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ON THE RIGIDITY OF POLYGONAL MESHES

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Dedicated to Prof. Pavel Pech at the occasion of his 60th birthday

ABSTRACT. A polygonal mesh is a connected subset of a polyhedral surface. We address the problem whether the intrinsic metric of a mesh, i.e., its development, can determine the exterior metric. If this is the case then the mesh is rigid. Among the non-rigid cases even flexible versions are possible. We concentrate on quadrangular meshes and in particular on a mesh with a flat pose in which the quadrangles belong to a tesselation. It is proved that this mesh admits a self-motion and that all its flexions represent discrete models of cylinders of revolution. These flexions can be generated from a skew line-symmetric hexagon by applying iterated coaxial helical motions.

INTRODUCTION

In the following we understand under a *polygonal mesh* a connected subset of any polyhedral surface in the Euclidean 3-space $\mathbb{E}^3$. This means, a polygonal mesh is a surface consisting of planar polygonal faces, edges and vertices. The edges are either *internal* when they are shared by two faces, or they belong to the boundary of the mesh. The term *combinatorial structure* of the mesh stands for the list of faces and the identification of those pairs $(\boldsymbol{f}_i, \boldsymbol{f}_j)$ of faces which share an internal edge e_{ij}. In this sense a *polyhedron* is a polygonal mesh with internal edges only.

Suppose that this mesh undergoes a transformation which acts on the faces as isometry and preserves their planarity. We call this mesh rigid when under this transformation also all dihedral angles between adjacent faces are preserved. The question whether the intrinsic metric of the mesh determines its spatial shape uniquely or not is also important for many engineering applications, e.g., for mechanical or constructional engineers, for biologists in protein modelling or for the analysis of isomers in chemistry.

Polygonal meshes, in particular quadrangular meshes, play an important role in discrete differential geometry and in new architecture where they serve as a discrete model of freeform surfaces.

In the following we present different kinds of rigidity and we characterize the flexions of Kokotsakis' flexible quadrangular tesselation of the plane. It should be noted that we only focus on geometric aspects of flexibility. We do not treat technical aspects like stiffness of faces and edges or clearances along hinges.

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1. The definitions of rigidity

Definition 1. A polyhedron or a polygonal mesh is called *"globally rigid"* when its development (unfolding) defines its spatial shape uniquely — apart from movements in space.

The *development* of the polygonal mesh defines its *intrinsic metric*, i.e., the true shape of the faces and the combinatorial structure. Global rigidity means that the intrinsic metric defines the exterior metric in space including the dihedral angle φ_{ij} at each internal edge e_{ij} shared by $\boldsymbol{f}_i$ and $\boldsymbol{f}_j$. Conversely, the development together with all dihedral angles defines the spatial shape completely as we can assemble the mesh face by face, provided angle φ_{ij} is signed with respect to a prescribed orientation of edge e_{ij}. Of course, the dihedral angles cannot be given independently; they must be compatible with the intrinsic metric.

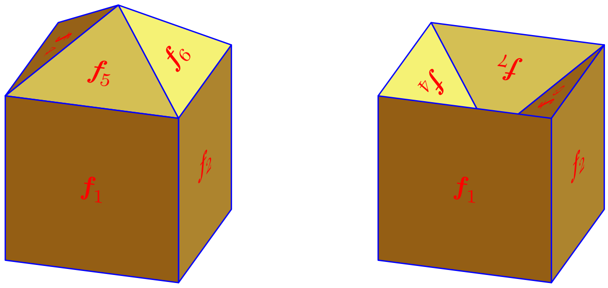

Figure 1. Two realizations of the same net.

From everybody's experience with assembling cardboard model of cubes, prisms or pyramids during school-days one tends to conjecture that each polyhedron is globally rigid. In fact, it is true, e.g., for a three-sided pyramid (tetrahedron) or for a cube or — more generally — for all polyhedra where at each vertex exactly three faces are meeting. However, the example in Fig. 1 shows two incongruent *realizations* with the same development. The polyhedron on the left-hand side is convex; it is built from a cube where the top face is replaced by pyramid. The polyhedron on the right-hand side contains edges along which the interior dihedral angle is $> \pi$.

When the height of the attached pyramid is sufficiently small, we can transform the convex version into the concave one by applying a slight force to the apex of the pyramid. In this case we speak of *"snapping"* polyhedra; we can vary the spatial shape between two possibilities when admitting small deformations in between, e.g., by slight bending of faces and edges. Theoretically, both realizations are locally rigid according to the following definition.

Definition 2. A polygonal mesh is called *"locally rigid"*, if its intrinsic metric admits no other realization with dihedral angles sufficiently close to that of the given one.

Surprisingly, there are examples of polyhedra where the development admits even infinitely many incongruent realizations.

Definition 3. A polygonal mesh is called *flexible* if there is a continuous family of mutually incongruent meshes sharing the intrinsic metric. In this case the mesh admits a *self-motion*; each pose obtained during this self-motion is called a *flexion*.

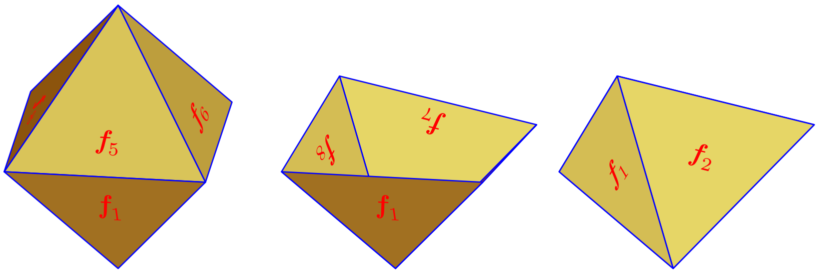

FIGURE 2. The regular octahedron and its re-assembled continuously flexible versions.

In Fig. 2 a trivial example of a flexible polyhedron is displayed. The intrinsic metric originates from that of a *regular* octahedron. We can re-assemble this polyhedron by putting one four-sided pyramid into the other. This gives a twofold covered quadratic pyramid without basis, which of course admits a self-motion. From a flat pose of this four-sided pyramid with congruent faces we can even switch to realizations consisting of a fourfold covered mesh of two triangles.

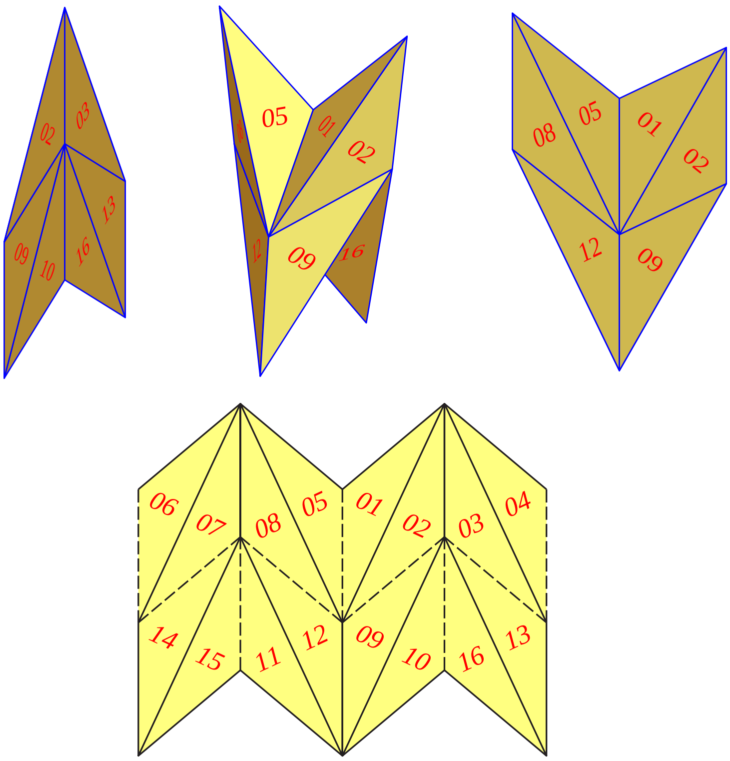

FIGURE 3. This polyhedron called *"Vierhorn"* is locally rigid, but snaps between its spatial shape and two flat realizations; Below: Development of the *"Vierhorn"*; dashes indicate valley folds.

It turns out that the computation of the spatial shape of any four-sided double-pyramid from given internal metric, i.e., from its 12 edge lengths, is an algebraic problem of degree 8. Hence, there are either up to 8 different realizations, or the edge lengths admit a flexible form.

One needs to be careful when any polygonal mesh looks like a flexible one. A famous example is described in C. Schwabe's and W. Wunderlich's article [12] on a polyhedron exposed at the science exposition "Phänomena" 1984 in Zürich (see Fig. 3). At that time it was falsely stated that this polyhedron is flexible, but it is only snapping between two different flat realizations and one spatial shape. The development reveals that all faces of this polyhedron are congruent isoceless triangles.

There are four mile-stones in the theory of flexible polyhedra:

- The first important result in the theory of rigidity claims that every convex polyhedron is rigid [2]. This is due to A. L. Cauchy 1813. The example presented in Fig. 2 is no contradiction since the convex form, the regular octahedron, is locally rigid.
- 1897 R. Bricard [1] classified all flexible octahedra, i.e., all flexible four-sided double-pyramids with a not necessarily coplanar equator. However, all these polyhedra have self-intersections. A real-world model can only be built either as a wireframe or as a cardboard polyhedron where two faces are omitted.
- R. Connelly detected 1977 the first flexible sphere-homeomorphic polyhedron without self-intersections and without twofold covered faces [3]. A simplified flexing sphere with 9 vertices was presented 1980 by K. Steffen [10] (compare [5, p. 347]) as a compound of two Bricard's polyhedra.
- 1996 I. Sabitov [6] proved the famous *Bellows Conjecture* stating that for every flexible polyhedron in $\mathbb{E}^3$ the oriented volume keeps constant during the self-motion [6]. This was a consequence of his generalization of Heron's formula: For any orientable polyhedron with triangular faces in $\mathbb{E}^3$ there exists a polynomial whose coefficients are polynomials in the squared edge-lengths over $\mathbb{Q}$ and which has the square V^2 of the volume as a root; this polynomial depends only on the combinatorial structure of the polyhedron. There is only an algorithm available for determining this polynomial.

If Sabitov's result had been known at the exposition of "Vierhorn" (Fig. 3), then it would have been evident that this is not really flexible because the volume changes drastically during the transition from the spatial form to the flat pose.

There is still another kind of flexibility placed between riditity and continuous flexibility: According to W. Whiteley's principle of averaging (see [11]) it can be seen as a limit of snapping polyhedra. In the following definition we make use of a standard notation of kinematics (see, e.g., [8]): Each infinitesimal rotation[1] about an oriented axis with normalized dual coordinate vector $\widehat{\mathbf{a}} \in \mathbb{R}^6$ and signed angular velocity ω can be represented by the *twist vector* $\omega\,\widehat{\mathbf{a}}$. The twist of the composition of two infinitesimal rotations is the sum of the two twist vectors.

Definition 4. Suppose that to each internal edge e_{ij} of a polygonal mesh we can assign an angular velocity ω_{ij} for the relative motion of the adjacent $\boldsymbol{f}_i$ against $\boldsymbol{f}_j$

[1]This is an affine map which appoints to each point in $\mathbb{E}^3$ a velocity vector.

in such a way that for each loop $\boldsymbol{f}_0, \dots, \boldsymbol{f}_s = \boldsymbol{f}_0$ of faces, where any two consecutive faces share an edge $e_{i-1\,i}$, $i = 1, \dots, s$, the sum of corresponding twists vanishes, i.e., $\sum_{i=1}^{s} \omega_{i-1\,i}\,\widehat{\mathbf{e}}_{i-1\,i} = \widehat{\mathbf{0}}$. Then the mesh is called *"infinitesimally flexible"*. When the only compatible appointment of angular velocities to all internal edges is the trivial one with $\omega_{ij} = 0$, then the mesh is called *infinitesimally rigid*.

A real-word model of an infinitesimally flexible mesh shows an apparent, but somehow confined flexibility. At the Vierhorn both flat positions are infinitesimally flexible. A classical extension of Cauchy's result states that each convex polytope is even infinitesimally rigid.

Remark 1. a) Infinitesimal flexibility is usually defined for frameworks, but this works only for triangular meshes. Before applying it, e.g., to quadrangular meshes, we have to split all quadrangles by a diagonal into two triangles and to build pyramids over them in order to guarantee the planarity of the quadrangle.
b) There are even different orders of infinitesimal rigidity to distinguish (e.g., [7]). But for the sake of brevity we focus here only on first-order infinitesimal flexibility.

2. Flexible polygonal meshes

We are now concentrating on quadrangular meshes and start with a *Kokotsakis mesh* (German: Vierflach), the compound of 3×3 planar quadrangles, which is named after A. Kokotsakis [4]. In Fig. 4, left, the scheme of a quadrangular Kokotsakis mesh is shown with a central face $\boldsymbol{f}_0$ and a belt of 8 quadrangles around it. On the right hand side a flexion is displayed.

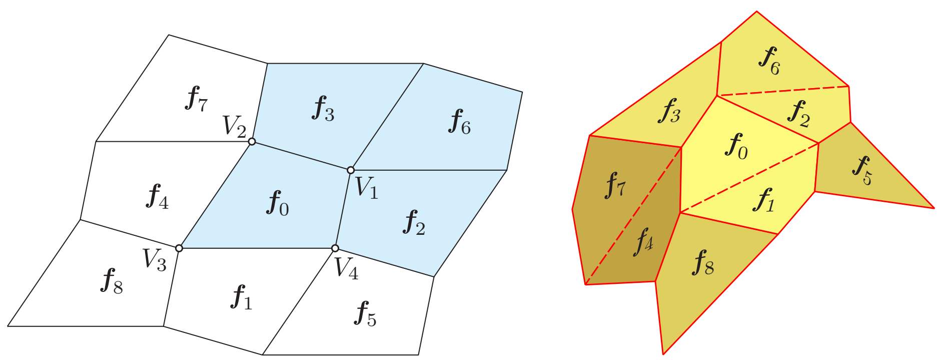

Figure 4. Left: Scheme of a quadrangular Kokotsakis mesh. Right: Flexion of a flexible version; dashes indicate valley folds.

A complete classification of all continuously flexible Kokotsakis meshes is still open (compare, e.g., [9]). However, the geometric characterization of infinitesimally flexible meshes has already been given in [4] (see Fig. 5). We follow Kokotsakis' ideas and use in our proof of this characterization standard results from Kinematics (e.g., [8]): For any two faces $\boldsymbol{f}_i, \boldsymbol{f}_j$ sharing an edge, this edge — here denoted by ij — is the axis of the relative motion. Due to the *Three-Pole-Theorem* for any three faces $\boldsymbol{f}_i, \boldsymbol{f}_j, \boldsymbol{f}_k$ with rotations as pairwise relative motions the three relative axes ij, ik and jk must be coplanar and share a point, which is denoted by ijk.

The angles α_j, α_k which jk enclosed with ij and ik, respectively, define the ratio of angular velocities of $\boldsymbol{f}_j$ and $\boldsymbol{f}_k$ against $\boldsymbol{f}_i$ by

$$\omega_{ji} : \omega_{ki} = \sin\alpha_k : \sin\alpha_j \, .$$

This implies, e.g., that at the Kokotsakis mesh (see Fig. 5) the axis *12* of the relative motion between $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ is the line of intersection between the planes spanned by $\boldsymbol{f}_5$ and $\boldsymbol{f}_0$. We call the line of intersection between the planes of $\boldsymbol{f}_i$ and $\boldsymbol{f}_0$ the *trace* of $\boldsymbol{f}_i$, $i = 1, \dots, 8$.

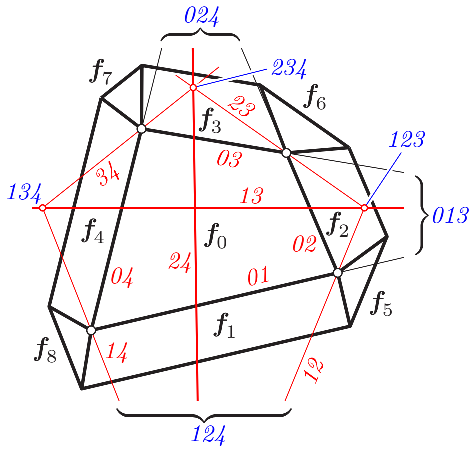

FIGURE 5. Infinitesimally flexible Kokotsakis mesh; the bounding faces $\boldsymbol{f}_1, \dots, \boldsymbol{f}_8$ are cut by a plane parallel to that of $\boldsymbol{f}_0$.

Theorem 5 (Kokotsakis, 1932)**.** *A Kokotsakis mesh is infinitesimally flexible if and only if the following three points are collinear: the points of intersection 013, 123 and 134 between the traces of the pairs of faces* $(\boldsymbol{f}_1, \boldsymbol{f}_3)$, $(\boldsymbol{f}_5, \boldsymbol{f}_6)$ *and* $(\boldsymbol{f}_7, \boldsymbol{f}_8)$*, respectively. This is equivalent to the statement that the points of intersection 024, 234 and 124 between the traces of the pairs of faces* $(\boldsymbol{f}_2, \boldsymbol{f}_4)$, $(\boldsymbol{f}_6, \boldsymbol{f}_7)$ *and* $(\boldsymbol{f}_5, \boldsymbol{f}_8)$*, resp., are aligned.*

Proof. Due to the Three-Pole-Theorem the traces *12* of $\boldsymbol{f}_5$ and *23* of $\boldsymbol{f}_6$ meet at point *123*; the traces *14* of $\boldsymbol{f}_8$ and *34* of $\boldsymbol{f}_7$ meet at point *134*; the traces *01* of $\boldsymbol{f}_1$ and *03* of $\boldsymbol{f}_3$ meet at point *013*. In the infinitesimal case the three points *013*, *123* and *134* must be located on the relative axis *13*. Also by Desargues' Theorem we can conclude that the collinearity of these three points is equivalent to the collinearity of points *024*, *124*, and *234*.

Conversely, the position of *13* defines the ratio of angular velocities of the faces $\boldsymbol{f}_1$ and $\boldsymbol{f}_3$ with respect to $\boldsymbol{f}_0$. The other relative axes define the angular velocities of the other faces. Hence, the stated collinearity implies a compatible assignment of angular velocities to all internal edges. □

An interesting continuously flexible quadrilateral mesh dates also back to Kokotsakis [4]. We start with a flat pose which consists of congruent quadrangles of a planar tessellation (Fig. 6). Any two quadrangles sharing a side (e.g., $\boldsymbol{f}_3$ and $\boldsymbol{f}_4$) change place under a rotation through 180° (= *half-turn*) about the midpoint (C)

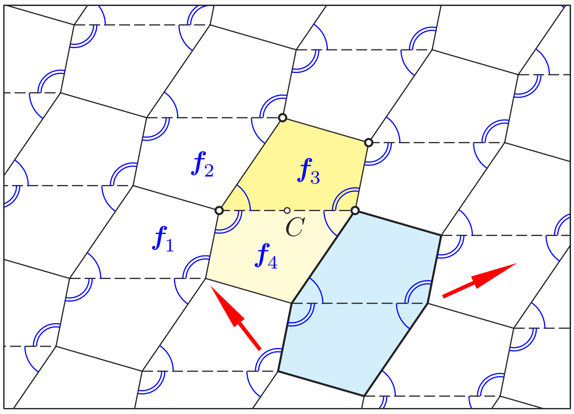

FIGURE 6. Kokotsakis' flexible tessellation.

of the common side. Any two adjacent quadrangles form a centrally symmetric hexagon, and the complete tesselation can also be generated by translations of this hexagon. The arrows in Fig. 6 indicate the directions of these translations.

When the quadrangles are convex, then this polygonal mesh is flexible (Kokotsakis [4, p. 647]. We call this mesh a *tesselation mesh*, for short.

In the following theorem we extend Kokotsakis' result by characterizing the flexions of this polygonal mesh. We are interested on constrained motions of the mesh. Therefore we exclude additional degrees of freedom of single faces by the request: Whenever the tesselation mesh includes three faces with a common vertex, then also the fourth face of this pyramid must be included.

On the other hand, when the basic quadrangle is a trapezoid, then there are aligned edges along which the mesh can be folded. We exclude these trivial flexes by requiring a generic basic quadrangle.

Theorem 6. *Let a polygonal mesh be extracted from the planar tesselation displayed in Fig. 6 in such a way, that with any three faces with a common vertex also the fourth face through this vertex is included.*

a) This quadrangular mesh is continuously flexible if and only if the initial quadrangle is convex.

b) In the generic flexible case, at each non-planar pose of a continuous self-motion all vertices are located on a cylinder of revolution (Figs. 8 and 9).

c) The faces of the flexion can be obtained from a line-symmetric hexagon composed from two adjacent quadrangles by applying iterated coaxial helical motions. In the flat pose these helical motions convert into the translations applied to a centrally symmetric hexagon in order to generate the planar tessellation.

Proof. First we pick out the four faces $\boldsymbol{f}_1, \dots, \boldsymbol{f}_4$ with the common vertex V_1 (Fig. 7). These congruent faces form a four-sided pyramid which is flexible, provided the fundamental quadrangle is convex. Otherwise, one interior angle of a face at V_1 would be greater than the sum of the other 3 interior angles so that the only realization is the flat pose.

Let any non-planar flexion of this pyramid be given (Fig. 7, left). For any pair $(\boldsymbol{f}_1, \boldsymbol{f}_2), \dots, (\boldsymbol{f}_4, \boldsymbol{f}_1)$ of adjacent faces there is a respective half-turn $\rho_1, \dots, \rho_4$

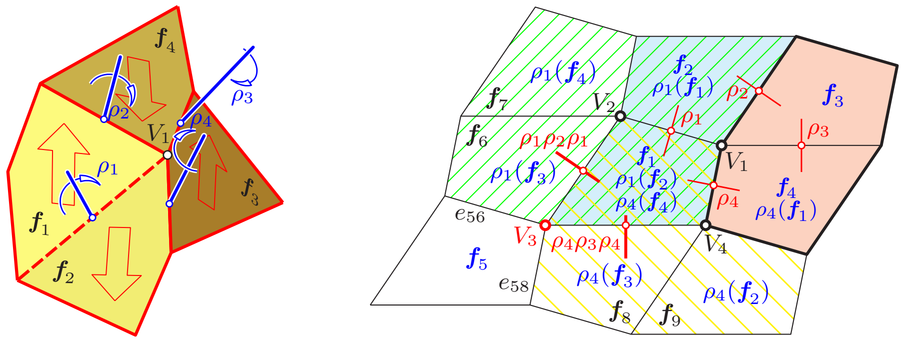

FIGURE 7. The complete flexion can be generated by applying iterated half-turns ρ_i to an initial face $\boldsymbol{f}_1$.

which swaps the two faces. So, e.g., $\boldsymbol{f}_2 = \rho_1(\boldsymbol{f}_1)$ and $\boldsymbol{f}_1 = \rho_1(\boldsymbol{f}_2)$. The axis of ρ_1 (see Fig. 7, left) is perpendicular to the common edge V_1V_2, and it is located in a plane which bisects the dihedral angle between $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$.

After applying all four half-turns $\rho_1, \dots, \rho_4$ consecutively to the quadrangle $\boldsymbol{f}_1$, this is mapped via $\boldsymbol{f}_2$, $\boldsymbol{f}_3$, and $\boldsymbol{f}_4$ onto itself; hence the product $\rho_4 \dots \rho_1$ equals the identity. (We indicate the composition of mappings by left multiplication.) Because of $\rho_i^{-1} = \rho_i$ we obtain

$$\rho_1\rho_2 = \rho_4\rho_3. \tag{2.1}$$

Lemma 7. *The product of two half-turns about non-parallel axes a_1, a_2 is a helical motion. Its axis is the common perpendicular of a_1 and a_2; its angle of rotation is twice the angle and the length of translation is twice the distance of the axes a_1, a_2.*

When our pyramid with apex V_1 is not flat, then the axes of the half-turns are pairwise skew; the common perpendicular for any two of these axes is unique. Hence (2.1) implies that the axes of the four half-turns have a common perpendicular s. The motions $\rho_1\rho_2 = \rho_4\rho_3$ and $\rho_1\rho_4 = \rho_2\rho_3$ are helical motions with the common axis s.

Now we extend the flexion of the pyramid with apex V_1 stepwise to our polygonal mesh by adding congruent copies of the initial pyramid without restricting the flexibility:

The rotation ρ_1 exchanges not only $\boldsymbol{f}_1$ with $\boldsymbol{f}_2$ but maps the pyramid with apex V_1 onto a congruent copy with apex V_2 sharing two faces with its preimage. This is the area which is hatched in Fig. 7, right. Analogously, ρ_4 generates a pyramid with apex V_4 and sharing the faces $\boldsymbol{f}_1$ and $\boldsymbol{f}_4$ with the initial pyramid.

Finally there are two ways to generate a pyramid with apex V_3. Either, we transform ρ_2 by ρ_1 and apply $\rho_1\rho_2\rho_1$, which exchanges $\rho_1(\boldsymbol{f}_2) = \boldsymbol{f}_1$ with $\rho_1(\boldsymbol{f}_3)$ and swaps V_2 and V_3. Or we proceed with $\rho_4\rho_3\rho_4$, which exchanges $\rho_4(\boldsymbol{f}_4) = \boldsymbol{f}_1$ with $\rho_4(\boldsymbol{f}_3)$ and swaps V_4 and V_3.

Thus we obtain mappings $(\rho_1\rho_2\rho_1)\rho_1 = \rho_1\rho_2$ and $(\rho_4\rho_3\rho_4)\rho_4 = \rho_4\rho_3$ with $\boldsymbol{f}_1 \mapsto \boldsymbol{f}_5$ and $V_1 \mapsto V_3$. Both displacements are equal by (2.1), and we notice

$$\rho_1\rho_2 = \rho_4\rho_3\colon\ \boldsymbol{f}_1 \mapsto \boldsymbol{f}_5,\ \boldsymbol{f}_2 \mapsto \rho_1(\boldsymbol{f}_3),\ \boldsymbol{f}_3 \mapsto \boldsymbol{f}_1,\ \boldsymbol{f}_4 \mapsto \rho_4(\boldsymbol{f}_3). \tag{2.2}$$

Hence each flexion of the initial pyramid with apex V_1 is compatible with a flexion of the complete 3×3 tesselation mesh. Is this the only flexion of this mesh induced by the given flex of the initial pyramid?

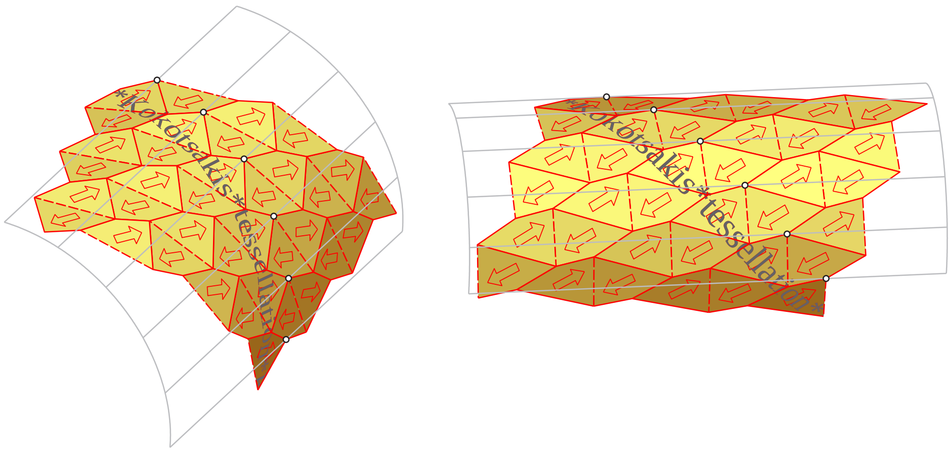

FIGURE 8. For each flexion of the first and second kind the vertices are placed on a cylinder of rotation; the marked points are located on a helical line.

Lemma 8. *A generic flexion of a 3×3 tesselation mesh is uniquely defined by the flexion of one included pyramid consisting of four faces with a common vertex V_1.*

Proof. At our previously defined flexion the pyramids with vertices V_2, V_3 and V_4 are congruent to that with vertex V_1. However, there is another possibility at V_2, which is compatible with that at V_1: We can reflect the two left faces $\boldsymbol{f}_6, \boldsymbol{f}_7$ (Fig. 7, right) in the plane spanned by the edges e_{27} and e_{16}. In the same way it is possible at V_4 to replace the faces $\boldsymbol{f}_8, \boldsymbol{f}_9$ by their mirrors with respect to the plane $e_{18} \vee e_{49}$. In total this gives 3 alternatives: Reflect either one pair of faces or both simultaneously. Does any of these alternatives keep place for inserting the last face $\boldsymbol{f}_5$ with given interior angle at V_3?

(i) When only $\boldsymbol{f}_6$ and $\boldsymbol{f}_7$ are reflected, the edge e_{56} is replaced by its mirror in the plane $e_{27} \vee e_{16}$. This preserves the angle with e_{58} only if the plane $e_{27} \vee e_{16}$ passes through e_{58}. After applying ρ_1, this is equivalent to the statement that $\rho_2(e_{14})$ is placed in $e_{23} \vee e_{14}$, which means that the diagonal plane $e_{23} \vee e_{14}$ of our initial pyramid is the exterior bisector of the faces $\boldsymbol{f}_2$ and $\boldsymbol{f}_3$.

(ii) When reflecting $\boldsymbol{f}_8$ and $\boldsymbol{f}_9$ only, then the mirror plane $e_{18} \vee e_{49}$ must pass through e_{56} in order to preserve the interior angle of $\boldsymbol{f}_5$ at V_3. After applying ρ_3, this is equivalent to the condition that $e_{12} \vee e_{43}$ is the exterior bisector of the faces $\boldsymbol{f}_3$ and $\boldsymbol{f}_4$.

(iii) When e_{56} is reflected in $e_{27} \vee e_{16}$ and e_{58} is reflected in $e_{18} \vee e_{49}$, then the angle between e_{56} and e_{58} is preserved if and only if there is a rotation mapping e_{56} and e_{58} onto their respective mirror images. The axis a if this rotation is the line of intersection between the two mirror planes. In the case of a rotation the axis a must span planes with e_{56} and e_{58} which enclose with $e_{27} \vee e_{16}$ and $e_{18} \vee e_{49}$, respectively, the same oriented angle.

Only in particular poses condition (i), (ii) or (iii) can be fulfilled, but we excluded this by the request for a generic pose. Hence the flexion of the 3×3 tesselation mesh is uniquely defined. □

By Lemma 8 we can uniquely extend the flexion of the initial pyramid to the 3×3 tesselation mesh and furtheron to the complete polygonal mesh, apart from particular poses. But under a continuous self-motion it is not possible to switch into one of the exceptional reflected positions listed above in (i), (ii) or (iii). Hence this mesh is continuously flexible, and all included four-sided pyramids are congruent.

We detect at the flexion also the spatial analogues of the translations in the plane: The product $\rho_1\rho_4 = \rho_2\rho_3$ maps the pyramid with apex V_1 onto that with apex V_2. On the other hand we have

$$\rho_1\rho_4 \colon\ \boldsymbol{f}_1 \mapsto \rho_1(\boldsymbol{f}_4),\ \boldsymbol{f}_4 \mapsto \boldsymbol{f}_2,\ \rho_4(\boldsymbol{f}_2) \mapsto \boldsymbol{f}_1,\ \rho_4(\boldsymbol{f}_3) \mapsto \rho_1(\boldsymbol{f}_3). \tag{2.3}$$

When $\boldsymbol{f}_3$ and $\boldsymbol{f}_4$ are glued together, we obtain a skew hexagon, one half of our initial pyramid with apex V_1. The half-turn ρ_4 maps this hexagon onto itself; hence it is line-symmetric. By (2.2) the helical motion $\rho_1\rho_2$ maps this hexagon onto the compound of $\boldsymbol{f}_1$ and $\rho_4(\boldsymbol{f}_3)$ and furthermore $\boldsymbol{f}_1$ onto $\boldsymbol{f}_5$. The inverse $\rho_2\rho_1$ is the spatial analogon of the translation indicated in Fig. 6 by the red arrow pointing upwards to the right. On the other hand, $\rho_4\rho_1$ maps the compound of $\boldsymbol{f}_1$ and $\rho_4(\boldsymbol{f}_3)$ onto $\rho_1(\boldsymbol{f}_4)$ and $\rho_1(\boldsymbol{f}_3)$. It these two helical motions act repeatedly on the line-symmetric hexagon, the complete flexion is obtained.

Since all vertices of the flexion arise from V_1 by motions which keep the common perpendicular s of the half-turn axes fixed, e.g., $V_2 = \rho_1(V_1)$, $V_3 = \rho_1\rho_2(V_1)$, $V_4 = \rho_4(V_1)$, they all have the same distance to s, i.e., they are located on a cylinder of revolution with axis s. □

Remark 2. a) When starting from the flat initial pose of the pyramid with apex V_1, there are two self-motions possible since there are two edges of the pyramid where the adjacent interior angles at V_1 have a sum smaller than $180°$. These edges become valley-folds in Fig. 8. Hence our polygonal mesh admits two kinds of differentiable self-motions. Figure 9 shows snapshots of these two self-motions.
b) In the case of a trapezoid $\boldsymbol{f}_1$ one kind of generating motion is a rotation about s, the other a translation along s. However, in this case we have a higher degree of freedom since the mesh can be bended along each serie of aligned edges.

There is also a direct way to get a flexion of a tesselation mesh: We can start with any point V_1 and with three half-turns ρ_1, ρ_2, ρ_3 such that the axes have a common perpendicular. However, the quadrangle $V_1 \dots V_4$ will be planar only if V_1 obeys an additional condition.

Conclusion

Not each polyhedron is uniquely defined by its development. We shed light on the different types of rigidity of polygonal meshes und characterize the flexions of Kokotsakis' tesselation meshes.

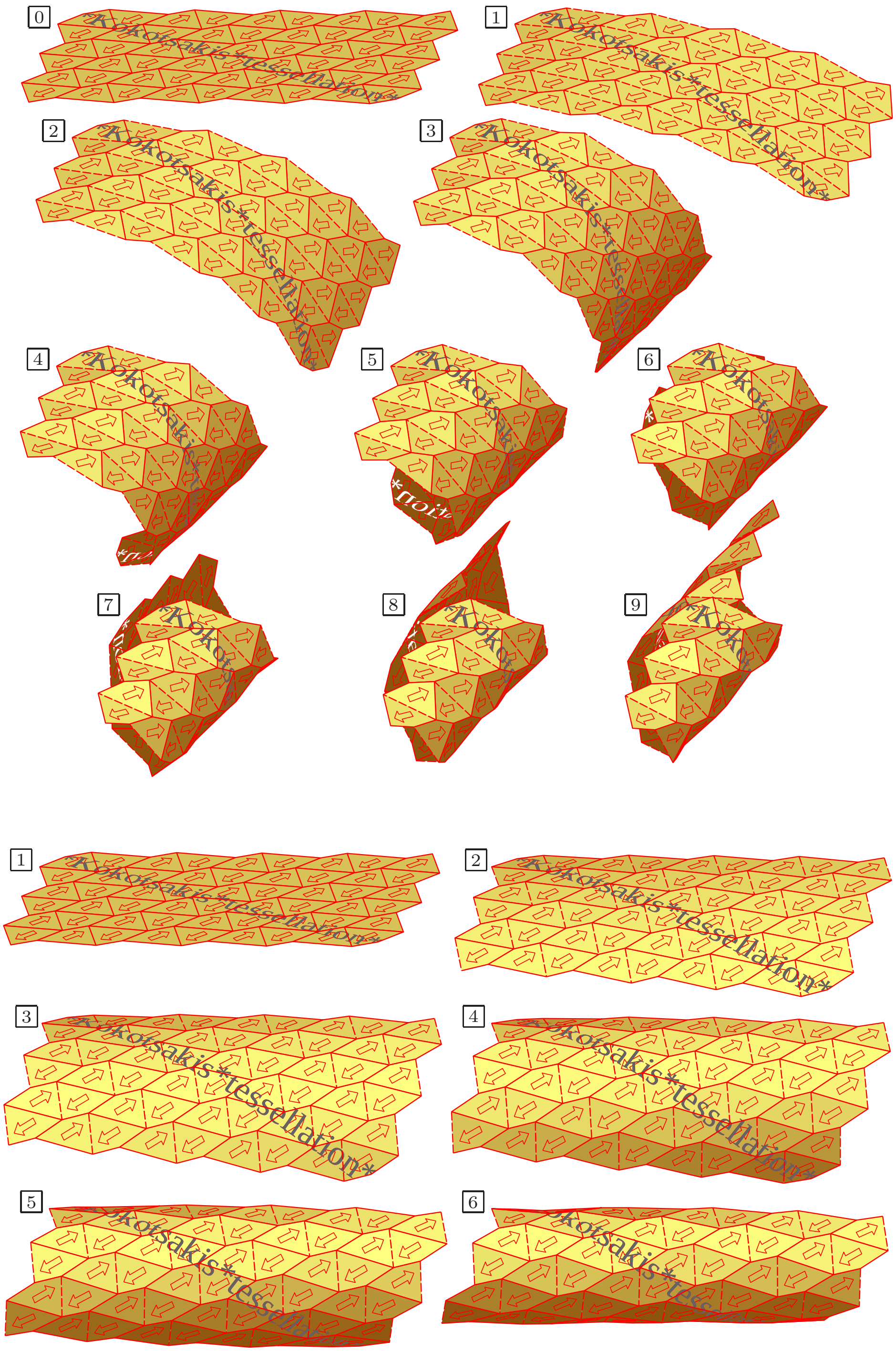

FIGURE 9. Kokotsakis tesselation: Flexions of first and second kind.

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