On Martin Disteli's Main Achievements in Spatial Gearing: Disteli's Diagram

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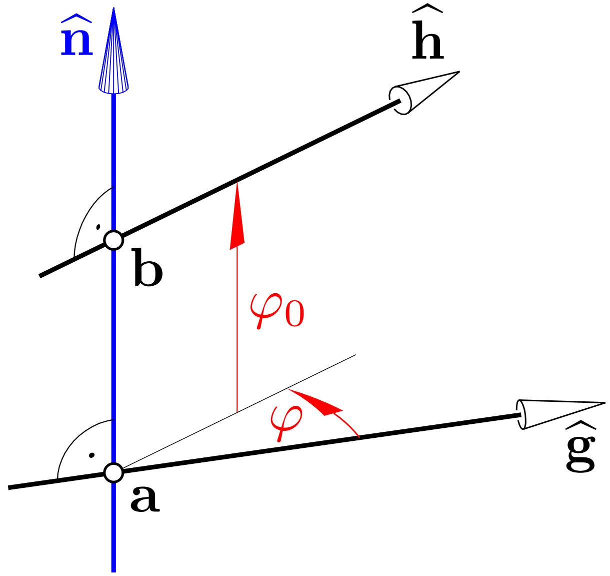
1. The relative axis for helical gears

We treat the relation between the angular velocities of a pair of skew gears and the position and pitch of their relative screw motion.

We use consistently dual vectors representing directed lines and screws.

We identify directed lines g with their dual unit vectors $\widehat{\mathbf{g}}$ ($\widehat{\mathbf{g}}\cdot\widehat{\mathbf{g}} = 1$) and note for the dual angle $\widehat{\varphi} = \varphi + \varepsilon\varphi_0$ between any two directed lines $\widehat{\mathbf{g}}$ and $\widehat{\mathbf{h}}$:



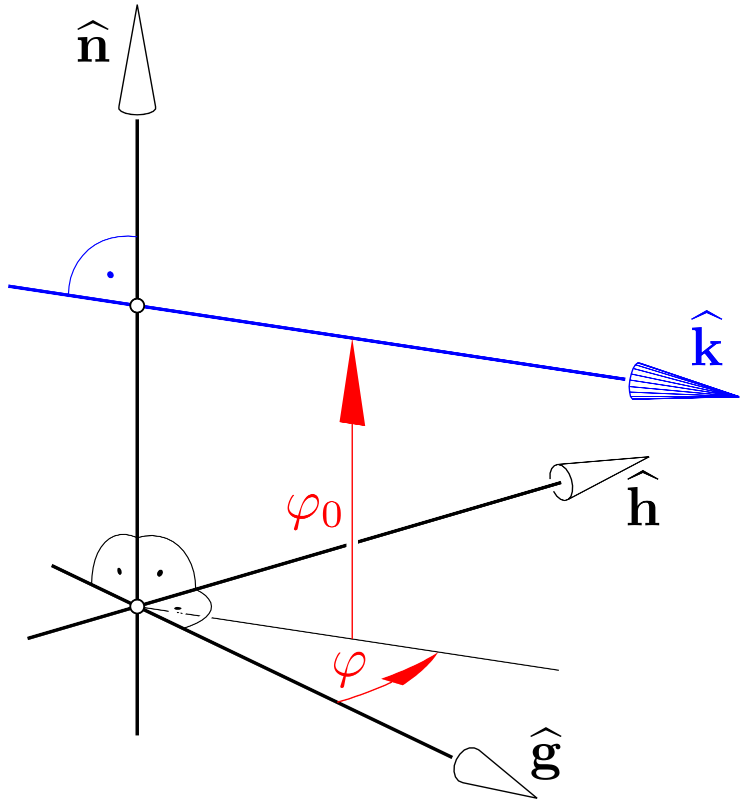
$$\begin{aligned}
\widehat{\mathbf{g}}\cdot\widehat{\mathbf{h}} &= \mathbf{g}.\mathbf{h} + \varepsilon(\mathbf{g}_0.\mathbf{h} + \mathbf{g}.\mathbf{h}_0) = \cos\varphi - \varepsilon\varphi_0\sin\varphi = \cos\widehat{\varphi} \\
\widehat{\mathbf{g}}\times\widehat{\mathbf{h}} &= \mathbf{g}\times\mathbf{h} + \varepsilon(\mathbf{g}_0\times\mathbf{h} + \mathbf{g}\times\mathbf{h}_0) = (\sin\varphi + \varepsilon\varphi_0\cos\varphi)(\mathbf{n} + \varepsilon\mathbf{n}_0) = \sin\widehat{\varphi}\,\widehat{\mathbf{n}}.
\end{aligned}$$

Example for ‘dual continuation’

Given: Two orthogonally intersecting spears $\widehat{\mathbf{g}}$ and $\widehat{\mathbf{h}}$, i.e., $\widehat{\mathbf{g}} \cdot \widehat{\mathbf{h}} = 0$, $\Longrightarrow$

$$\widehat{\mathbf{k}} := \cos\,\widehat{\varphi}\,\widehat{\mathbf{g}} + \sin\,\widehat{\varphi}\,\widehat{\mathbf{h}}$$

is the image of $\widehat{\mathbf{g}}$ under the helical motion along $\widehat{\mathbf{n}} = \widehat{\mathbf{g}} \times \widehat{\mathbf{h}}$ with angle φ and translatory length φ_0 .



Proof: $\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}} = 1$, $\widehat{\mathbf{k}} \cdot \widehat{\mathbf{n}} = \det(\cos\,\widehat{\varphi}\,\widehat{\mathbf{g}} + \sin\,\widehat{\varphi}\,\widehat{\mathbf{h}},\ \widehat{\mathbf{g}},\ \widehat{\mathbf{h}}) = 0$, $\widehat{\mathbf{k}} \cdot \widehat{\mathbf{g}} = \sin\,\widehat{\varphi}$. □

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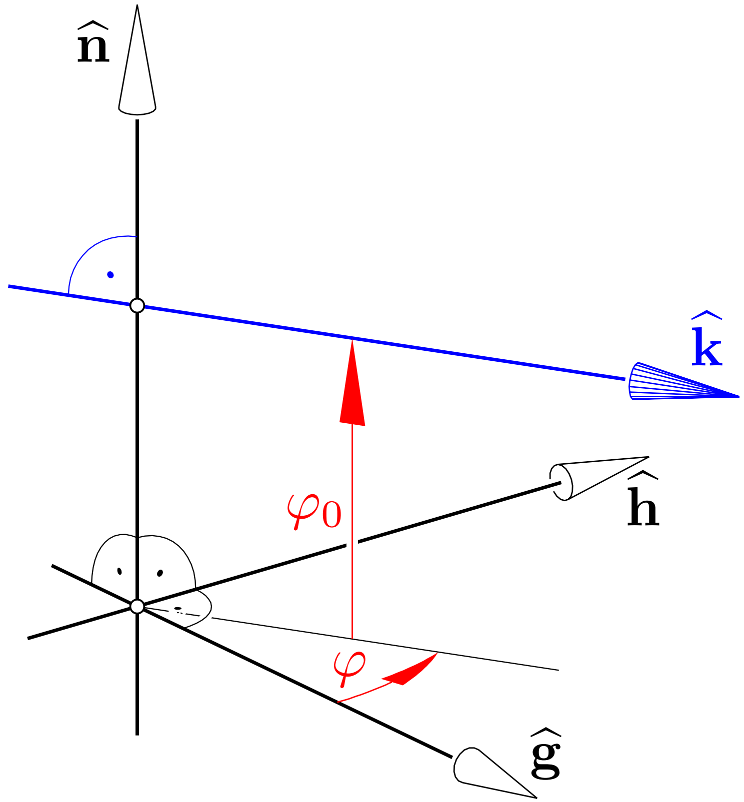


Proof: $\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}} = 1, \quad \widehat{\mathbf{k}} \cdot \widehat{\mathbf{n}} = \det(\cos\,\widehat{\varphi}\,\widehat{\mathbf{g}} + \sin\,\widehat{\varphi}\,\widehat{\mathbf{h}},\ \widehat{\mathbf{g}},\ \widehat{\mathbf{h}}) = 0, \quad \widehat{\mathbf{k}} \cdot \widehat{\mathbf{g}} = \sin\,\widehat{\varphi}.$

$\square$

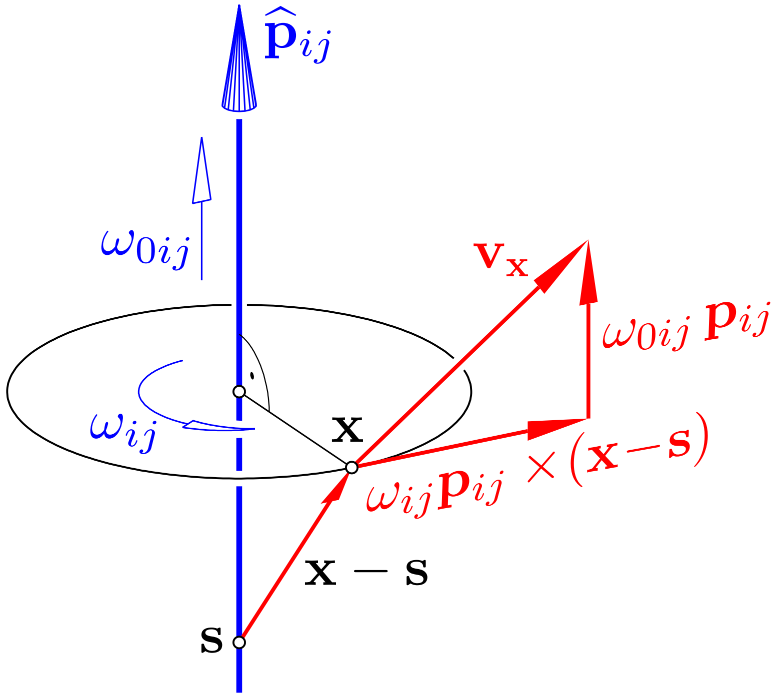
Example for ‘dual continuation’

In the same way the screw $\widehat{\mathbf{q}}_{ij}$ stands for the instant screw of the relative motion between the two frames Σ_i and Σ_j.

This screw is represented by the dual vector

$$\widehat{\mathbf{q}}_{ij} = \widehat{\omega}_{ij}\, \widehat{\mathbf{p}}_{ij} = (\omega_{ij} + \varepsilon\omega_{0ij})(\mathbf{p}_{ij} + \varepsilon\mathbf{p}_{0ij}).$$

- $\widehat{\mathbf{p}}_{ij}$ is the screw axis,
- $\widehat{\omega}_{ij} = \omega_{ij} + \varepsilon\omega_{0ij}$ is the dual *amplitude* of the twist, with *signed magnitudes* of the angular velocity ω_{ij} and the point velocity ω_{0ij} along the screw axis.
- $h_{ij} := \omega_{0ij}/\omega_{ij}$ is the pitch of this screw.



3D Version of the Three-Pole-Theorem

Spatial Three-Pole-Theorem: (Aronhold, Kennedy, ...)
Given three systems $\Sigma_1, \Sigma_2, \Sigma_3$, *let* $\widehat{\mathbf{q}}_{21}$, $\widehat{\mathbf{q}}_{31}$ *be the instantaneous screws of* Σ_2/Σ_1, Σ_3/Σ_1, *resp., then*

$$\widehat{\mathbf{q}}_{32} = \widehat{\mathbf{q}}_{31} - \widehat{\mathbf{q}}_{21}$$

is the instantaneous screw of the relative motion Σ_2/Σ_1 .

Expressed in terms of the screw axes and the velocities we obtain

$$\widehat{\omega}_{32}\,\widehat{\mathbf{p}}_{32} = \widehat{\omega}_{31}\,\widehat{\mathbf{p}}_{31} - \widehat{\omega}_{21}\,\widehat{\mathbf{p}}_{21}\,.$$

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$$\widehat{\omega}_{32}\,\widehat{\mathbf{p}}_{32} = \widehat{\omega}_{31}\,\widehat{\mathbf{p}}_{31} - \widehat{\omega}_{21}\,\widehat{\mathbf{p}}_{21}\,.$$

Relations in the dual triangle

There are many formulas expressing relations between the angles α_i and distances α_{0i} of the relative axis $\widehat{\mathbf{p}}_{32}$ and the axes $\widehat{\mathbf{p}}_{21}$ and $\widehat{\mathbf{p}}_{31}$,

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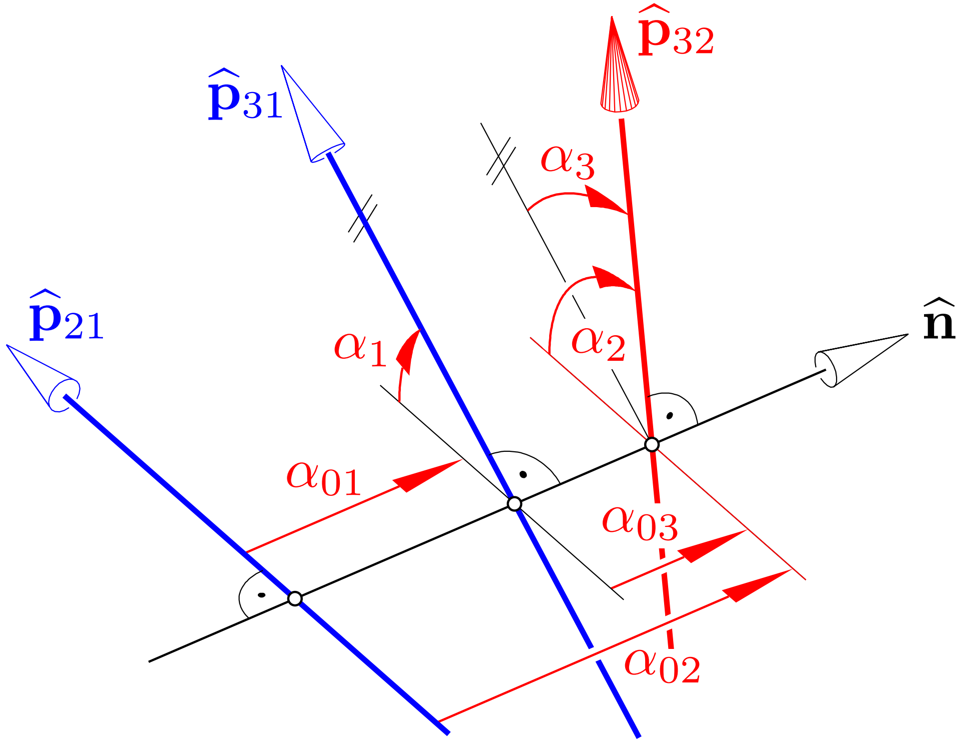
even in the rotational case $\widehat{\omega}_{21}, \widehat{\omega}_{31} \in \mathbb{R}$, i.e.,

$$\omega_{021} = \omega_{031} = 0\,.$$



E.g., formulas presented in J. Phillips: *General Spatial Involute Gearing*. Springer Verlag, New York 2003, 498 p

$$(1)\quad \theta = \Sigma/2 - \arctan\,[(k\sin\Sigma)/(1 + k\cos\Sigma)]$$
$$(2)\quad r_2 = C/2\,[1 - \operatorname{cosec}\Sigma\,\sin 2\theta]$$
$$(3)\quad r_3 = C - r_2$$

Phillips [1]

$$(4)\quad r_2 = C\,[(k^2 + k\cos\Sigma)/(1 + 2k\cos\Sigma + k^2)]$$
$$(5)\quad r_3 = C\,[(1 + k\cos\Sigma)/(1 + 2k\cos\Sigma + k^2)]$$

Konstantinov [9]

$$(6)\quad \Sigma = \psi_2 + \psi_3$$
$$(7)\quad k = (\sin\psi_2)/(\sin\psi_3)$$
$$(8)\quad r_2/r_3 = (\tan\psi_2)/(\tan\psi_3)$$

Steeds [4]

$$(9) \quad p = (C/2) \cot \Sigma - (C/2) \operatorname{cosec} \Sigma \cos 2\theta$$
$$(1) \quad \theta = \Sigma/2 - \arctan [(k \sin \Sigma)/(1 + k \cos \Sigma)]$$

Phillips [1]

$$(4) \quad r_2 = C[(k^2 + k \cos \Sigma)/(1 + 2k \cos \Sigma + k^2)]$$
$$(5) \quad r_3 = C[(1 + k \cos \Sigma)/(1 + 2k \cos \Sigma + k^2)]$$
$$(10) \quad \psi_2 = \arctan [(k \sin \Sigma)/(1 + k \cos \Sigma)]$$
$$(11) \quad \psi_3 = \arctan [(\sin \Sigma)/(k + \cos \Sigma)]$$
$$(12) \quad p = - C[(k \sin \Sigma)/(1 + 2k \cos \Sigma + k^2)]$$

$$(13) \quad r_2 = C[k^2/(1 + k^2)]$$
$$(14) \quad r_3 = C[1/(1 + k^2)]$$
$$(15) \quad \psi_2 = \arctan k$$
$$(16) \quad \psi_3 = \arctan (1/k)$$
$$(17) \quad p = - C[k/(1 + k^2)]$$

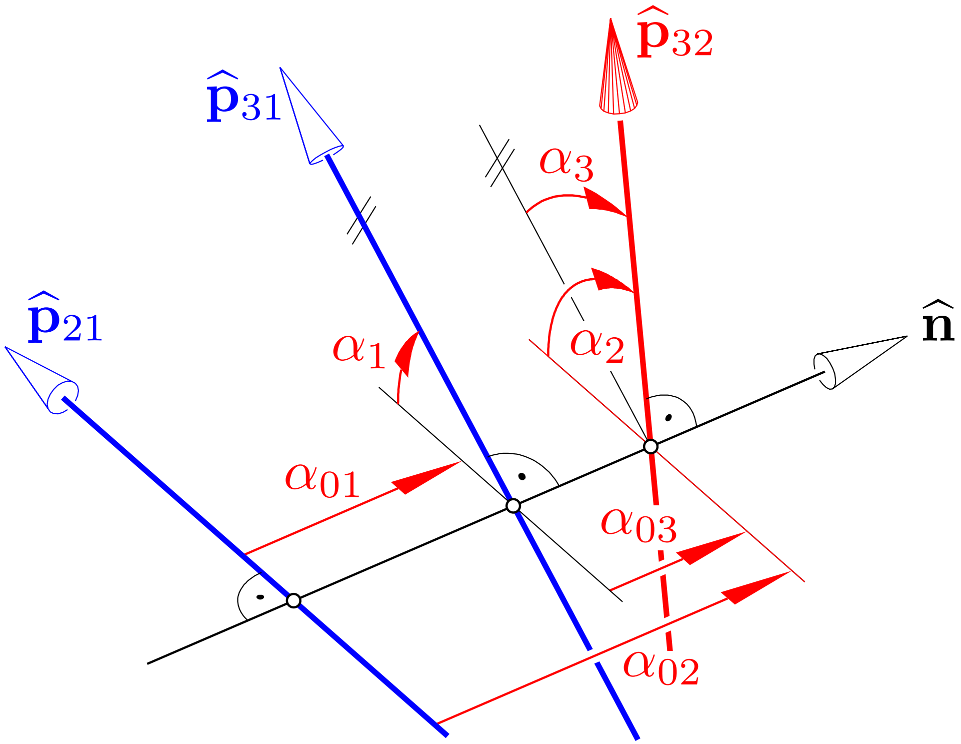
Relations in the dual triangle

dual Cosine-Theorem:

$$\widehat{\omega}_{32}^2 = \widehat{\omega}_{31}^2 + \widehat{\omega}_{21}^2 - 2\,\widehat{\omega}_{21}\,\widehat{\omega}_{31}\,\cos\,\widehat{\alpha}_1\,.$$

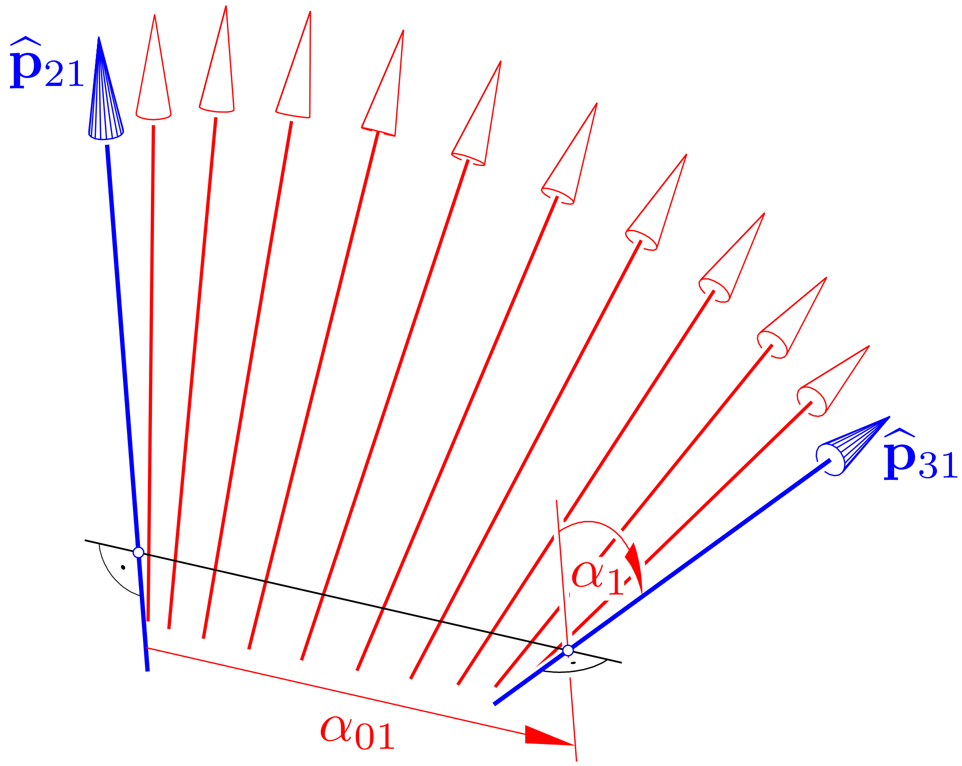
dual Sine-Theorem:

$$\frac{\sin\,\widehat{\alpha}_2}{\widehat{\omega}_{31}} = \frac{\sin\,\widehat{\alpha}_3}{\widehat{\omega}_{21}} = \frac{\sin\,\widehat{\alpha}_1}{\widehat{\omega}_{32}}\,.$$



Rotational case with variable $\omega_{31} : \omega_{21}$

What happens with the relative axis $\widehat{\mathbf{p}}_{31}$ when in the rotational case the transmission ratio $i := \omega_{31} : \omega_{21}$ varies?



Special case

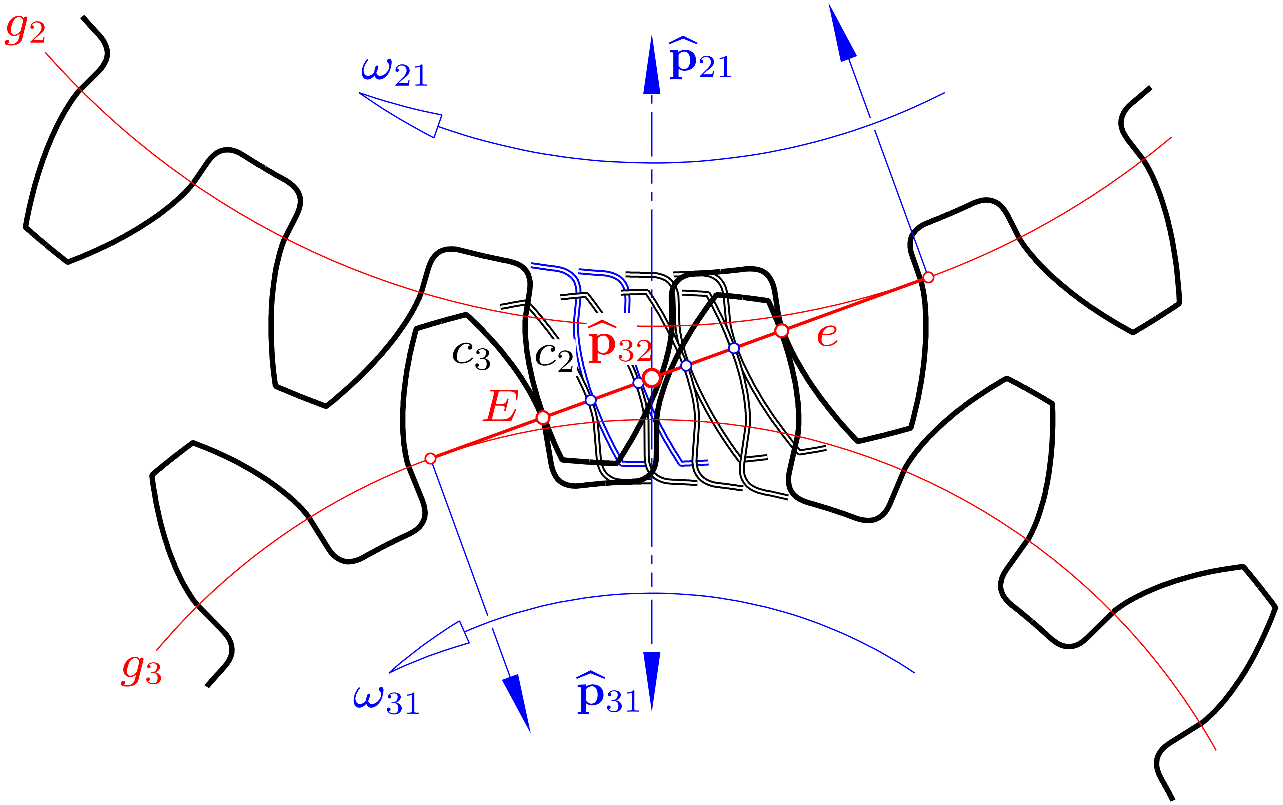
planar gearing $\widehat{\mathbf{p}}_{21} \parallel \widehat{\mathbf{p}}_{31}$:

The axes are seen in a point view.

$\widehat{\mathbf{p}}_{32}$ divides $\widehat{\mathbf{p}}_{21}\widehat{\mathbf{p}}_{31}$ in the ratio $\omega_{31} : \omega_{21}$.

For variable $\omega_{31} : \omega_{21}$ the "pitch point" $\widehat{\mathbf{p}}_{32}$ sweeps out the whole line.

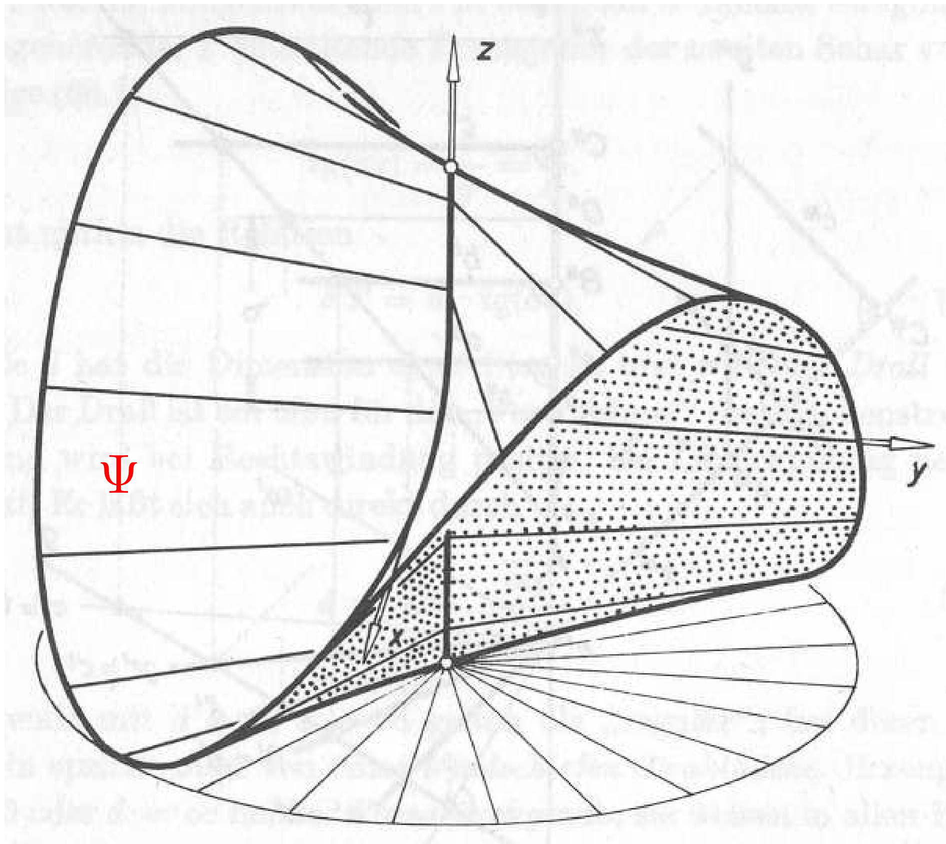
Analogously, in the spherical case for bevel gears the relative axis $\widehat{\mathbf{p}}_{32}$ performs a full rotation.



Rotational case with variable $\omega_{31} : \omega_{21}$

Given: Rotations Σ_2/Σ_1 and Σ_3/Σ_1 with angular velocities ω_{21}, ω_{31} about skew $\widehat{\mathbf{p}}_{21}$ and $\widehat{\mathbf{p}}_{31}$.

If $\omega_{21}, \omega_{31} \in \mathbb{R}$ vary, the axis $\widehat{\mathbf{p}}_{32}$ of the relative motion Σ_3/Σ_2 constitutes a cylindroid = Plücker's conoid Ψ.



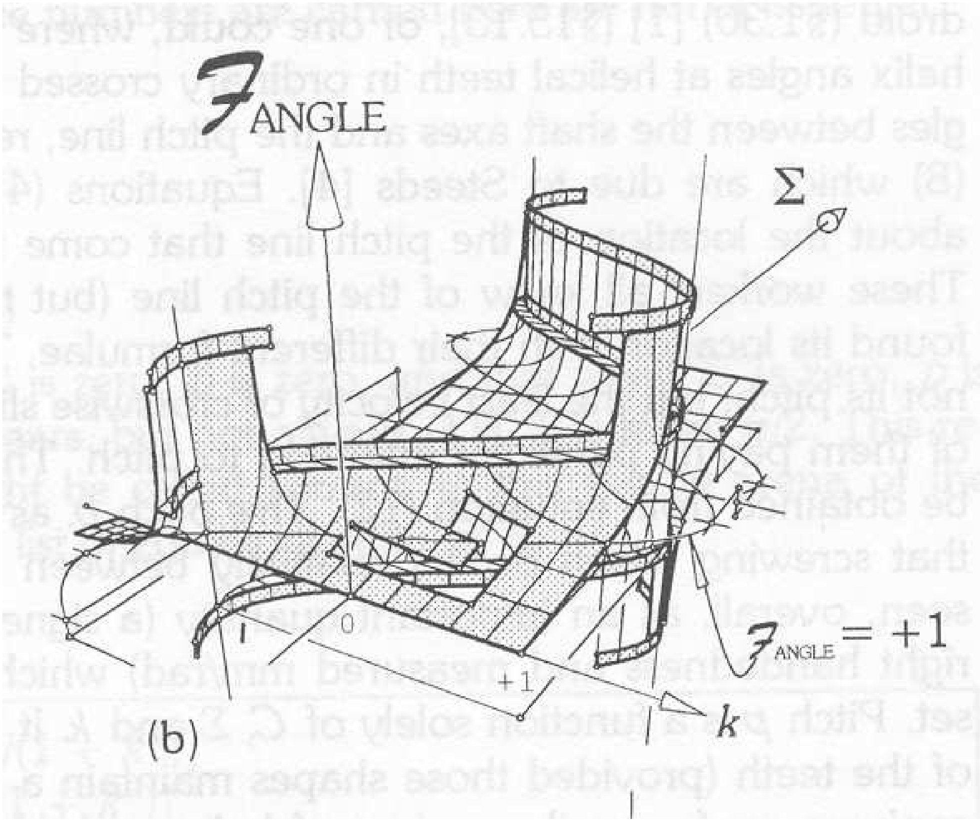
Cylindroid

Rotational case with variable $\omega_{31} : \omega_{21}$

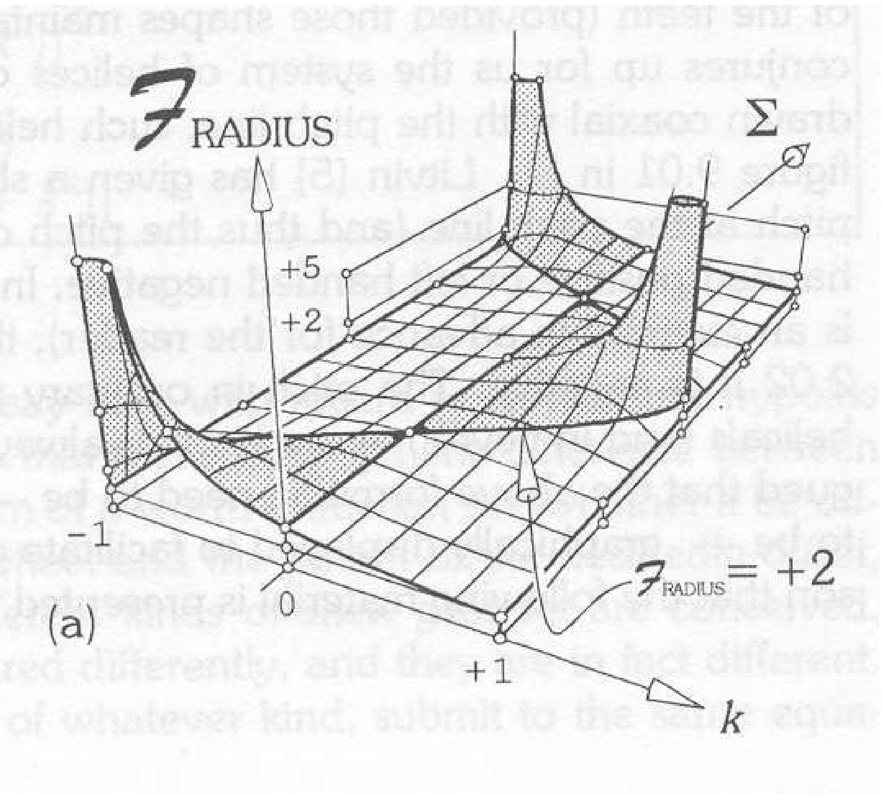
How to visualize the relation between the mutual position of the axes $\widehat{\mathbf{p}}_{21}$ and $\widehat{\mathbf{p}}_{31}$, the angular velocities $\omega_{21} : \omega_{31}$ of the pair of skew gears

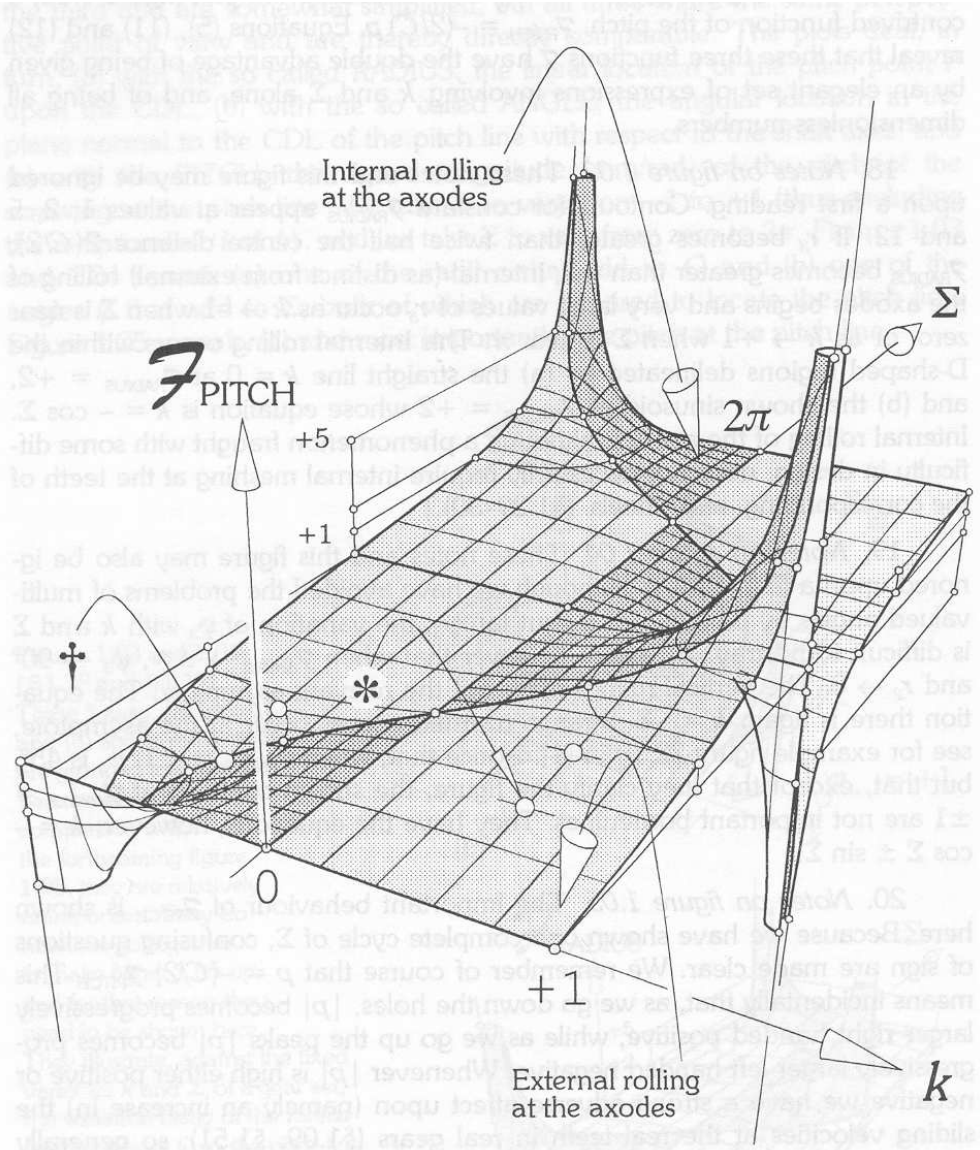
and the axis $\widehat{\mathbf{p}}_{32}$ and the pitch $h_{32} = \omega_{032}/\omega_{32}$ of the relative screw motion between the two gears?

Diagrams presented in J. PHILLIPS: *General Spatial Involute Gearing.* Springer Verlag, New York 2003



$\Sigma = \alpha_1$, $k = \omega_{21} : \omega_{31}$, angle $= \alpha_2$





$\Sigma = \alpha_1$, $k = \omega_{21} : \omega_{31}$, radius $= \alpha_{02}$

pitch $= h_{32}$ $\longrightarrow$

2. DISTELI's diagram for pure rotations

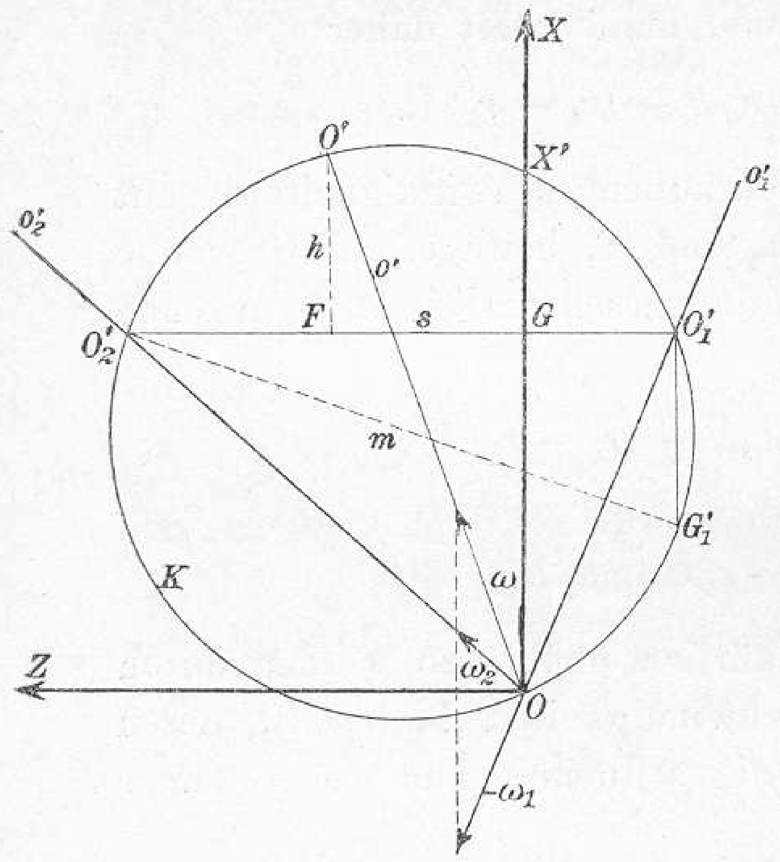
In

M. DISTELI: *Über die Verzahnung der Hyperboloidräder mit geradlinigem Eingriff.*
Z. Math. Phys. **59**, 244–298 (1911)

this problem is solved with one single diagram, based on a circle.

This diagram seems to have been overlooked by the community of kinematicians.

Fig. 2.



On Martin Disteli

DISTELI's papers are hard to read because

1. their lack of vector notation, which leads to lengthy expressions from where little information can be drawn, and

2. the use of rather uncommon left-hand frames. Although Disteli used screw theory, he described screws only explicitly, by listing their six coordinates (p, q, r, u, v, w).

3. Moreover, the up-to-six different frames occurring in the paper are not identified by subscripts, but rather by different characters.

Martin DISTELI, (1862–1923)
Dresden, Karlsruhe

Schriftenverzeichnis.

1. Die Steinerschen Schließungsprobleme nach darstellend-geometrischer Methode. Inaug.-Diss. Zürich 1888.
2. Zur Konfiguration der Wendepunkte der allgemeinen ebenen Kurve dritter Ordnung. Vierteljahrsschrift der Züricher Naturf.-Ges. XXXV, 1890.
3. Die Metrik der zirkularen ebenen Kurven dritter Ordnung im Zusammenhange mit geometrischen Lehrsätzen Jakob Steiners. Ebenda XXXVII, 1891.
4. Über Stellen innigster Berührung einer ebenen dritter Ordnung mit einer ebenen Kurve n-ter Ordnung. Zeitschr. f. Math. u. Phys. 38, 1893.
5. Über Rollkurven und Rollflächen. Ebenda 43, 1898.
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7. Über instantane Schraubengeschwindigkeiten und die Verzahnung der Hyperboloidräder. Ebenda 51, 1904.
8. Über einige Sätze der kinematischen Geometrie, welche der Verzahnungslehre zylindrischer und konischer Räder zugrunde liegen. Ebenda 56, 1908.
9. Über die Verzahnung der Hyperboloidräder mit geradlinigem Eingriff. Ebenda 59, 1911.
10. Über das Analogon der Savaryschen Formel und Konstruktion in der kinematischen Geometrie des Raumes. Ebenda 62, 1913.

Inhalt.

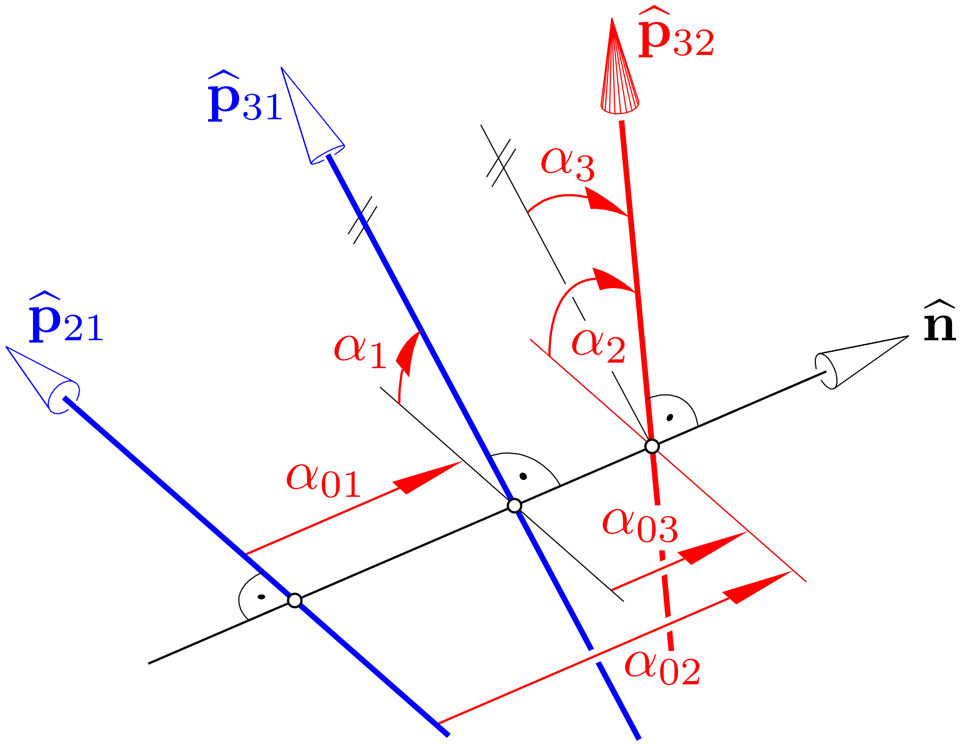
On Martin Disteli

We follow the final comment in his obituary, which reads (F. SCHUR):

"It would be desirable that somebody rewrites DISTELI*'s arguments, thus making them much more understandable".*

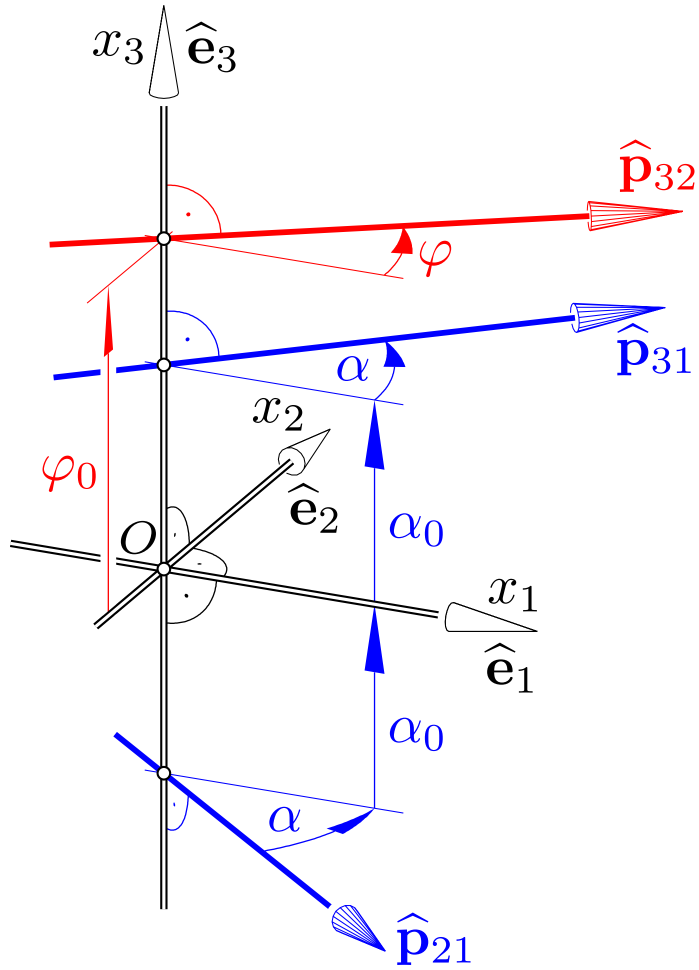
Arbeiten die Axoide erledigt, so gilt es weiter die Verzahnung der zugehörigen Räderpaare durchzuführen. Dies geschieht in den folgenden Abhandlungen im Anschlusse an die Ballsche Schraubentheorie. Wir können in Rücksicht auf den uns zugemessenen Raum schon deshalb nicht auf Einzelheiten eingehen, weil die Darstellung hier vielfach insofern nicht ganz glücklich ist, als bei der Zusammenfassung der schönen und neuen Resultate die darin vorkommenden Gebilde nur durch Buchstaben bezeichnet werden, deren Bedeutung oft auf einer verwickelten Erklärung beruht. Es wäre wünschenswert, wenn diese Teile einer verständlicheren Bearbeitung unterzogen würden. Wir bemerken zum Schlusse nur noch, daß in der letzten Schrift die sogenannte Savarysche Formel der ebenen Bewegungslehre, welche den augenblicklichen Krümmungskreis der Bahnkurve irgendeines Punktes des bewegten Systems zu bestimmen erlaubt, falls die augenblicklichen Krümmungskreise der beiden Polbahnen bekannt sind, auf die hier in Betracht kommenden räumlichen Bewegungen ausgedehnt werden. An die Stelle des Krümmungskreises tritt dann die augenblickliche Striktionsschraubenfläche der Axoide. Gerade in der Behandlung der zuletzt skizzierten Probleme zeigte sich Distelis besondere Stärke.

Disteli's diagram



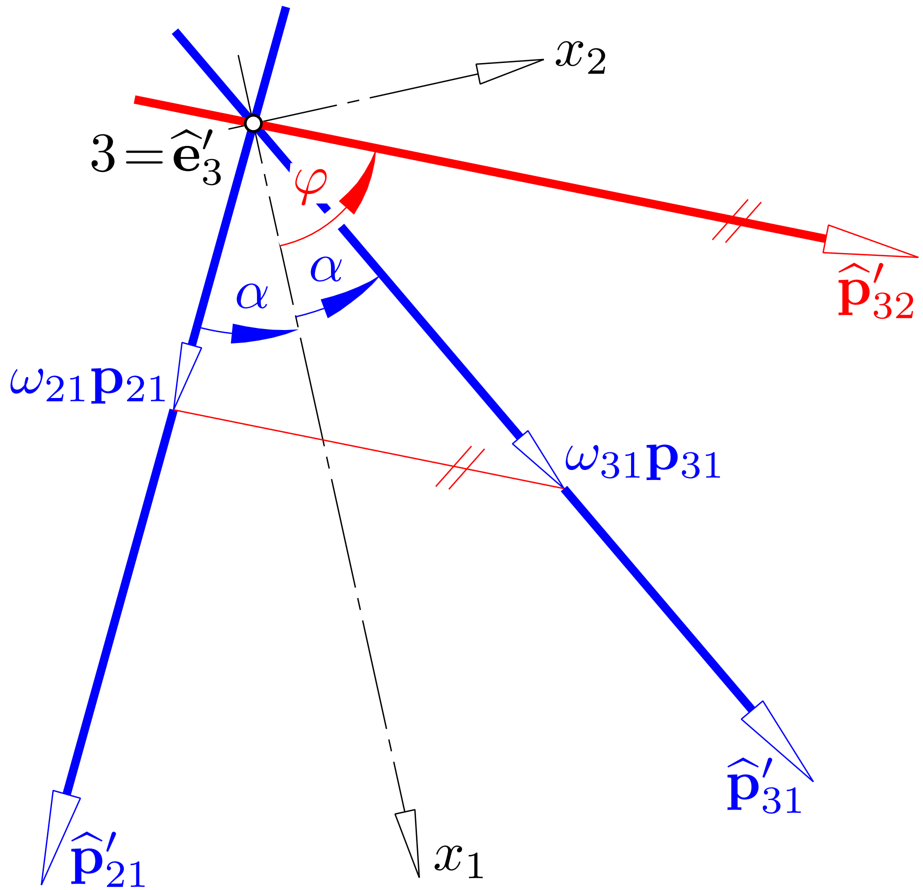
We change the notation to $\longrightarrow$

and look in direction of $\hat{\mathbf{n}} = \hat{\mathbf{e}}_3$:



Disteli's diagram

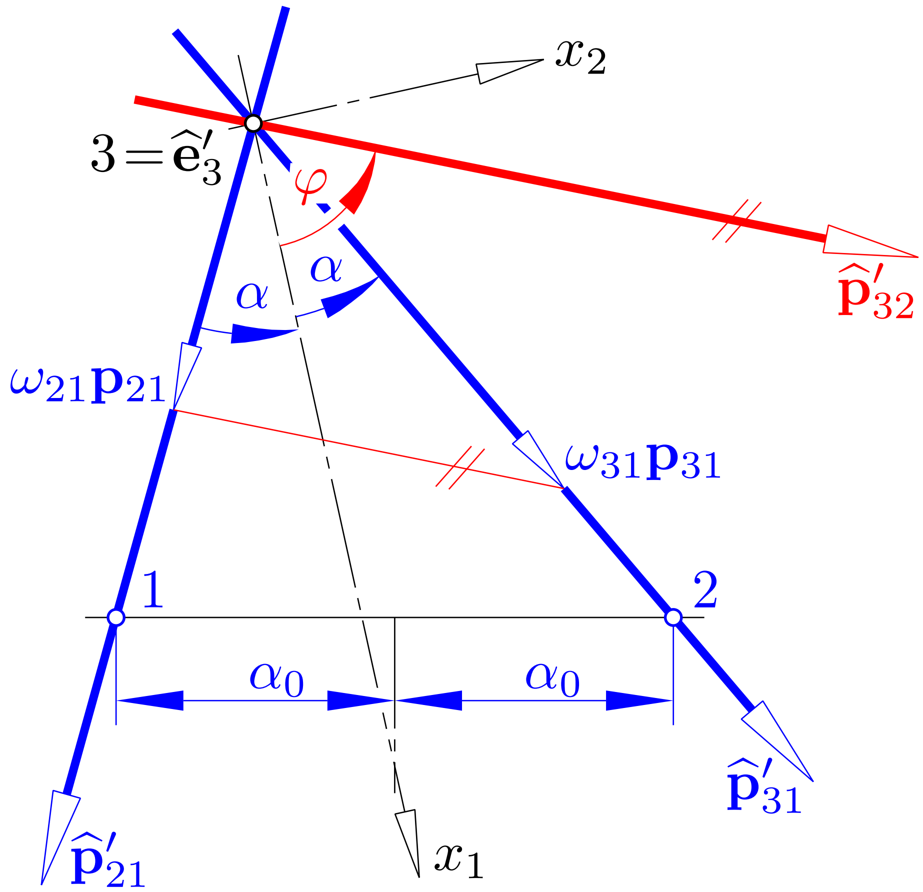
Step 1: The ratio $\omega_{31} : \omega_{21}$ of angular velocities defines the direction of the relative axis $\widehat{\mathbf{p}}_{32}$ and the angle φ:



Disteli's diagram

Step 2: We specify points $1 \in \widehat{\mathbf{p}}_{21}$ and $2 \in \widehat{\mathbf{p}}_{31}$ such that the distance $\overline{12} = 2\alpha_0$.

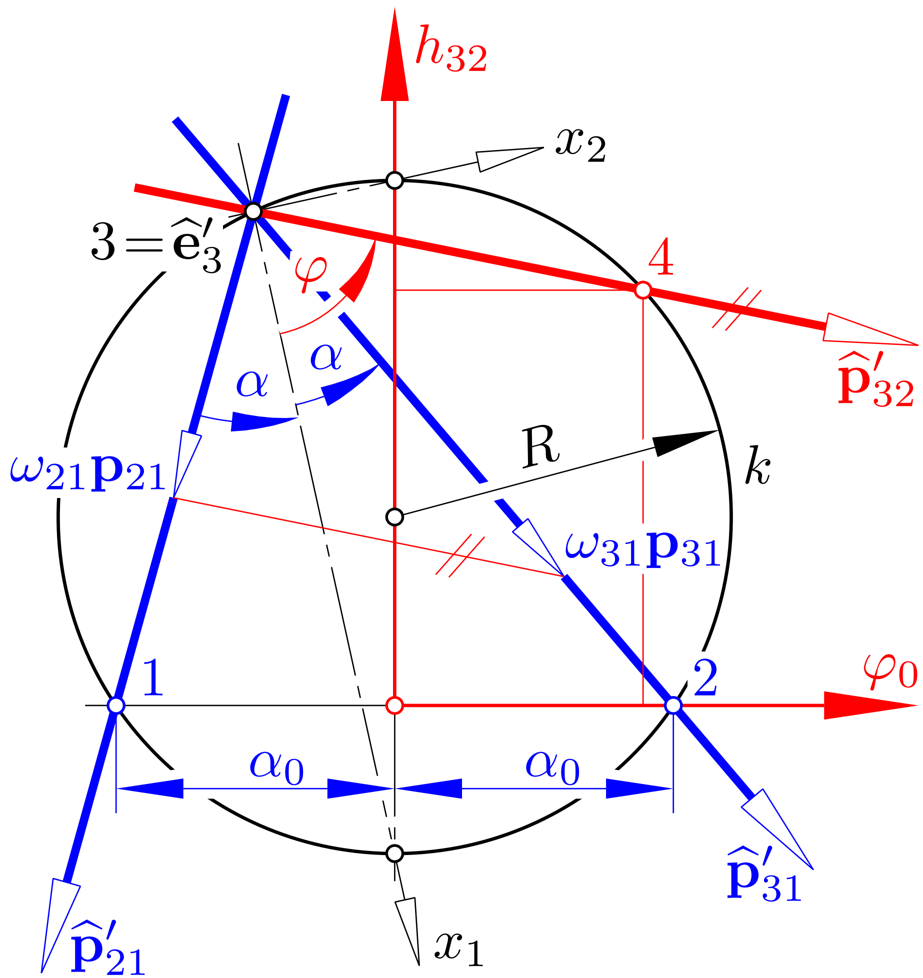
There are ∞ solutions for 1 and 2.



Disteli's diagram

Step 3: We draw the *circumcircle* k of points 123.

Then the point 4 of intersection between $\widehat{\mathbf{p}}_{32}$ and k defines the height φ_0 and the pitch h_{32} of the relative motion.



Proof for DISTELI's diagram

We consequently use dual vectors:

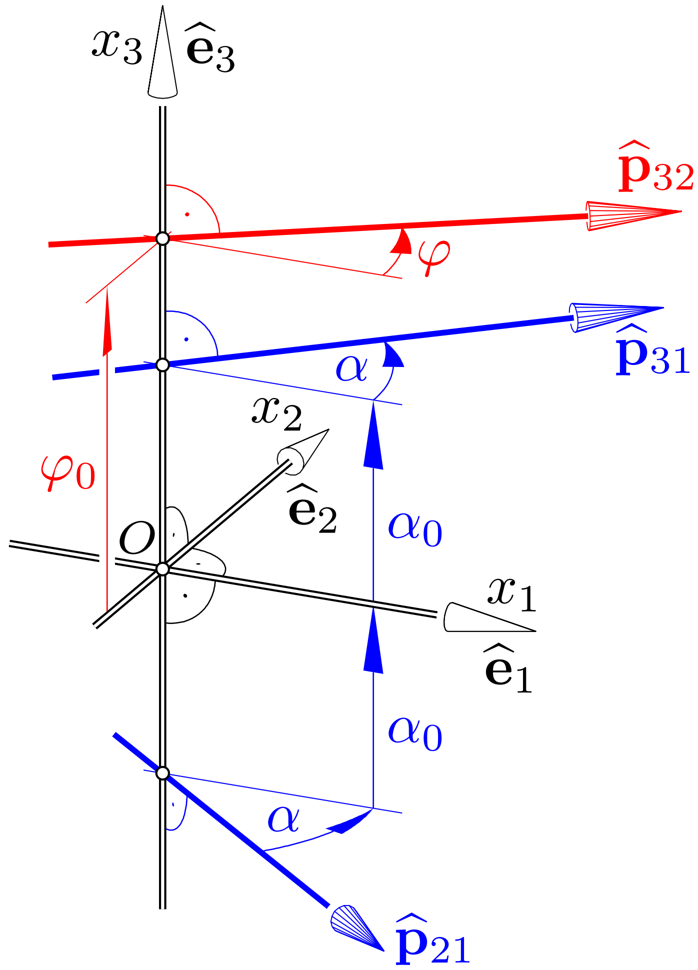
$$\begin{aligned} \widehat{\mathbf{p}}_{21} &= \widehat{\mathbf{e}}_1 \cos\widehat{\alpha} - \widehat{\mathbf{e}}_2 \sin\widehat{\alpha} \\ \widehat{\mathbf{p}}_{31} &= \widehat{\mathbf{e}}_1 \cos\widehat{\alpha} + \widehat{\mathbf{e}}_2 \sin\widehat{\alpha}\,. \end{aligned}$$

The spatial Three-Pol-Theorem implies

$$\widehat{\mathbf{q}}_{32} = \widehat{\omega}_{32}\,\widehat{\mathbf{p}}_{32} = \widehat{\omega}_{32}\,(\widehat{\mathbf{e}}_1 \cos\widehat{\varphi} + \widehat{\mathbf{e}}_2 \sin\widehat{\varphi})$$
$$\omega_{31}\,\widehat{\mathbf{p}}_{31} - \omega_{21}\,\widehat{\mathbf{p}}_{21}.$$

Comparison of coefficients $\Longrightarrow$

$$\begin{aligned} \widehat{\omega}_{32} \cos\widehat{\varphi} &= (\omega_{31} - \omega_{21}) \cos\widehat{\alpha} \\ \widehat{\omega}_{32} \sin\widehat{\varphi} &= (\omega_{31} + \omega_{21}) \sin\widehat{\alpha}\,. \end{aligned}$$



Proof for DISTELI's diagram

We eliminate $\widehat{\omega}_{32}$ and obtain

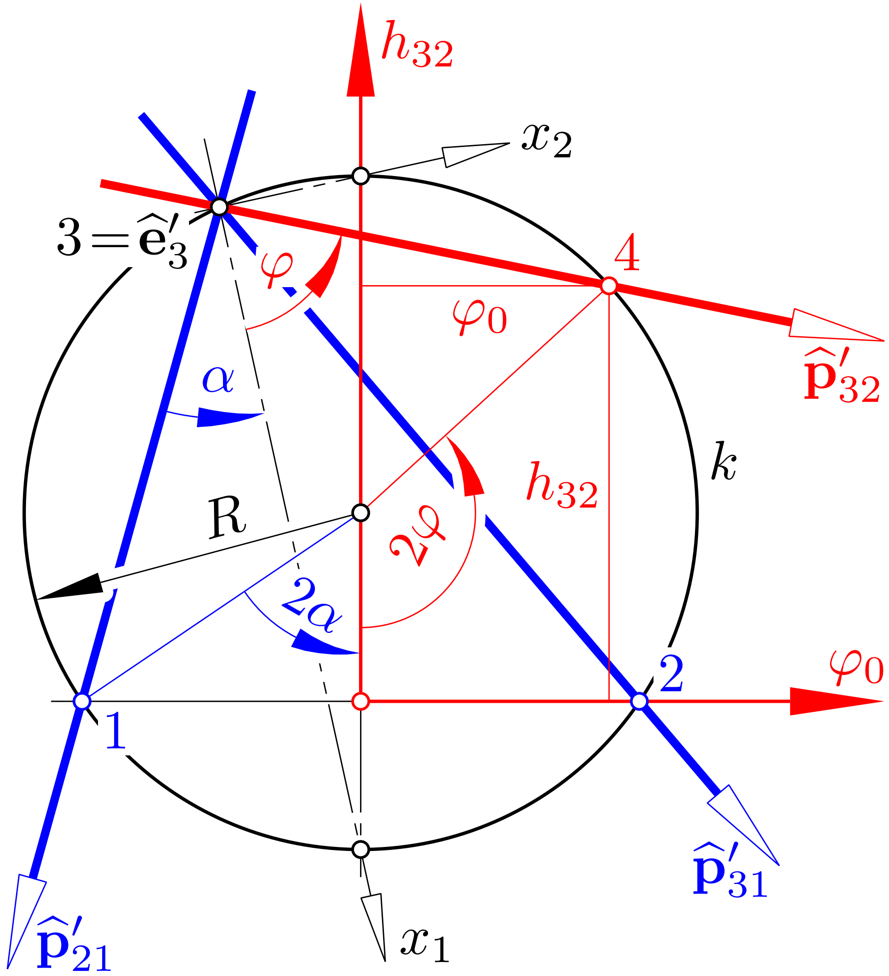
$$\omega_{21} \sin(\widehat{\alpha} + \widehat{\varphi}) + \omega_{31} \sin(\widehat{\alpha} - \widehat{\varphi}) = 0,$$

The real part gives

$$\varphi_0 = \frac{\alpha_0}{\sin 2\alpha} \sin 2\varphi = R \sin 2\varphi$$

with R as radius of k. On the other hand we obtain after some computation

$$h_{32} = \frac{\omega_{032}}{\omega_{32}} = R(\cos 2\alpha - \cos 2\varphi).$$

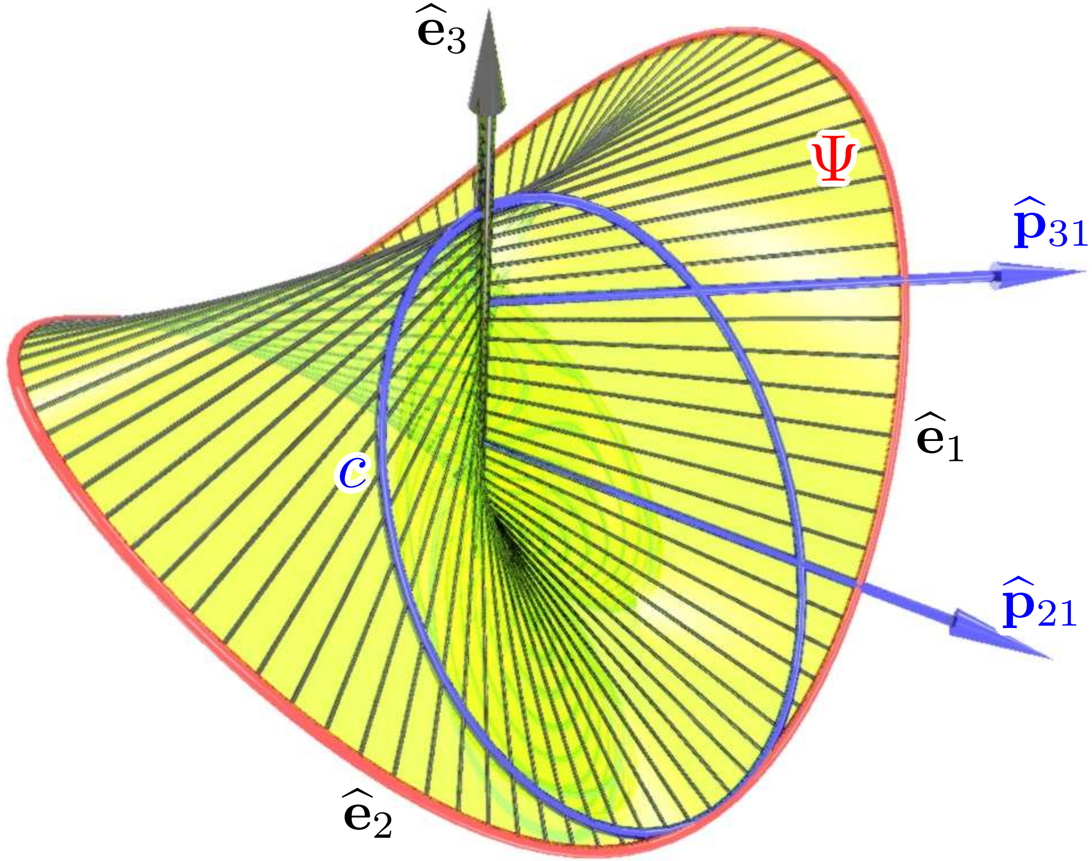


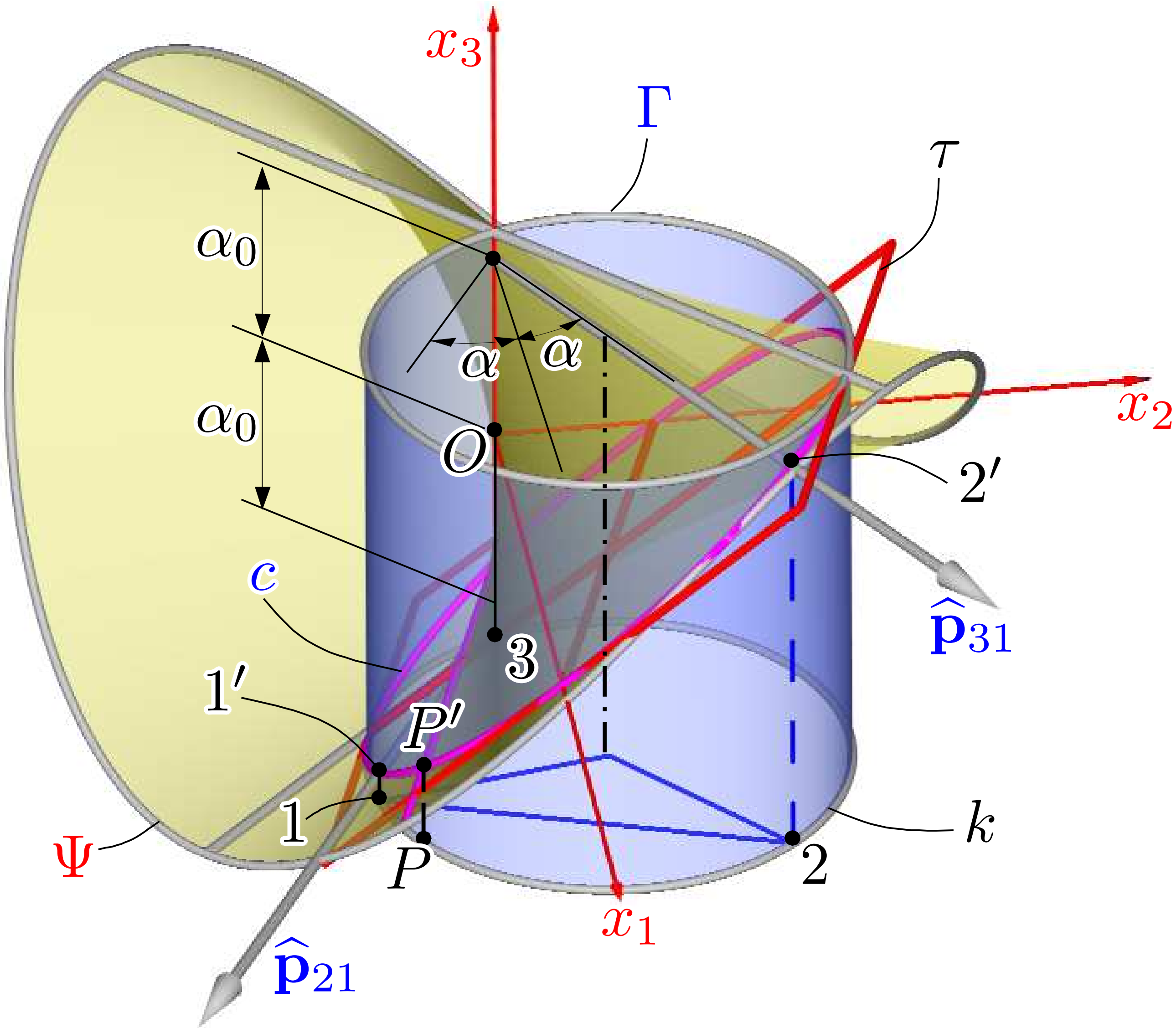
Proof for DISTELI's diagram

Alle relative axes $\widehat{\mathbf{p}}_{32}$ are located on the cylindroid Ψ.

Tangent planes τ of Ψ intersect along conics c which are located on cylinders Γ of rotation.

We concentrate on tangent planes τ with $45°$ inclination:

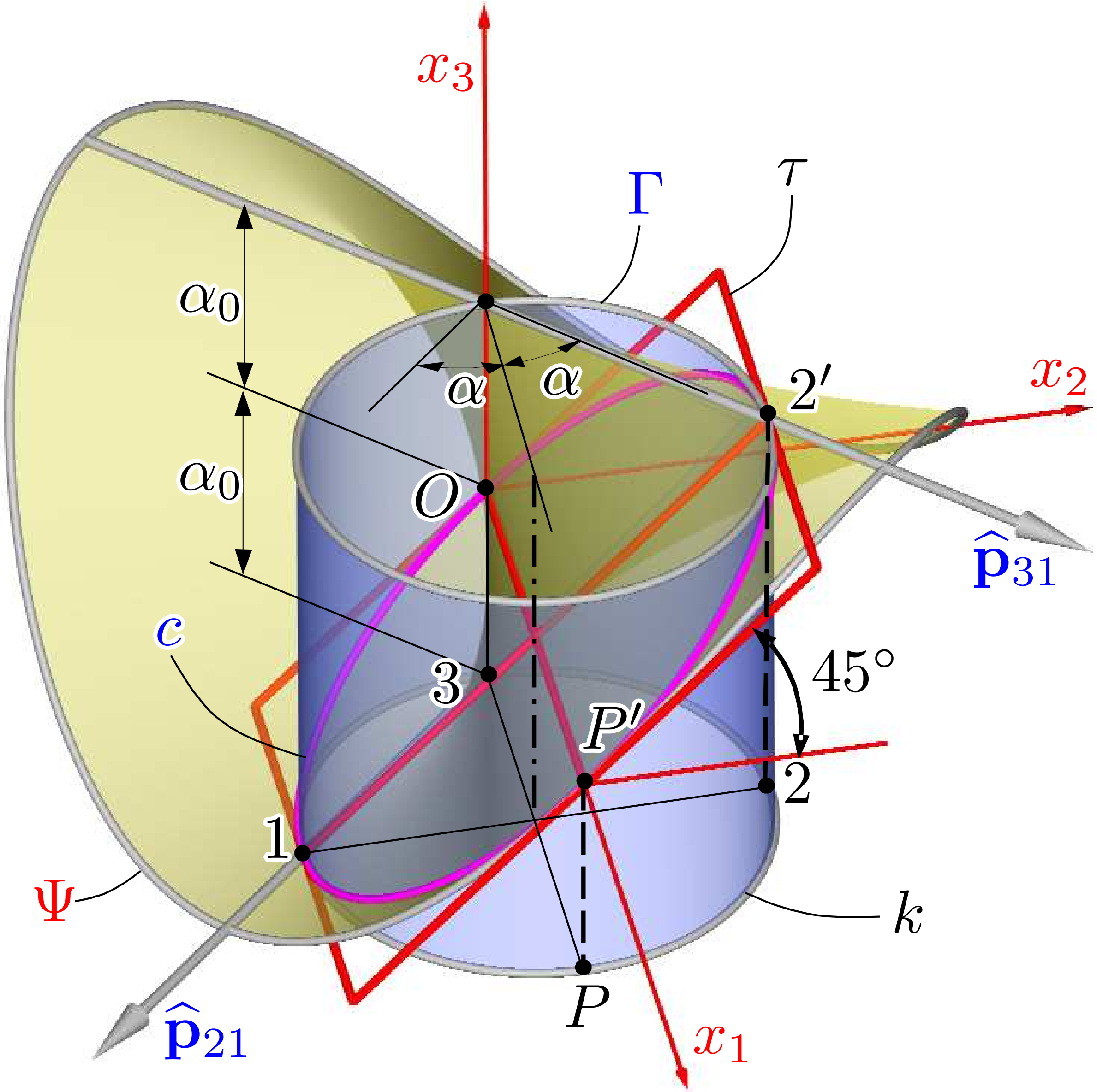




Three-Dimensional view of DISTELI's diagram

The difference in height can be seen in the top view of the slope line.

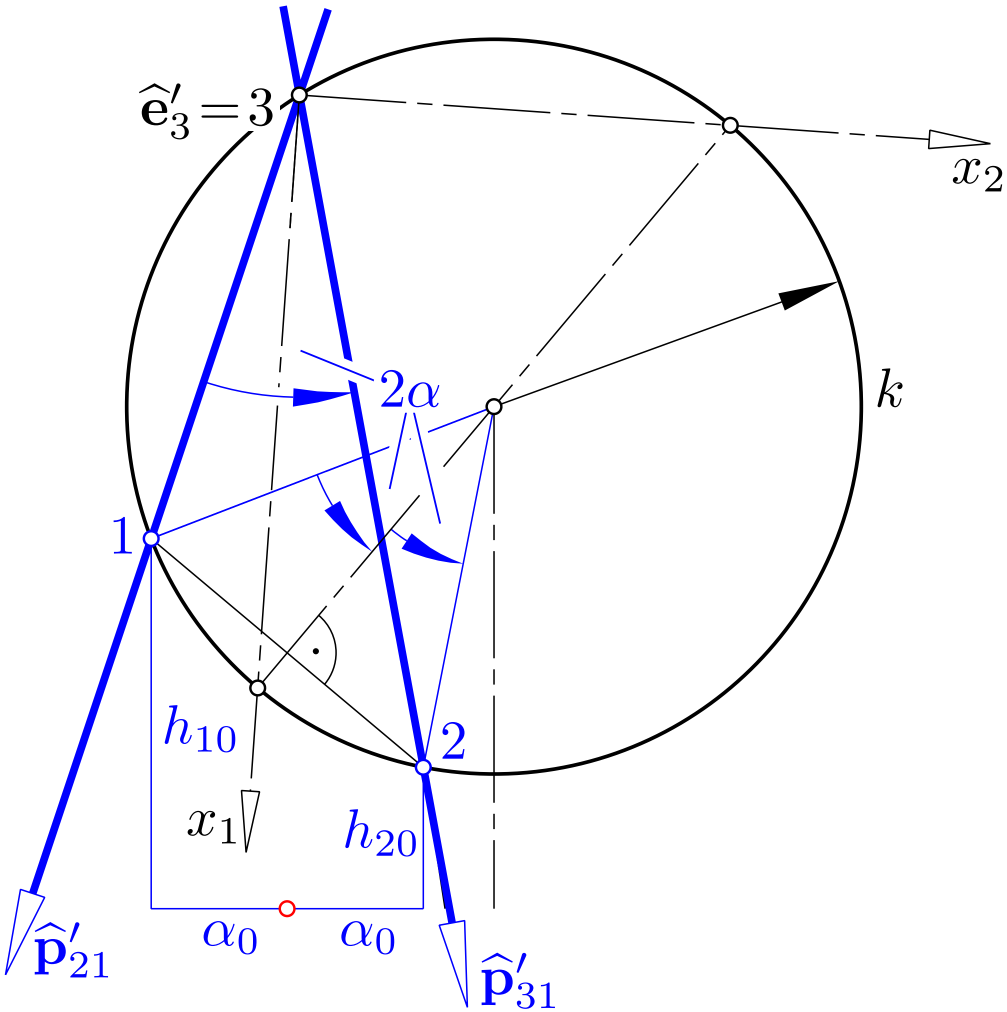
P' is the point of contact beetween τ and Ψ.



Particular view ot the three-dimensional view of DISTELI's diagram. $\widehat{\mathbf{p}}_{21}$ and $\widehat{\mathbf{p}}_{31}$ are perpendicular and therefore torsal generators of Ψ.

The difference in height can be seen in the top view of the slope line.

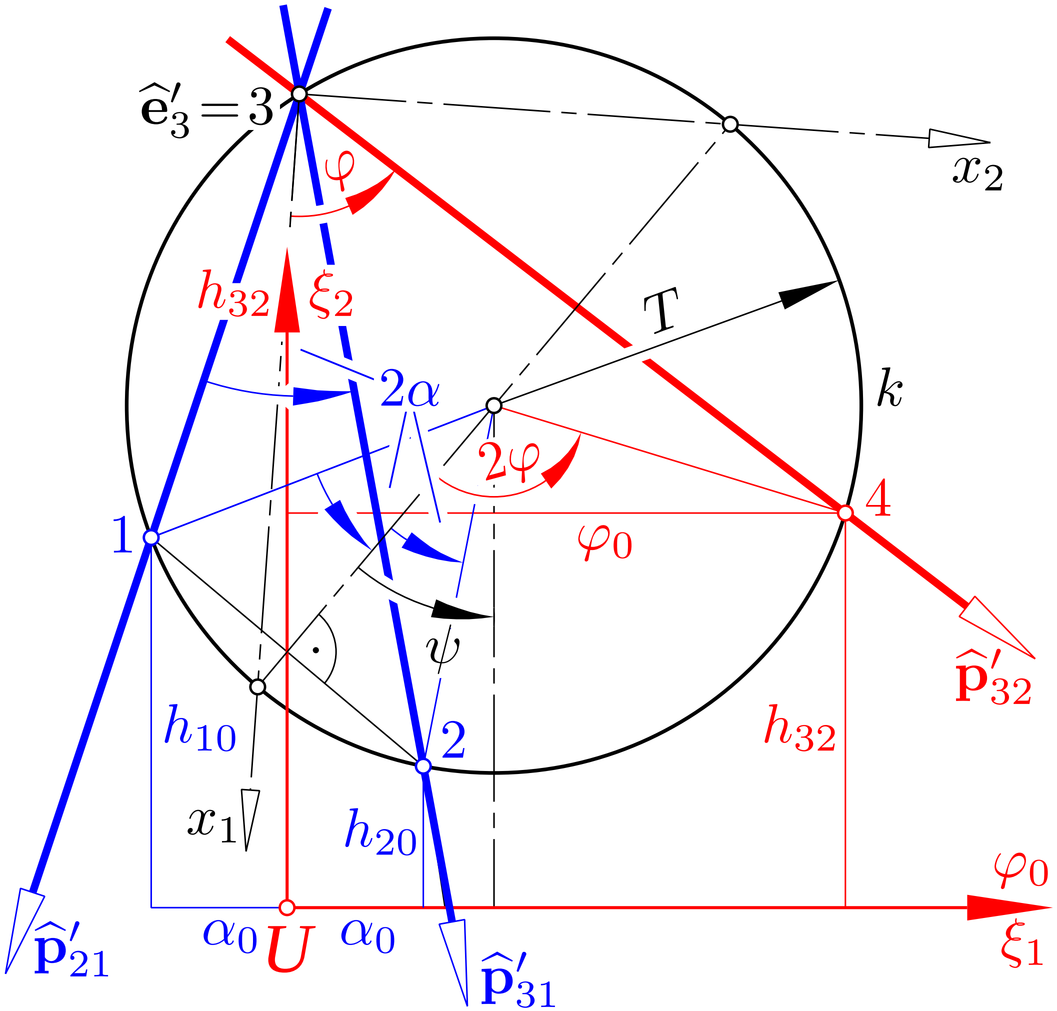
3. DISTELI's diagram: general case



DISTELI's diagram is also valid, when $\widehat{\mathbf{q}}_{21}$ and $\widehat{\mathbf{q}}_{31}$ are helical screws.

Step 1: With given α_0 und pitches h_{21} and h_{31} draw the points 1 und 2 and the circle k such that the center angle over 12 is 4α.

3. DISTELI's diagram: general case

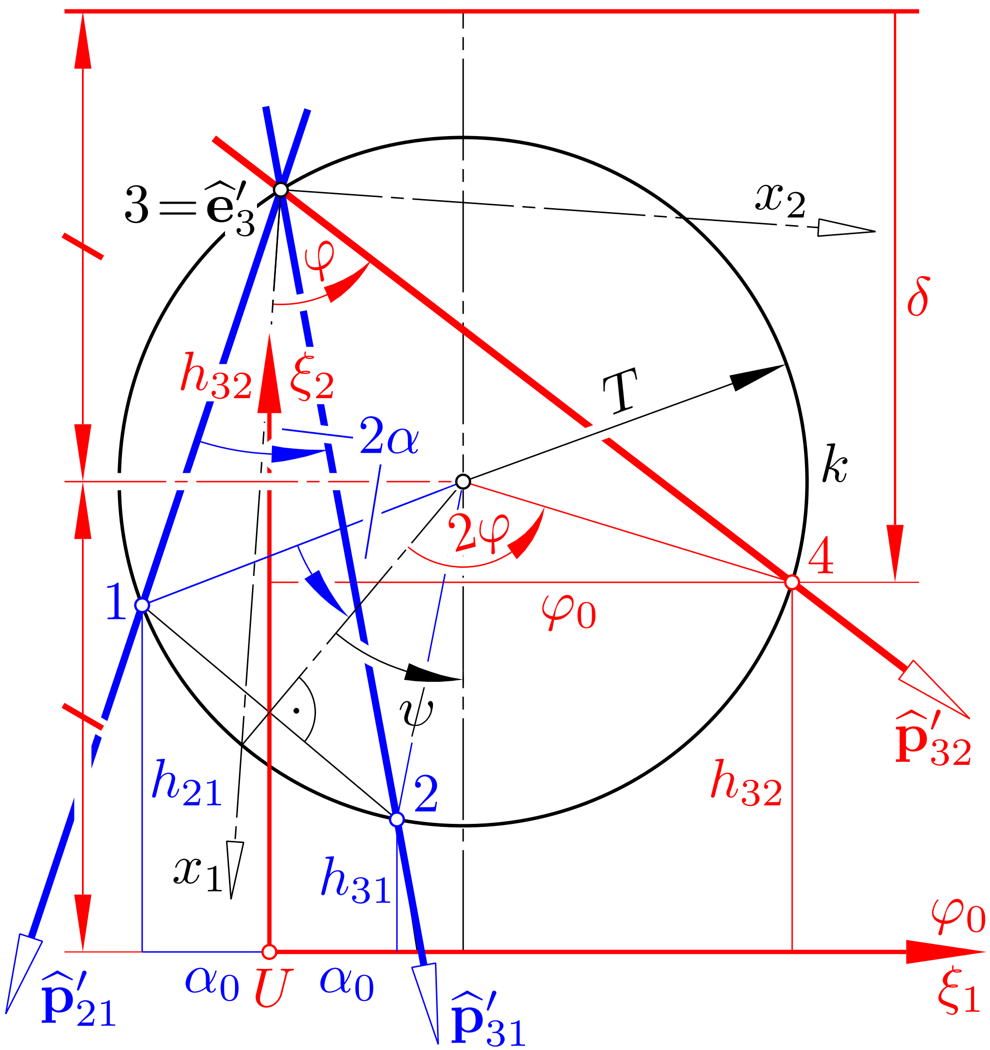


Step 2: The residual point 4 of intersection $k \cap \widehat{\mathbf{p}}'_{32}$ defines φ_0 and h_{32}.

$$\begin{aligned} \varphi_0 &= T\sin(2\varphi - \psi) \\ &\quad + T\cos 2\alpha \sin\psi, \\ h_{32} &= -T\cos(2\varphi - \psi) \\ &\quad + T\cos 2\alpha \cos\psi \\ &\quad + \frac{1}{2}(h_{31} + h_{21}) \end{aligned}$$

with constants T and ψ.

Further information taken from the diagram



δ is the distribution parameter of the axodes of Σ_3/Σ_2.

$$\delta = T\cos(2\varphi - \psi) + T\cos 2\alpha\cos\psi + \frac{1}{2}(h_{31} + h_{21}).$$

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