sides of $c_{\partial\mathbf{P}}$ can be understood better when the following term is used which dates back to Tuschel [7]:

2.1. The helical projection

It is well known from kinematics that, in each moment of a differentiable one-parameter motion, the distribution of velocity vectors is that of a helical motion with fixed axis and fixed helical parameter, the *instantaneous helical motion.* Its axis p is called *pole axis.*

We now see all the oriented helical curves of the instantaneous helical motion as "rays of sight" of a projection. Suppose that there is an image plane π perpendicular to the axis p. Then, for non-vanishing helical parameter, all spatial points X can be projected along these curved fibres onto points $X^s \in \pi$ (Fig. 3). This mapping is called a *helical projection.*

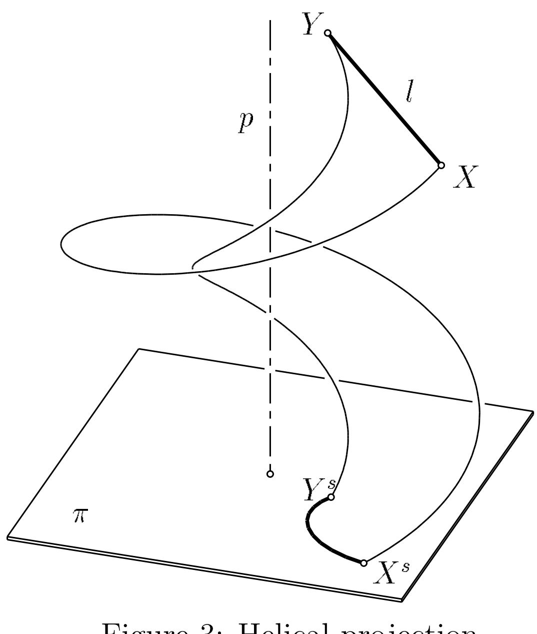

Figure 3: Helical projection

2.2. Characterization of $c_{\partial\mathbf{P}}$

The characteristic $c_{\partial\mathbf{P}}$ coincides with the silhouette of $\partial\mathbf{P}$ with respect to the instantaneous helical projection. $X \in \partial\mathbf{P}$ is a point of $c_{\partial\mathbf{P}}$, if and only if the helix s_X through X touches the boundary locally just at point X.

[2]New results on the characteristic curves of moving smooth surfaces can be found in [4].

This definition is ambiguous only on lines or in planes which are moved into themselves under the instantaneous helical motion. This can only happen in the limiting cases of a pure translation or a pure rotation. In the first case our projection is an orthogonal projection.

2.3. The instantaneous screw

For an analytic treatment let $\mathfrak{p}$ denote the unit vector of the pole axis p, and $\widehat{\mathfrak{p}} := \mathfrak{x}\times\mathfrak{p}$ for $\mathfrak{x} \in p$ the so-called momentum vector (cf. [3]). It is convenient to combine both vectors in a *dual* vector $\underline{\mathfrak{p}} := \mathfrak{p} + \varepsilon\widehat{\mathfrak{p}}$ where the dual unit ε obeys the rule $\varepsilon^2 = 0$. Thus the oriented line p is represented by a dual *unit vector* $\underline{\mathfrak{p}}$ as $\underline{\mathfrak{p}}\cdot\underline{\mathfrak{p}} = \mathfrak{p}\cdot\mathfrak{p} + 2\varepsilon\,\mathfrak{p}\cdot\widehat{\mathfrak{p}} = 1$. In the same way we combine the *angular velocity* ω and the *translation velocity* $\widehat{\omega}$ of the instantaneous helical motion in the dual number $\underline{\omega} := \omega + \varepsilon\widehat{\omega}$. Then

$$\underline{\mathfrak{q}} := \mathfrak{q} + \varepsilon\widehat{\mathfrak{q}} = \underline{\omega}\,\underline{\mathfrak{p}} = \omega\mathfrak{p} + \varepsilon(\omega\widehat{\mathfrak{p}} + \widehat{\omega}\mathfrak{p}) \tag{1}$$

is the so-called *instantaneous screw.* It allows us to express the velocity vector $\mathfrak{v}_\mathfrak{x}$ of point X with coordinate vector $\mathfrak{x}$ by

$$\mathfrak{v}_\mathfrak{x} = \widehat{\mathfrak{q}} + (\mathfrak{q}\times\mathfrak{x}). \tag{2}$$

The first derivative gives the instantaneous acceleration vector

$$\mathfrak{a}_\mathfrak{x} = \dot{\widehat{\mathfrak{q}}} + (\dot{\mathfrak{q}}\times\mathfrak{x}) + (\mathfrak{q}\times\mathfrak{v}_\mathfrak{x}) = \dot{\widehat{\mathfrak{q}}} + (\mathfrak{q}\times\widehat{\mathfrak{q}}) + (\dot{\mathfrak{q}}\times\mathfrak{x}) + (\mathfrak{q}\cdot\mathfrak{x})\mathfrak{q} - \omega^2\mathfrak{x}\,. \tag{3}$$

2.4. Sides of the characteristic curve $c_{\partial\mathbf{P}}$ in faces of P

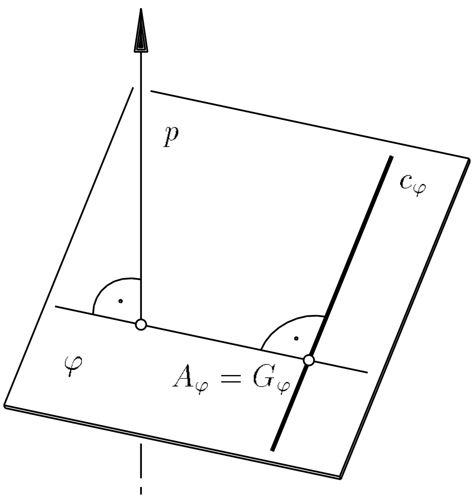

Figure 4: Characteristic line c_φ of φ

For a given face f of $\mathbf{P}$ with $\varphi = span(f)$, let $X \in f$ be a point with tangent line t_X. Then

(C1) $X \in f$ is a point of $c_{\partial\mathbf{P}}$ if and only if $t_X \subset \varphi$.

All points of φ with this property form a line, the *characteristic line* c_φ of φ (see Fig. 4, cf. [8], p. 168 or [3], p. 164).
For the plane φ with equation

$$\mathfrak{n}\cdot\mathfrak{x} = d \ \text{ with } \ \|\mathfrak{n}\| = 1\,, \tag{4}$$

the condition $\mathfrak{n}\cdot\mathfrak{v}_\mathfrak{x} = 0$ is equivalent to

$$(\mathfrak{q}\times\mathfrak{n})\cdot\mathfrak{x} = \mathfrak{n}\cdot\widehat{\mathfrak{q}}\,. \tag{5}$$

Under $\mathfrak{q}\times\mathfrak{n} \neq \mathfrak{o}$ the left sides of these two equations are linearly independent. Then the line c_φ of intersection has the direction of

$$\mathfrak{c} = \mathfrak{n}\times(\mathfrak{q}\times\mathfrak{n}) = \mathfrak{q} - (\mathfrak{n}\cdot\mathfrak{q})\mathfrak{n} \tag{6}$$

and we get a point $\mathfrak{x}$ of this line according to

$$\mathfrak{x} = \begin{cases} \mathfrak{f} - \dfrac{(\mathfrak{q}\cdot\widehat{\mathfrak{q}})(\mathfrak{n}\times\mathfrak{q})}{(\mathfrak{n}\cdot\mathfrak{q})\|\mathfrak{n}\times\mathfrak{q}\|^2} & \text{for } \ \mathfrak{n}\cdot\mathfrak{q} \neq 0 \\ \mathfrak{s} + [d - (\mathfrak{n}\cdot\mathfrak{s})]\,\mathfrak{n} & \text{for } \ \mathfrak{n}\cdot\mathfrak{q} = 0 \end{cases} \quad \text{with} \quad \mathfrak{f} := \frac{d\,\mathfrak{q} + (\mathfrak{n}\times\widehat{\mathfrak{q}})}{\mathfrak{n}\cdot\mathfrak{q}},\ \ \mathfrak{s} := \frac{\mathfrak{q}\times\widehat{\mathfrak{q}}}{\mathfrak{q}\cdot\mathfrak{q}}\,. \tag{7}$$

Under $\mathfrak{n}\cdot\mathfrak{q} = 0$ and $\omega \neq 0$ the plane φ is parallel to the pole axis p. In planes orthogonal to p there is no finite characteristic line.

During the motion β, the plane φ with eq. (4) envelops a *torse* Γ_0. The different positions of the characteristic lines c_φ are *generators* of Γ_0 and – in general – tangent to the *cuspidal curve* g of Γ_0. The point G_φ of contact between c_φ and g, the so-called *central point* of c_φ, obeys the equations (4), (5) ($\Leftrightarrow \dot{\mathfrak{n}}\cdot\mathfrak{x} = \dot{d}$) and $\ddot{\mathfrak{n}}\cdot\mathfrak{x} = \ddot{d}$, where

$$\begin{aligned} \dot{\mathfrak{n}} &= \mathfrak{q}\times\mathfrak{n}\,, & \dot{d} &= \mathfrak{n}\cdot\widehat{\mathfrak{q}}\,,\\ \ddot{\mathfrak{n}} &= (\dot{\mathfrak{q}}\times\mathfrak{n}) + (\mathfrak{q}\cdot\mathfrak{n})\mathfrak{q} - (\mathfrak{q}\cdot\mathfrak{q})\mathfrak{n}\,, & \ddot{d} &= \det(\mathfrak{q},\mathfrak{n},\widehat{\mathfrak{q}}) + (\mathfrak{n}\cdot\dot{\widehat{\mathfrak{q}}})\,. \end{aligned} \tag{8}$$

Hence the third equation for G_φ reads

$$\big[(\dot{\mathfrak{q}}\times\mathfrak{n}) + (\mathfrak{n}\cdot\mathfrak{q})\mathfrak{q} - \omega^2\mathfrak{n}\big]\cdot\mathfrak{x} = (\mathfrak{n}\cdot\dot{\widehat{\mathfrak{q}}}) - \det(\mathfrak{n},\mathfrak{q},\widehat{\mathfrak{q}}) \;\text{ or }\; (\mathfrak{n}\cdot\mathfrak{a}_\mathfrak{x}) - 2\det(\mathfrak{n},\mathfrak{q},\mathfrak{v}_\mathfrak{x}) = 0\,. \tag{9}$$

This shows that, in general, the central point G_φ differs from the *osculating point* A_φ of φ which is defined by $\mathfrak{n}\cdot\mathfrak{a}_\mathfrak{x} = 0$. Just for A_φ the osculating plane of the path under β is equal to φ. However, in the particular case of a helical motion with constant axis and parameter, the osculating point A_φ, as well as the central point G_φ, coincide with the pedal point of the common perpendicular between c_φ and the pole axis p (see Fig. 4), provided $p \not\subset \varphi$.

2.5. Sides of the characteristic curve $c_{\partial\mathbf{P}}$ on edges of P

For a given edge e of $\mathbf{P}$ let φ_1, φ_2 be the planes spanned by the neighbouring faces f_1, f_2. For a point $X \in e$ with path tangent t_X there are exactly two cases where X is a point of the characteristic $c_{\partial\mathbf{P}}$:

(C2a) t_X is located in φ_1 or φ_2. Then X belongs to the corresponding characteristic lines c_{φ_1} or c_{φ_2} too.

(C2b) All interior points of f_1 and f_2 in a certain neighbourhood of X are on the same side of the plane spanned by edge e and the tangent line t_X.

Points of type (C2b) form open segments on e bounded by points of type (C2a). It is therefore sufficient to compute the points of type (C2a) on e. Then at most three intervals on e must be tested in order to discover whether their points fulfil condition (C2b) or not.

If for $i = 1, 2$ the vector $\mathfrak{n}_i$ is orthogonal to f_i and pointing outside, then the conditions (C2a) and (C2b) can be combined by

$$(\mathfrak{n}_1\cdot\mathfrak{v}_\mathfrak{x})(\mathfrak{n}_2\cdot\mathfrak{v}_\mathfrak{x}) \leq 0\,. \tag{10}$$

3. The solid model characteristic $c_\mathbf{P}$

$c_\mathbf{P}$ denotes the curve of contact between an intermediate $\partial\mathbf{P}$ and the boundary $\partial\mathbf{S}_0$ of the swept solid $\mathbf{S}_0$ (see Fig. 5, right).

3.1. Necessary conditions for $c_\mathbf{P}$

When $X \in \partial\mathbf{P}$ is a boundary point of $\mathbf{S}_0$ too, then in each neighbourhood of X there exists an exterior point Y which can't be reached by X or any adjacent point of $\mathbf{P}$ under the motion β. There are again two cases to distinguish:

(C2') For X on edge e, the condition $X \in c_\mathbf{P}$ implies that the plane spanned by the tangent

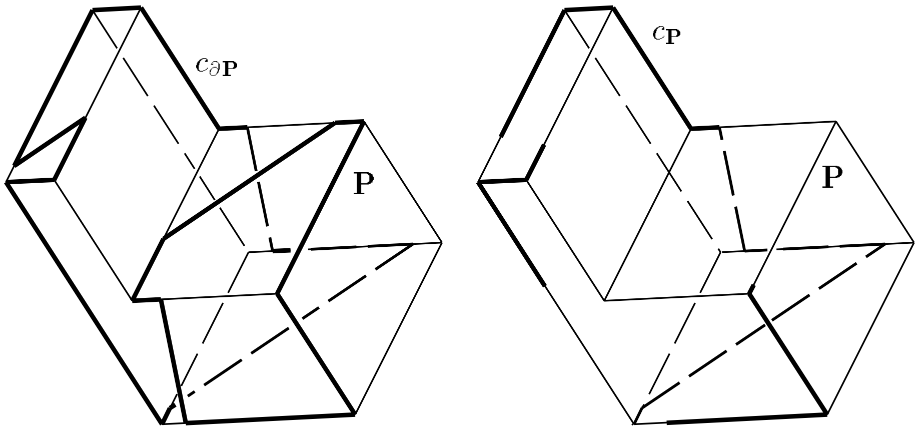

Figure 5: Characteristics $c_{\partial\mathbf{P}}$ and $c_{\mathbf{P}}$

line t_X and e shares locally just points of e with the wedge enclosed by the neighbour faces of e. This means that additional to 10 edge e *must be convex*.

(C1') For a point X in face f, the condition $X \in c_{\mathbf{P}}$ implies that locally the torse Γ_0 enveloped by $\varphi = span(f)$ under β is disjoint to the interior of $\mathbf{P}$. Therefore only the points of one half-line starting at the central point G_φ of c_φ can belong to $c_{\mathbf{P}}$.

Proof: There is an osculating (double) cone of revolution of Γ_0 at c_φ. Only at points $X \in c_\varphi$ on one half-line ending at the apex G_φ of this cone contains locally no interior points of $\mathbf{P}$.

In the following we deduce an analytic condition which is equivalent to (C1'): The motion β of $\mathbf{P}$ can be combined with such a self-motion of φ that the product ("absolute motion") is a pure rolling of φ on Γ_0. Hence at points $X \in c_\varphi$ the absolute velocity $\mathfrak{v}_\mathfrak{x}^a$ must vanish. This implies that the relative velocity $\mathfrak{v}_\mathfrak{x}^r$ of X reduces the vehicular velocity $\mathfrak{v}_\mathfrak{x}^f = \mathfrak{v}_\mathfrak{x}$ to zero, since $\mathfrak{v}_\mathfrak{x}^a = \mathfrak{v}_\mathfrak{x}^f + \mathfrak{v}_\mathfrak{x}^r$.

When φ is rolling on Γ_0, then the absolute acceleration vector $\mathfrak{a}_\mathfrak{x}^a$ at $X \in c_\varphi$ is just opposite to the vector pointing from X to the corresponding finite principal curvature centre of Γ_0. Condition (C1') is therefore equivalent to $\mathfrak{n}\cdot\mathfrak{a}_\mathfrak{x}^a < 0$ as $\mathfrak{n}$ points outside. The well known formula

$$\mathfrak{a}_\mathfrak{x}^a = \mathfrak{a}_\mathfrak{x}^f + \mathfrak{a}_\mathfrak{x}^r + \mathfrak{a}_\mathfrak{x}^c$$

expresses $\mathfrak{a}_\mathfrak{x}^a$ as the sum of the vehicular, the relative and the Coriolis acceleration. From

$$\mathfrak{n}\cdot\mathfrak{a}_\mathfrak{x}^r = 0\,, \quad \mathfrak{a}_\mathfrak{x}^c = 2(\mathfrak{q}\times\mathfrak{v}_\mathfrak{x}^r) = -2(\mathfrak{q}\times\mathfrak{v}_\mathfrak{x}^f) \text{ and } \mathfrak{a}_\mathfrak{x}^f = \mathfrak{a}_\mathfrak{x}\,,$$

we deduce

$$(\mathfrak{n}\cdot\mathfrak{a}_\mathfrak{x}) - 2\det(\mathfrak{n},\mathfrak{q},\mathfrak{v}_\mathfrak{x}) < 0 \tag{11}$$

as analytic equivalent to (C1') (compare (9)).

However, the conditions (C1') and (C2') are *not sufficient*, as points of $c_{\partial\mathbf{P}}$ obeying (C1') or (C2') might be located in the interior of previous or following positions of $\mathbf{P}$. The curve $c_{\mathbf{P}}$ is a subset of $c_{\partial\mathbf{P}}$ (see Fig. 5) and need not consist of closed curves on $\partial\mathbf{P}$.

3.2. The initial position

From now on β is supposed to be a helical motion with *fixed* axis and parameter. Otherwise no general statements could be made. Which boundary points of the initial position $\mathbf{P}^a := \beta(a)\mathbf{P}$

(see Fig. 1 or Fig. 2) are boundary points of the swept solid $\mathbf{S}_0$ too?

When all oriented helical curves of β are seen as "rays of light", then each point $X \in (\partial\mathbf{P}^i \cap \partial\mathbf{S}_0)$ must be lighted. Again, this condition is not sufficient, not even when β is restricted to a very small portion. This is a consequence of the following unexpected phenomenon which appears at faces f that contain the central point G_φ of $\varphi = span(f)$ in the interior.

Each point X located at the open half-line c_φ obeying (11) traces a helix under β which at the beginning takes its course in the outside half-space of φ. It touches φ at X. But it turns out that there is an additional point $\overline{X}$ of intersection with φ (see Fig. 6). The full characteristic line c_φ of φ traces a helical torse Γ_0 under β. This torse intersects the tangent plane φ in the generator c_φ and in an additional curve $\overline{c}_\varphi$. This second branch $\overline{c}_\varphi$ of $\Gamma_0 \cap \varphi$ touches c_φ at the central point G_φ, since all points of the cuspidal curve g are uniplanar singular points of Γ_0. Due to a theorem of PIRONDINI (1889) (cf. [2]), the ratio of the curvature radii of $\overline{c}_\varphi$ and of the cuspidal helix g (= path of G_φ under β) reads 4 : 3.

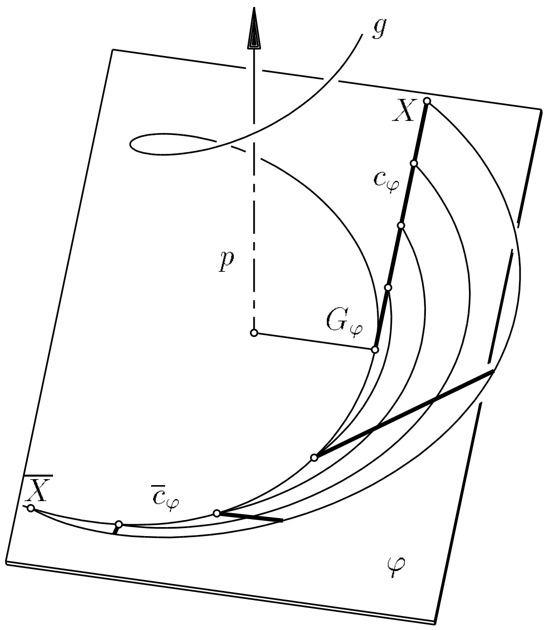

Figure 6: Shade and shadow lines in φ

Using the particular "illumination" mentioned at the beginning of this section we may say: Apart from the shade on one side of the characteristic line c_φ, the face f produces a shadow onto itself. This shadow is bounded by c_φ and by $\overline{c}_\varphi$ (see Fig. 6).

The nearer $X \in c_\varphi$ is to the central point G_φ, the nearer is X to the piercing point $\overline{X}$. This reveals that condition (C1') is not even locally sufficient: For each position of $\mathbf{P}$ different from the initial position $\mathbf{P}^a$ the half-line of c_φ matching (11) contains interior points of $\mathbf{P}^a$. Therefore the actual half-line $c_\varphi \cap c_\mathbf{P}$ can at the earliest start at the piercing point $\overline{X}$ between c_φ and the initial position φ^a of φ (compare Fig. 2). This is the reason *why actually only small portions of torses appear at the boundary* $\partial\mathbf{S}_0$ (see Fig. 2), less than one would expect. The first point of $c_{\partial\mathbf{P}}$ at the half-line c_φ is the infimum of points $X \in c_\varphi$ for which the helix s_X does not meet any other position of $\mathbf{P}$.

4. The global method

The given helical motion $\beta(t), a \leq t \leq b$ with $\omega\widehat{\omega} \neq 0$ can be divided into at most three phases, the *initial*, the *intermediate* and the *final phases*. For the definitions of these phases we embed β into the complete one-parameter group $\overline{\beta}(t), -\infty < t < \infty$. Then there is a parameter $t_1 > a$ such that, for all following positions $\overline{\beta}(t)\mathbf{P}$, $t > t_1$, the characteristic curves $c_{\partial\mathbf{P}}$ are disjoint from all positions of $\mathbf{P}$ for $t < a$. This means that at t_1 ends the influence of the fact that no position previous to the initial $\mathbf{P}^a$ is reached under β. Analogously, there is a $t_2 < b$ such that all previous positions $\overline{\beta}(t)c_{\partial\mathbf{P}}$, $t < t_2$, of the surface characteristic $c_{\partial\mathbf{P}}$ have an empty intersection with all positions of $\mathbf{P}$ for $t > b$.

4.1. The intermediate phase

If $t_1 < t_2$, then there is an intermediate phase of β for $t_1 \leq t \leq t_2$.[3] For this phase $c_\mathbf{P}$ is constant with respect to $\mathbf{P}$ and we can formulate the following algorithm:

– Determine $c_{\partial\mathbf{P}}$ according to the rules (C1) and (C2a,b). Under helical projection into the transverse plane π (Fig. 3) the closed polygons of $c_{\partial\mathbf{P}}$ are projected onto closed curves of $c^s_{\partial\mathbf{P}}$.[4]

– These curves define an area, the helical shadow $\mathbf{P}^s$ of $\mathbf{P}$. Now in an algorithmic way the parts of $c^s_{\partial\mathbf{P}}$ in the interior of $\mathbf{P}^s$ must be eliminated. When $\mathbf{P}$ is not simply connected, then this shadow need not be simply connected. There might be "holes" in it. In order to figure out the boundaries of holes it is necessary to compute for each component of $c^s_{\partial\mathbf{P}}$ whether the interior of $\mathbf{P}^s$ is on the left or right side. Particular attention has to be paid to cusps as the interior side changes.

– The remaining boundary of $\mathbf{P}^s$ is exactly the helical view $c^s_\mathbf{P}$ of the required $c_\mathbf{P}$. The boundary $\partial\mathbf{S}_0$ for the intermediate part is swept by $c_\mathbf{P}$ under $\beta(t)$ for $t_1 \leq t \leq t_2$. Note that different components might have endpoints whose paths cover the same helix.

The line segments of $c_\mathbf{P}$ trace portions of ruled helical surfaces Ψ_0. These are either torses or skew ruled surfaces. For the case of torses we learned from section 3.2 (see Fig. 6) that in any face f the included segment of $c_\mathbf{P}$ will never start at the central point but at the first point X whose helix s_X hits $\mathbf{P}$ only at boundary points. In order to obtain for a skew ruled surface Ψ_0 a fair polyhedral approximation which doesn't look too crumpled some additional investigations in the sense of [5] are necessary. The interested reader is referred to the first-named author's thesis.

Finally, it should be pointed out that also in the initial and final phases there can be parameter intervals for which the characteristic curve $c_\mathbf{P}$ remains fixed with respect to the moving polyhedron $\mathbf{P}$. A sufficient condition is that all corresponding positions of $c_{\partial\mathbf{P}}$ are disjoint from the initial and the final positions.

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