On the Flexibility of Kokotsakis Meshes

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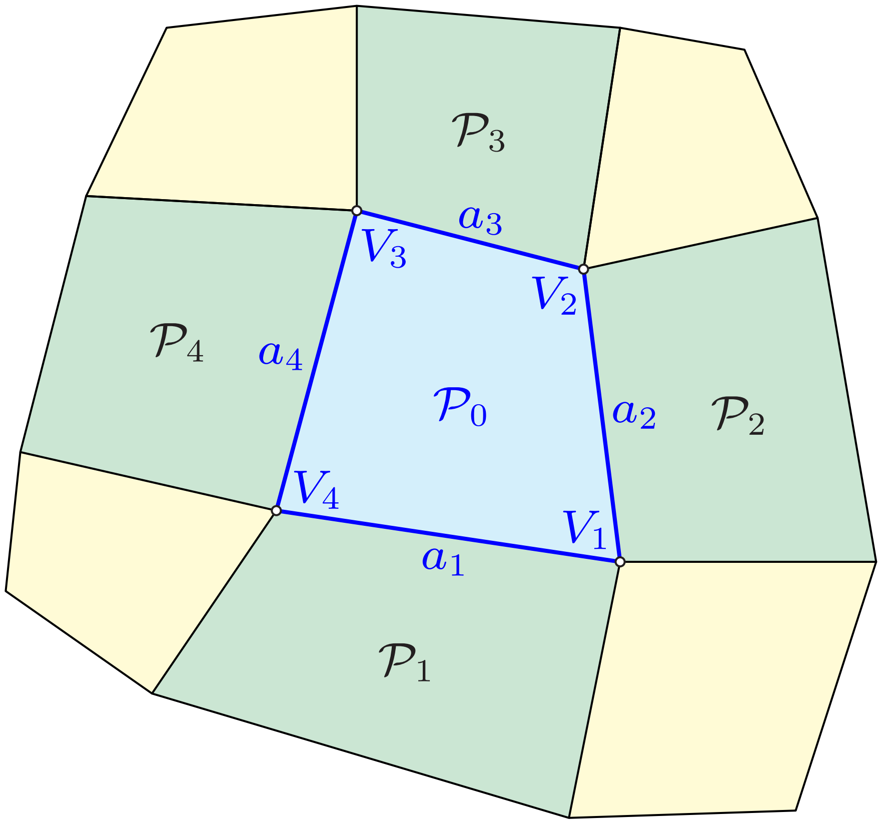
Table of contents

1. Introduction



Special case: $n = 4$

A Kokotsakis mesh is a polyhedral structure consisting of an n-sided central polygon $\mathcal{P}_0$ surrounded by a belt of polygons.

Each side a_i , $i = 1, \ldots, n$, of $\mathcal{P}_0$ is shared with a polygon $\mathcal{P}_i$. Each vertex V_i of $\mathcal{P}_0$ is the meeting point of four faces.

Each face is seen as a rigid body; only the dihedral angles can vary. Under which conditions a Kokotsakis mesh is continuously flexible ?

1. Introduction

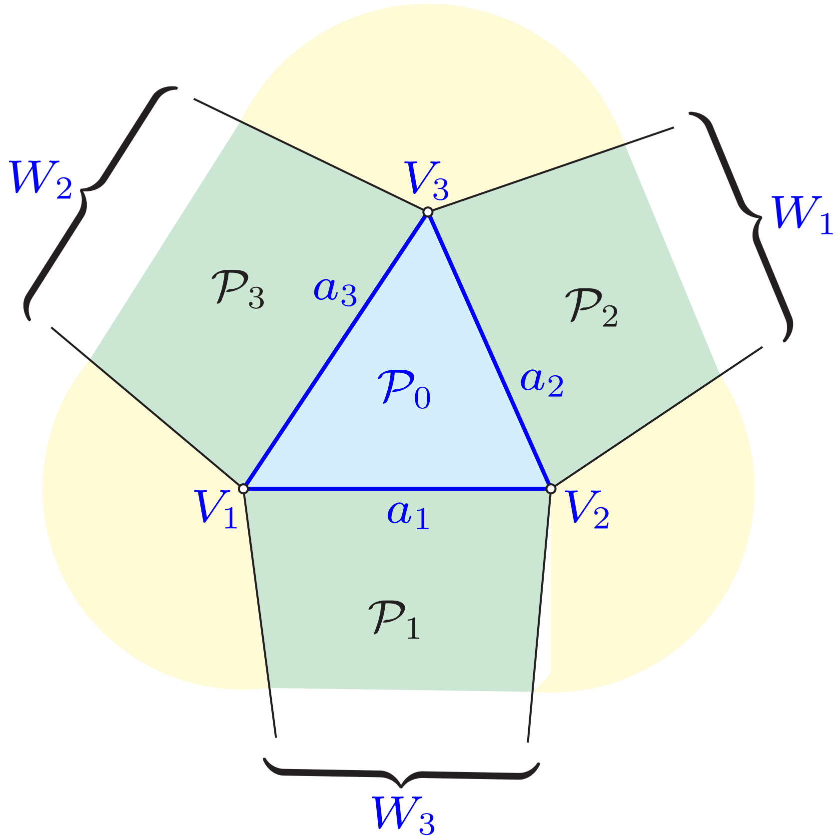
Antonios Kokotsakis
1899–1964

He was born on the island Crete in Greece. As a precocious child, he was accepted at the Department of Civil Engineering of Technical University of Athens already in the age of 16 .

After graduation he was appointed a lecturer in the Department of Descriptive and Projective Geometry. He finished his PhD-thesis entitled *"About flexible polyhedra"* under the supervision of K. Caratheodori in Munich/Germany.

His list of publications contains not more than 5 titles.

1. Introduction



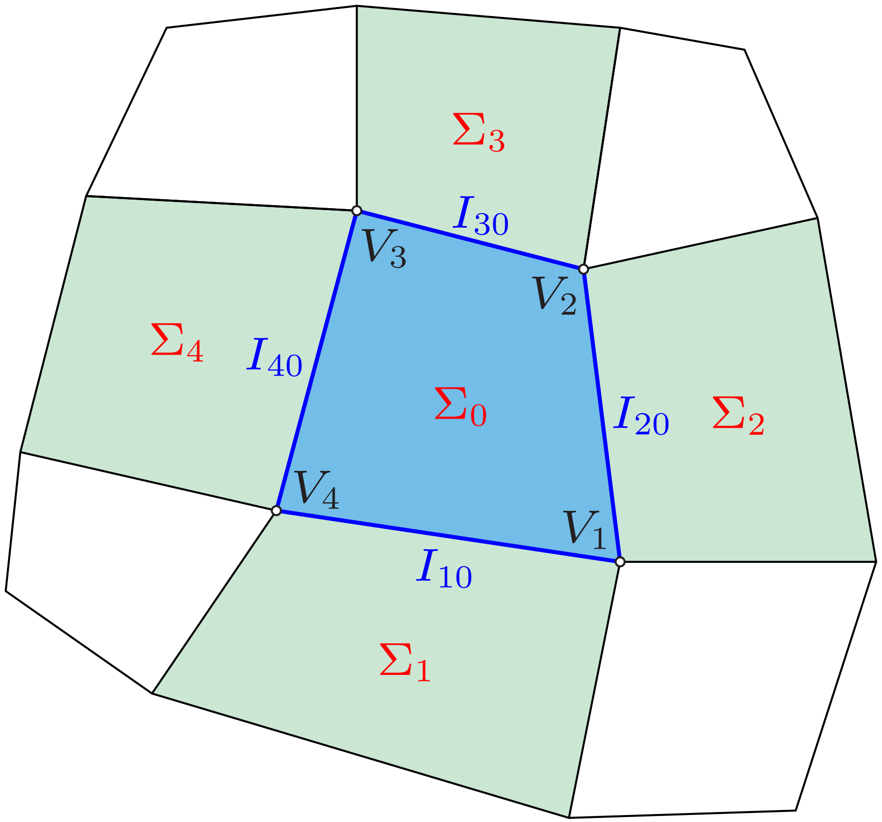
Special case: $n = 3$

Kokotsakis mesh for $n = 4$
$=$ Vierflach [German] (Kokotsakis 1931, Sauer 1932)
$=$ four-flat [English]

For $n = 3$ the Kokotsakis mesh is equivalent to an octahedron with $V_1V_2V_3$ and $W_1W_2W_3$ as opposite triangular faces.

This offers an alternative approach to R. Bricard's *flexible octahedra*.

1. Introduction



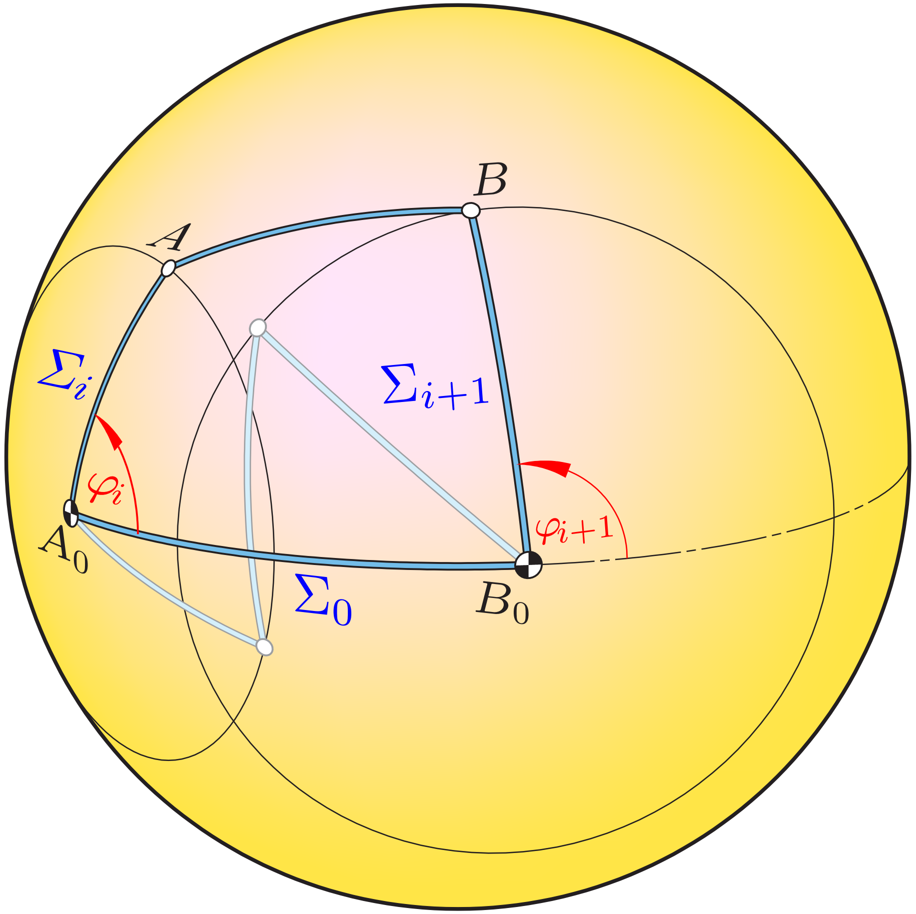
The polygons need not be planar

Kinematic interpretation:

The polygons represent different systems $\Sigma_0, \ldots, \Sigma_n$.
The sides a_i of $\mathcal{P}_0$ are instantaneous axes I_{i0} of the relative motions Σ_i/Σ_0 .

The relative motions Σ_{i+1}/Σ_i between consecutive systems are spherical four-bars.

1. Introduction

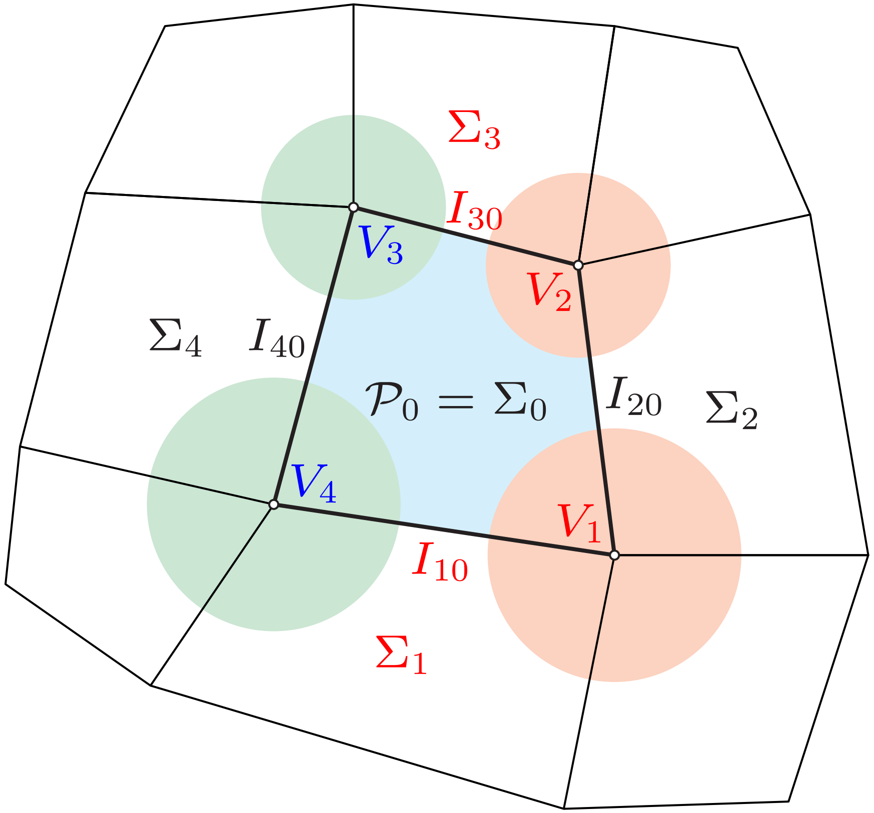


To recall:

A spherical four-bar transmits the rotation about the center A_0 by the coupler AB non-uniformly to the rotation about B_0.

The arms A_0A and B_0B represent consecutive systems Σ_i , Σ_{i+1} .

1. Introduction



The edge lengths $\overline{V_1V_2}, \ldots, \overline{V_4V_1}$ of the central polygon $\mathcal{P}_0$ have no influence on the flexibility $\Longrightarrow$

Theorem: A Kokotsakis-mesh for $n = 4$ is flexible if and only if the composition of the two four-bar (V_1, V_2) on the right hand side is the same as via (V_3, V_4) on the left hand side.

(. . . provided, a self-intersection of the quadrangle in Σ_3 is admitted)

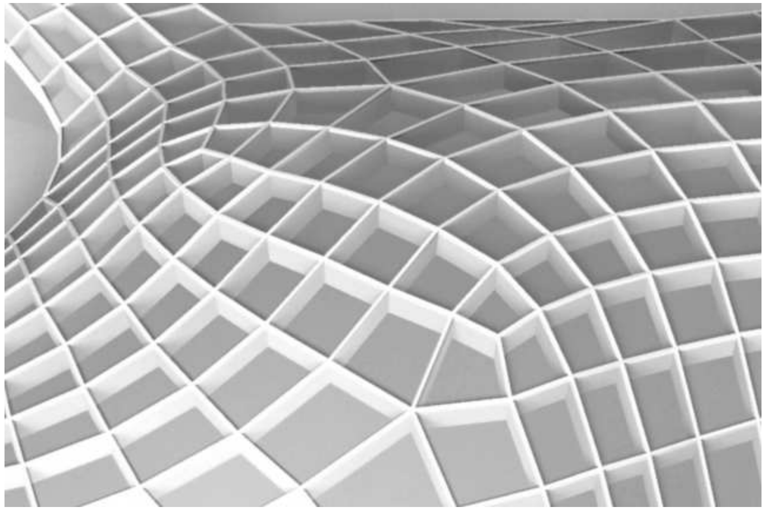
1. Introduction

In *discrete differential geometry* there is a interest on polyhedral structures composed of quadrilaterals (quadrilateral surfaces). When all quadrilaterals are planar, they constitute a *discrete conjugate net*.

Theorem: [Bobenko, Hoffmann, Schief 2008]

A discrete conjugate net in general position is continuously flexible $\iff$ all its 3×3 complexes are continuously flexible.

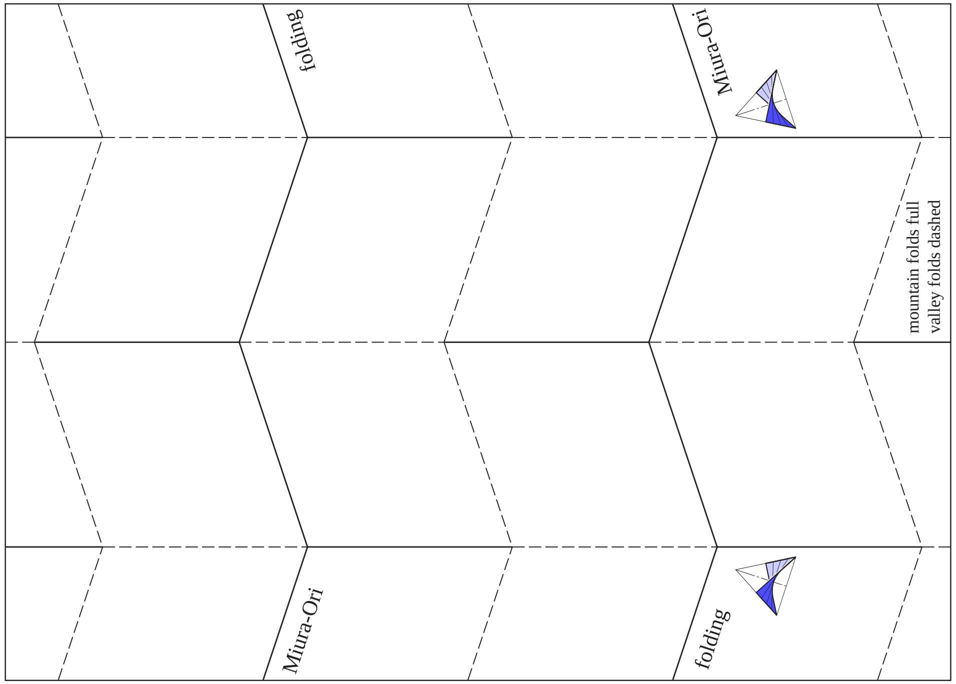
Bobenko et al., 2008:



H. Pottmann, Y. Liu, J. Wallner, A. Bobenko, W. Wang: *Geometry of Multi-layer Freeform Structures for Architecture.* ACM Trans. Graphics **26** (3) (2007), SIGGRAPH 2007

"... the complete classification of flexible discrete conjugate nets has not been achieved yet"

2. Flexible Kokotsakis meshes



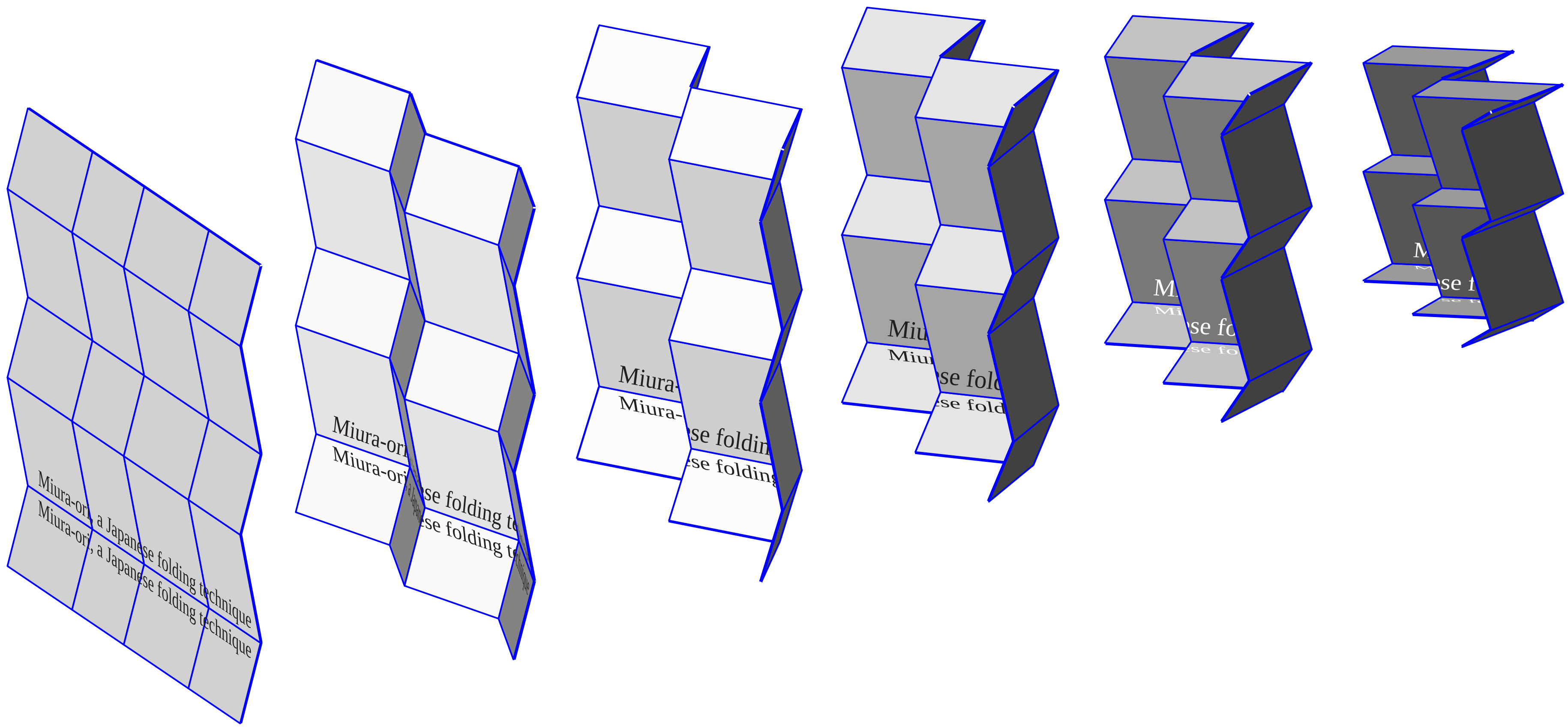
Unfolded miura-ori;
dashs are *valley folds*,
full lines are *mountain folds*

Miura-ori is a Japanese folding technique named after Prof. Koryo Miura, The University of Tokyo.

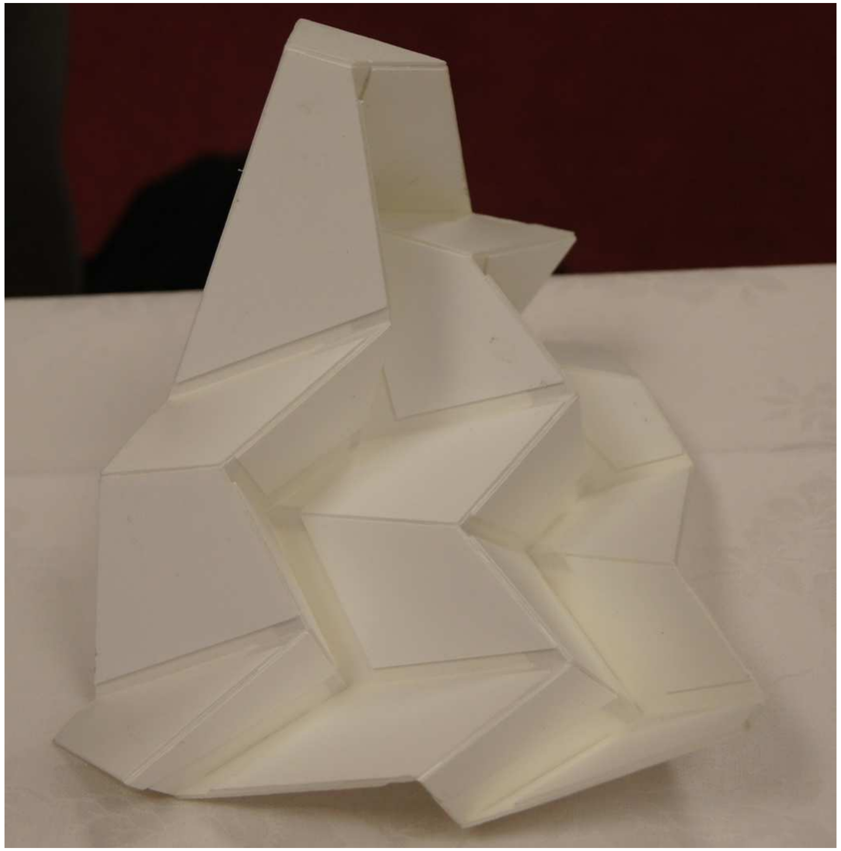
It is used for solar panels because it can be unfolded into its rectangular shape by pulling on one corner only.

On the other hand it is used as kernel to stiffen sandwich structures.

2. Flexible Kokotsakis meshes



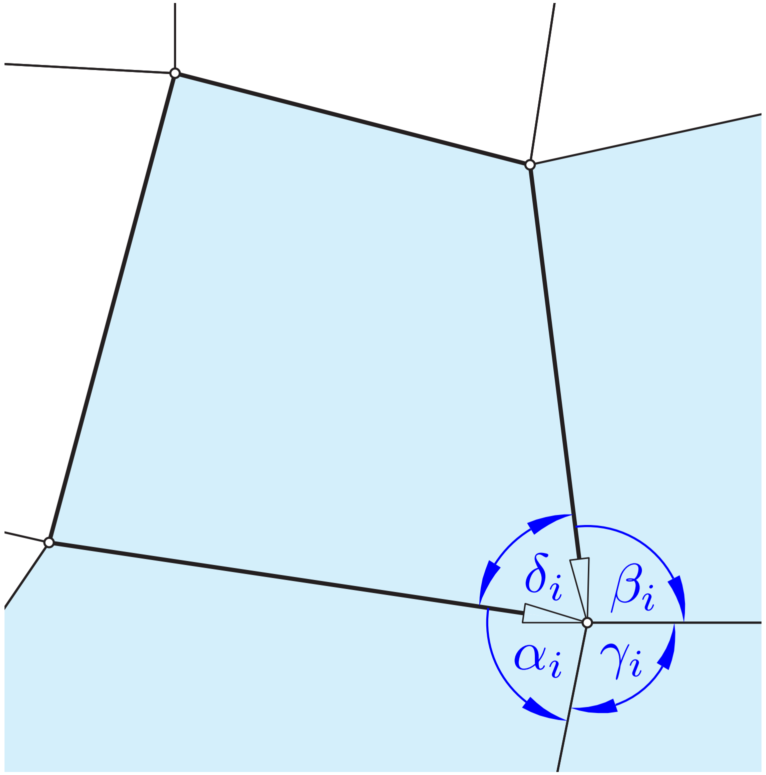
2. Flexible Kokotsakis meshes



Tomihiro TACHI (The University of Tokyo) designed an algorithm to compute numerically flexible quad-meshes with a flat initial pose.

Freeform Variations of Origami
Proc. 14th Internat. Conf. on Geometry and Graphics, Kyoto 2010, no. 221

2. Flexible Kokotsakis meshes



Miura-ori is a special case of

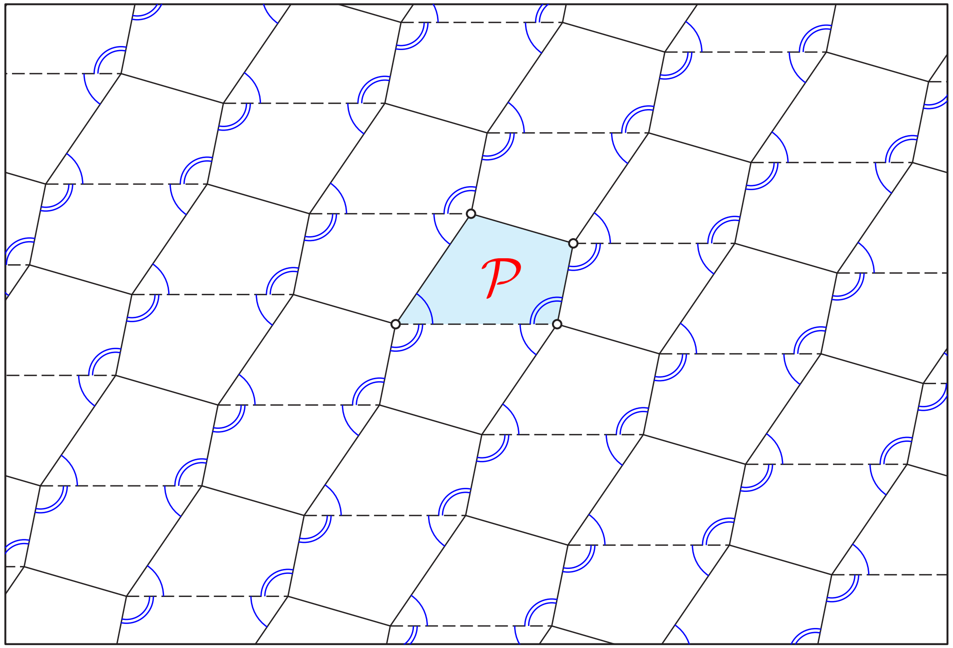
Theorem: [Kokotsakis 1932]
A Kokotsakis mesh is flexible if at each vertex V_i opposite angles are either equal or supplementary, i.e.,

$$\alpha_i = \beta_i, \quad \gamma_i = \delta_i \ \text{ or}$$
$$\alpha_i = \pi - \beta_i, \quad \gamma_i = \pi - \delta_i\,.$$

A discrete conjugate net where all vertices are of this type is called Voss surface:

- Its folds are geodesics,
- it is continuously flexible.

2. Flexible Kokotsakis meshes



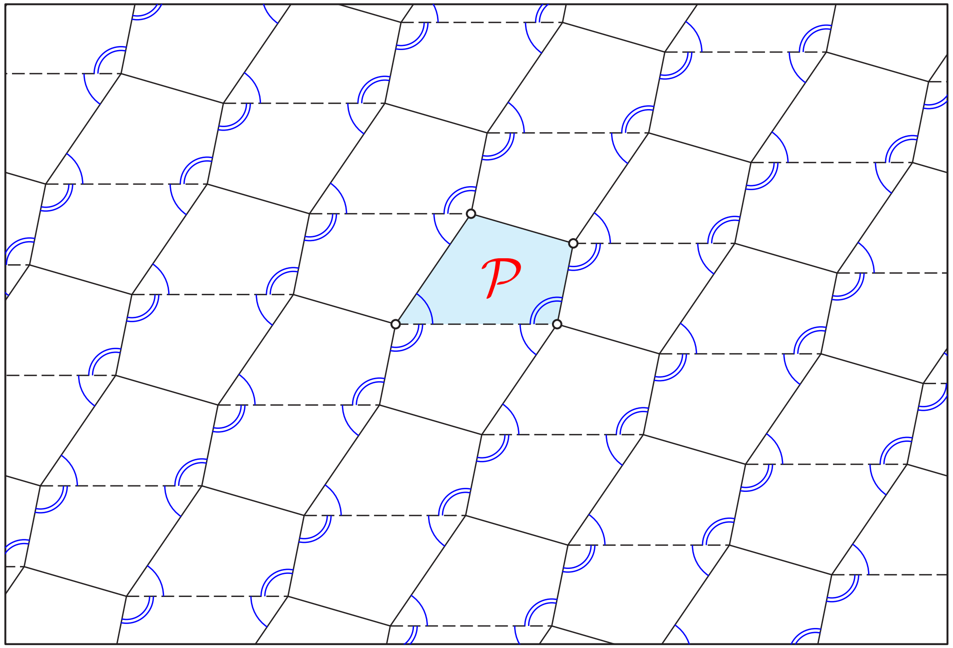
A. Kokotsakis, 1932
Athens

Any arbitrary plane quadrangle is a tile for a regular tessellation of the plane.

It is obtained by applying iterated 180°-rotations about the midpoints of the sides of an initial quadrangle $\mathcal{P}$.

For convex $\mathcal{P}$ this polyhedral structure is flexible

2. Flexible Kokotsakis meshes



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Any plane quadrangle is a tile for a regular tessellation of the plane.

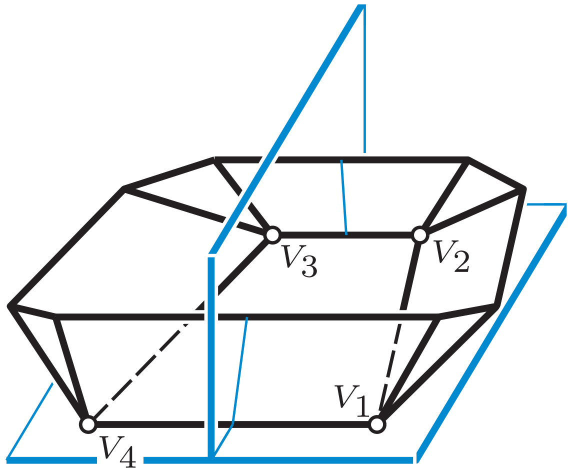
It is obtained by applying iterated $180°$-rotations about the midpoints of the sides of an initial quadrangle $\mathcal{P}$.

For convex $\mathcal{P}$ this polyhedral structure is continuously flexible

2. Flexible Kokotsakis meshes

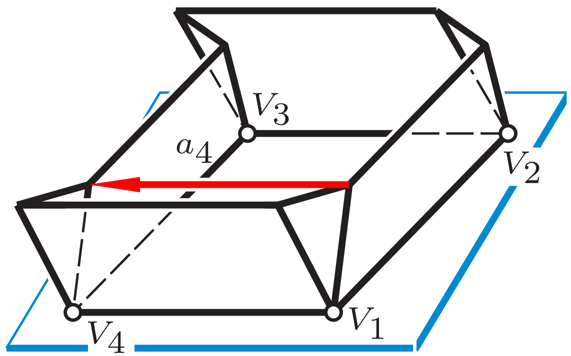
I. Planar-symmetric type (Kokotsakis 1932):

The reflection in the plane of symmetry of V_1 and V_4 maps each horizontal fold onto itself while the two vertical folds are exchanged.



II. Translational type:

There is a translation $V_1 \mapsto V_4$ and $V_2 \mapsto V_3$ mapping the three faces on the right hand side onto the triple on the left hand side.



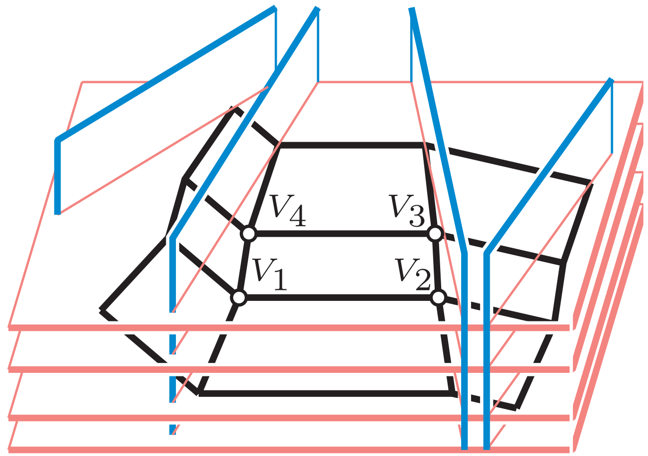
2. Flexible Kokotsakis meshes

III. Isogonal type (KOKOTSAKIS 1932):

At each vertex opposite angles are congruent = 3×3 complex of a Voss surface.

IV. Orthogonal type (GRAF, SAUER 1931):

Here the horizontal folds are located in parallel (say: horizontal) planes, the vertical folds in vertical planes. $\mathcal{P}_0$ is a trapezoid.

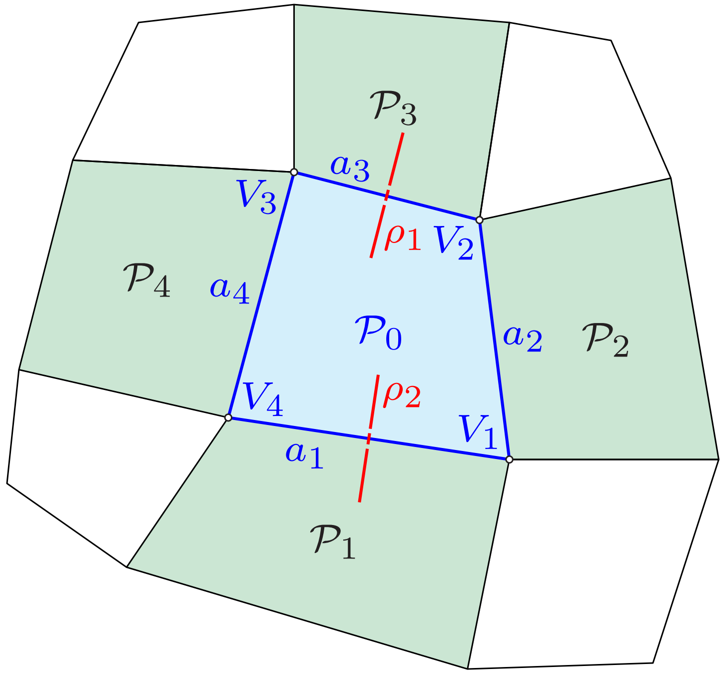


2. Flexible Kokotsakis meshes

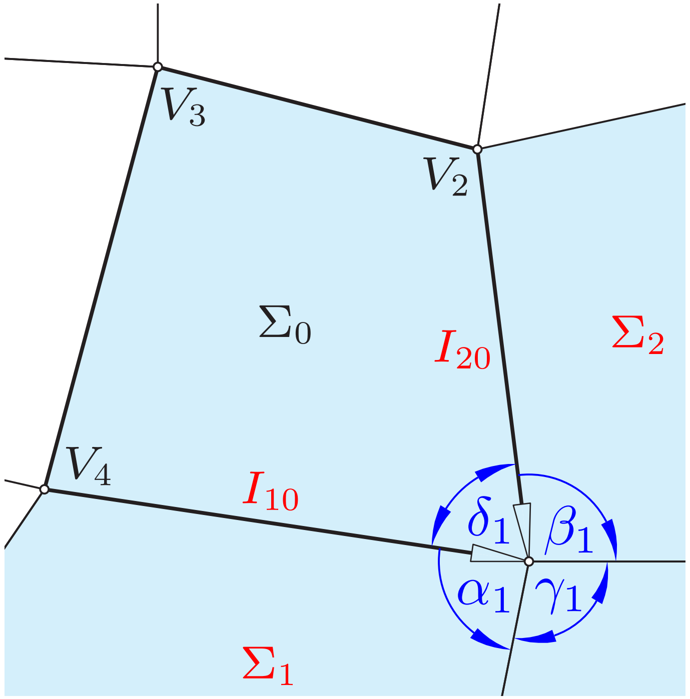
V. Line-symmetric type (H.S. 2009):

A line-reflection maps the pyramide at V_1 onto that of V_4; another one exchanges the pyramides at V_2 and V_3.

This includes Kokotsakis' example of a flexible tessellation.

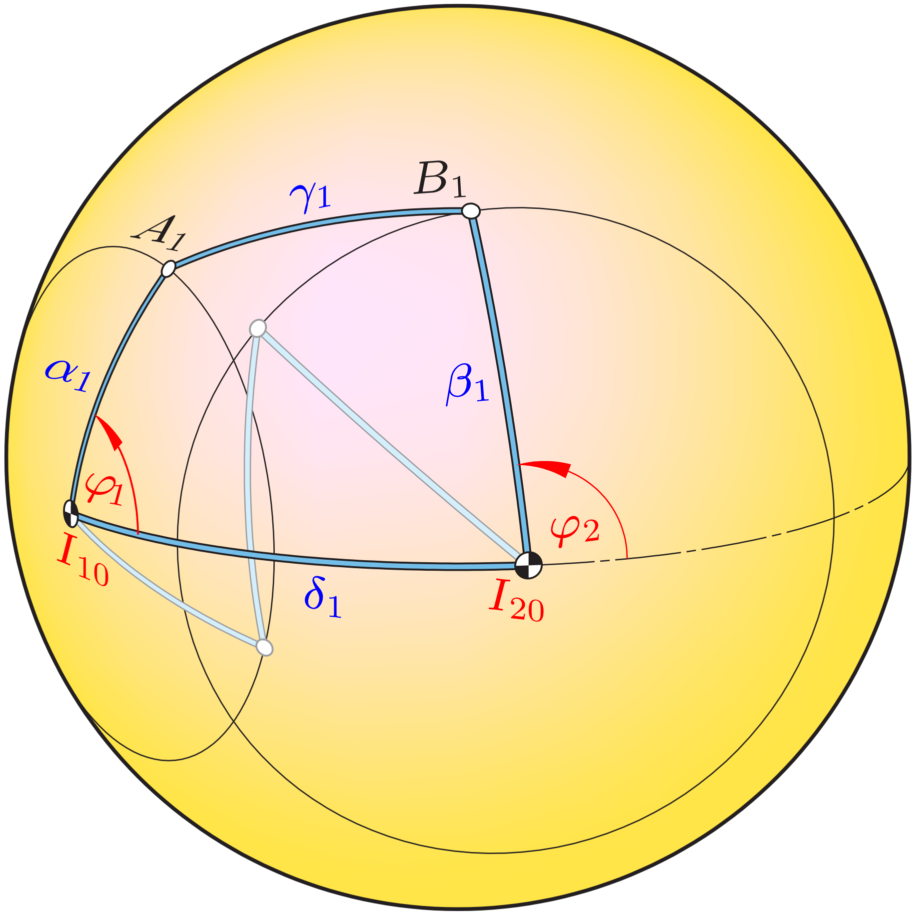


3. Transmission by one spherical four-bar

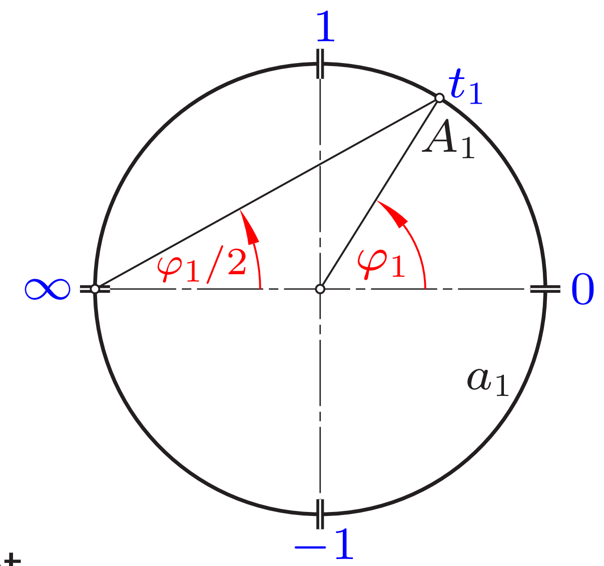


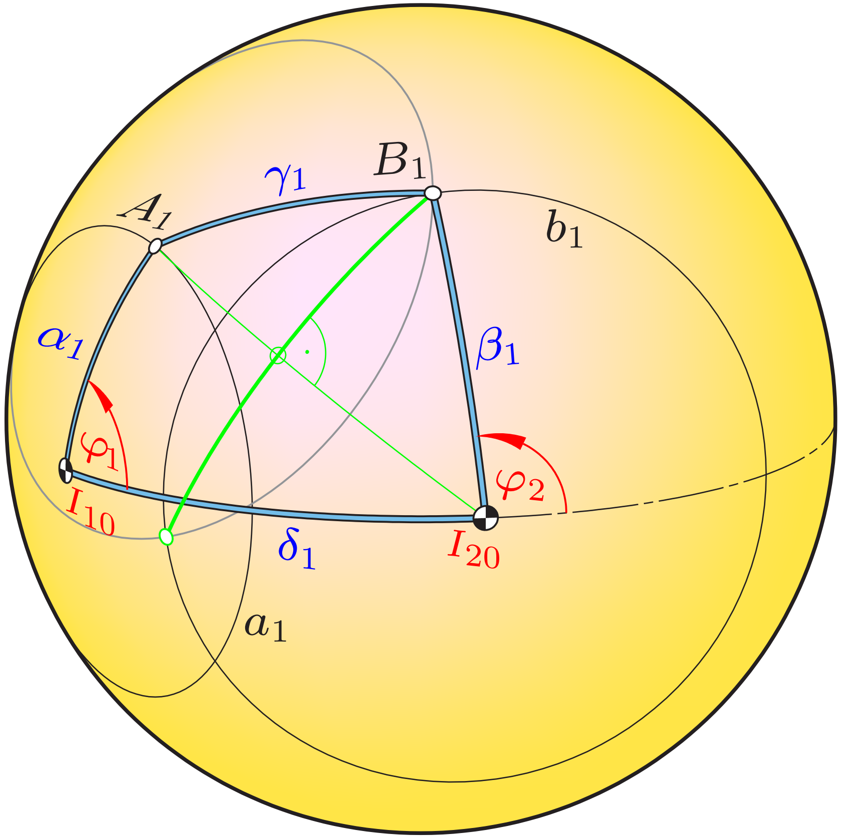
Four-bar motion Σ_2/Σ_1 and its spherical image

$$0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < 180^\circ$$



3. Transmission by one spherical four-bar





We set

$$t_1 := \tan\frac{\varphi_1}{2}, \quad t_2 := \tan\frac{\varphi_2}{2}.$$

t_1, t_2 are projective coordinates on the path circles a_1, b_1 of A_1 and B_1, resp., and obtain

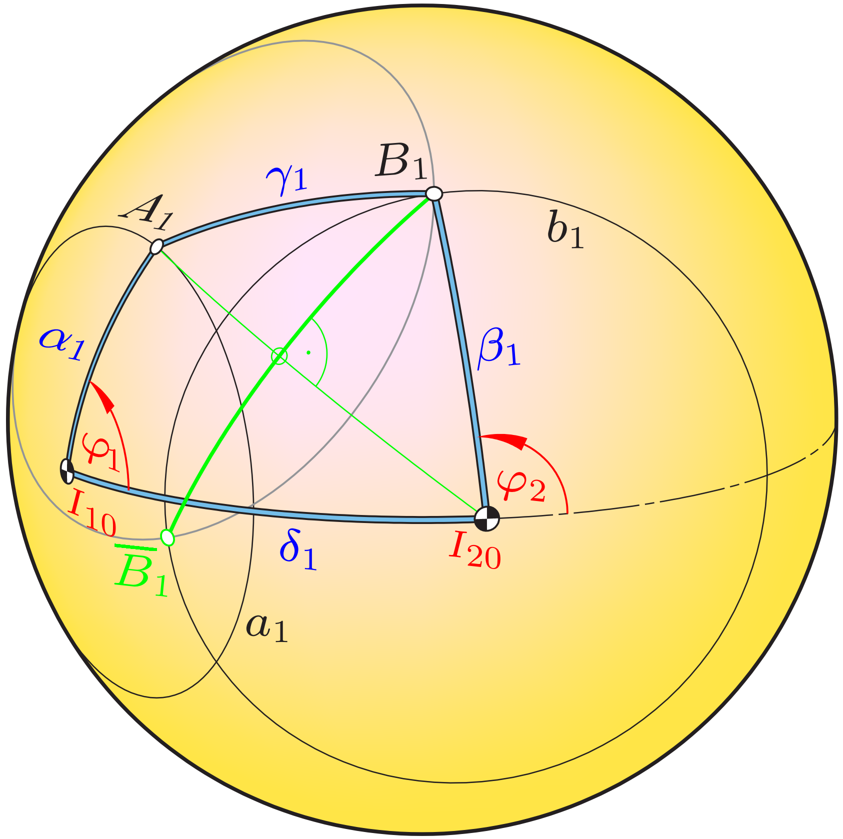
$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0 \quad \text{with} \quad c_{ik} = f(\alpha_1, \dots, \delta_1)$$

3. Transmission by one spherical four-bar

The transmission by the four-bar motion defines a 2-2-correspondance between the circles a_1 and b_1:

$$c_{22}t_1^2t_2^2+c_{20}t_1^2+c_{02}t_2^2+c_{11}t_1t_2+c_{00}=0$$

$$\left(t_1 := \tan\frac{\varphi_1}{2}, \quad t_2 := \tan\frac{\varphi_2}{2}.\right)$$



3. Transmission by one spherical four-bar

Coefficients in the biquadratic equation $c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$:

$$c_{22} = \sin\tfrac{\alpha_1-\beta_1+\gamma_1+\delta_1}{2}\ \sin\tfrac{\alpha_1-\beta_1-\gamma_1+\delta_1}{2}\,,$$

$$c_{20} = \sin\tfrac{\alpha_1+\beta_1+\gamma_1+\delta_1}{2}\ \sin\tfrac{\alpha_1+\beta_1-\gamma_1+\delta_1}{2}\,,$$

$$c_{11} = -2\ \sin\alpha_1\ \sin\beta_1 \neq 0$$

$$c_{02} = \sin\tfrac{\alpha_1+\beta_1+\gamma_1-\delta_1}{2}\ \sin\tfrac{\alpha_1+\beta_1-\gamma_1-\delta_1}{2}\,,$$

$$c_{00} = \sin\tfrac{\alpha_1-\beta_1+\gamma_1-\delta_1}{2}\ \sin\tfrac{\alpha_1-\beta_1-\gamma_1-\delta_1}{2}$$

The coefficients c_{ik} are algebraically dependent. c_{11} is a root of a 6th-degree polynomial with coefficients depending on $c_{00}, c_{02}, c_{20}, c_{22}$.

Particular cases of the transmission

The 2-2-correspondance between a_1 and b_1 splits into two projectivities $\iff$ the quadrangle is a spherical isogram, i.e., $\beta_1 = \alpha_1$ and $\delta_1 = \gamma_1$ $(c_{00} = c_{22} = 0)$.

In this case (. . . isogonal type)

$$t \;\mapsto\; t_2 = \frac{\sin\alpha_1 \pm \sin\gamma_1}{\sin(\alpha_1 - \gamma_1)}\, t_1 \quad \text{with} \quad 0 \mapsto 0,\ \infty \mapsto \infty \quad \text{for} \quad \alpha_1 \neq \gamma_1,\ \pi - \gamma_1.$$

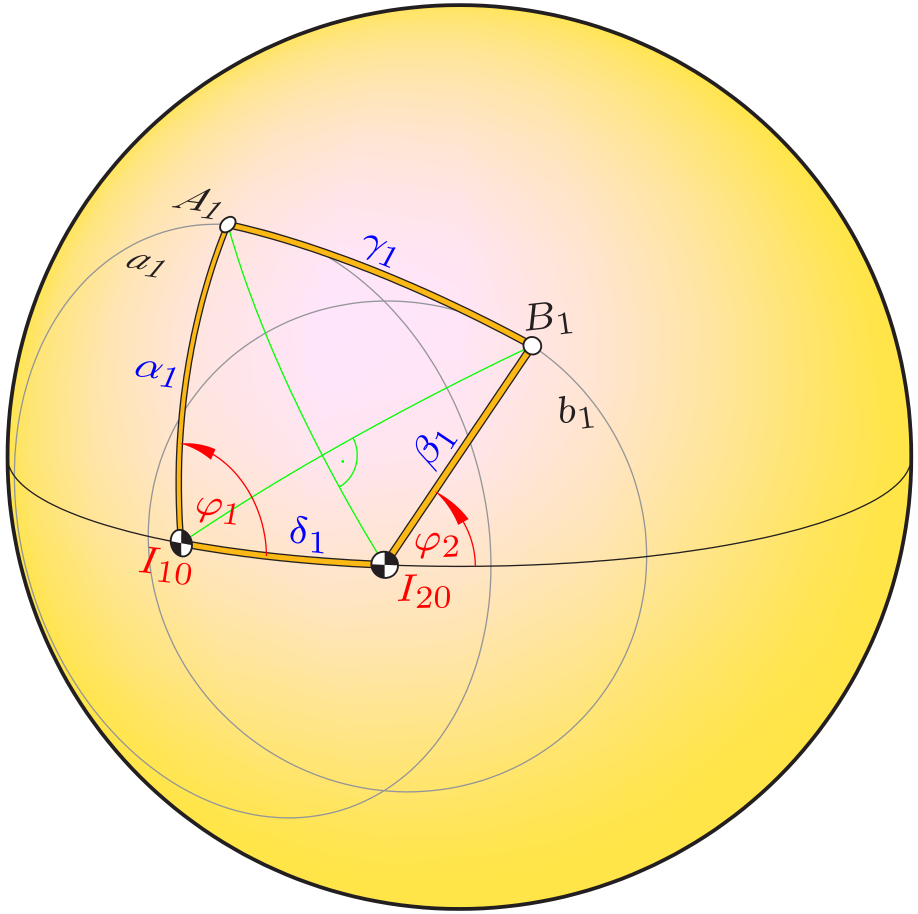
Particular cases of the transmission

Under the condition

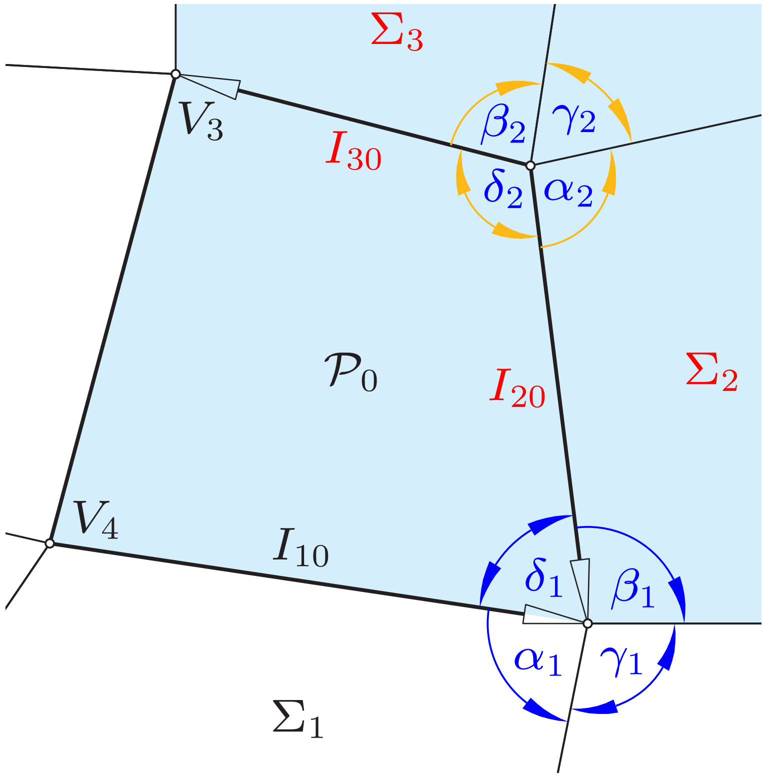
$$\cos\alpha_1\ \cos\beta_1 = \cos\gamma_1\ \cos\delta_1$$

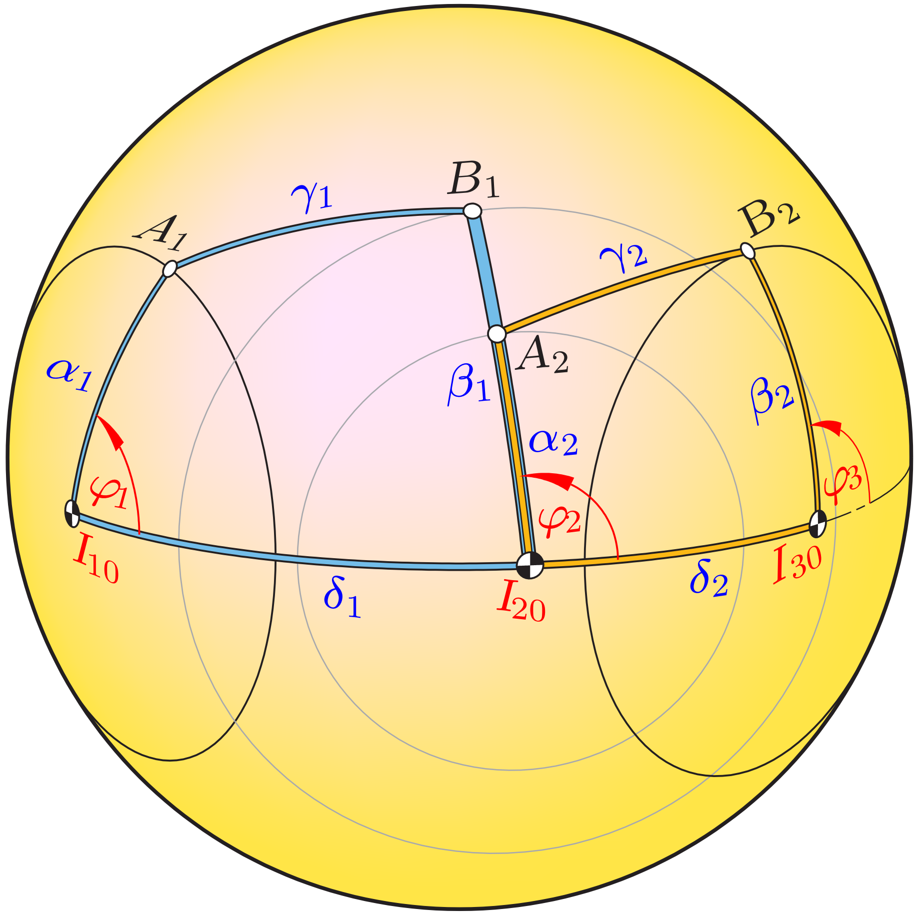
(equivalent to $\det(c_{ik}) = 0$) each quadrangle has orthogonal diagonals (... orthogonal type).

The 2-2-correspondance maps pairs of points on a_1 aligned with I_{20} onto pairs of points on b_2 located on the orthogonal line through I_{10} .



3. Transmission by two spherical four-bars





four-bar motions Σ_2/Σ_1 and Σ_3/Σ_1
and their spherical images

3. Transmission by two spherical four-bars

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$
$$d_{22}t_1^2t_2^2 + d_{20}t_1^2 + d_{02}t_2^2 + d_{11}t_1t_2 + d_{00} = 0$$

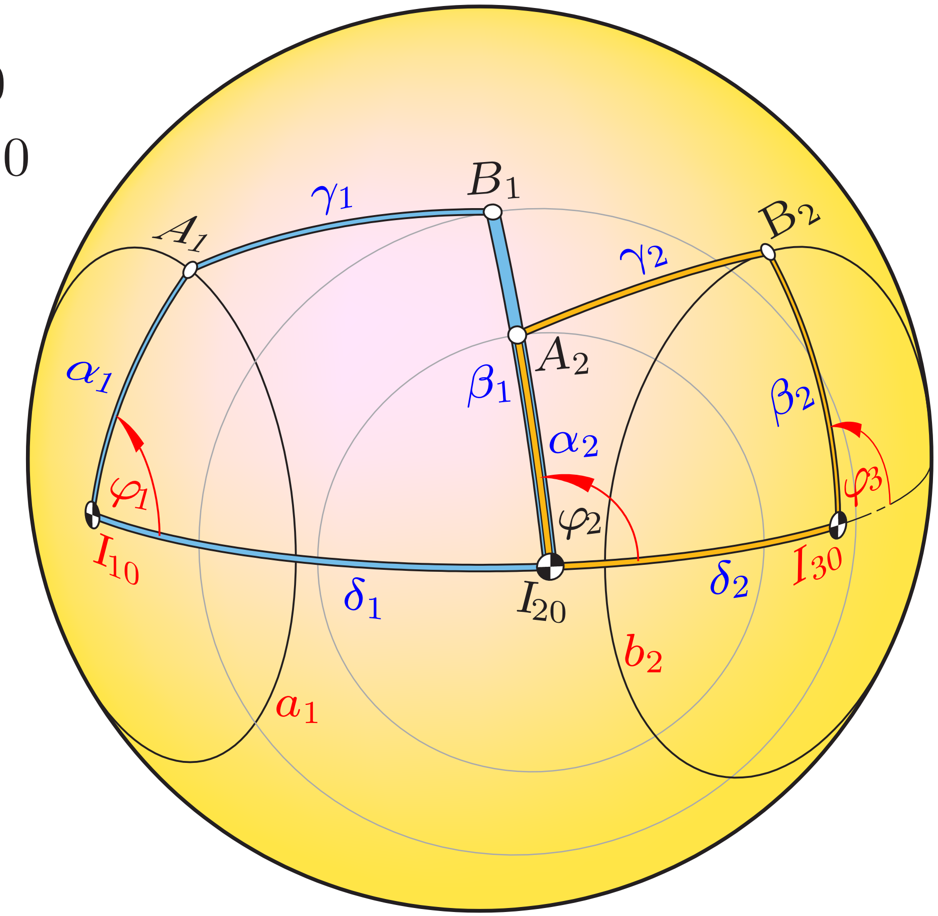
The four-bar transmissions are equivalent to these two bilinear equations.

We eliminate t_2 by computing the resultant with respect to t_2.

We obtain a biquartic equation in

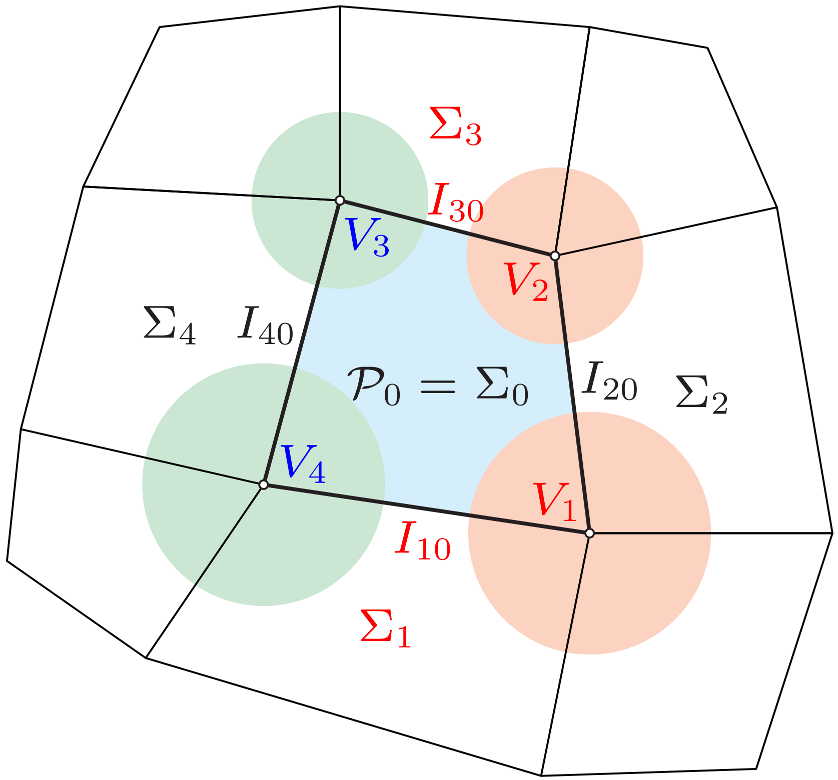
$$t_1 = \tan\frac{\varphi_1}{2} \text{ and } t_3 = \tan\frac{\varphi_3}{2},$$

i.e., a 4-4-correspondance between $A_1 \in a_1$ and $B_2 \in b_2$.



Twofold decomposition of 4-4-correspondance

Continuous flexibility of a Kokotsakis mesh for $n = 4$ means:
The 4-4-correspondance or – in the reducable case – one component can be decomposed in two different ways.



Kokotsakis (Graf, Sauer):
In the isogram case $(n \geq 4)$

$$\alpha_1 = \beta_1,\ \gamma_1 = \delta_1,\ \alpha_2 = \beta_2,\ \gamma_2 = \delta_2$$

the composition of two projectivities is a projectivity with $0 \mapsto 0$ and $\infty \mapsto \infty$.

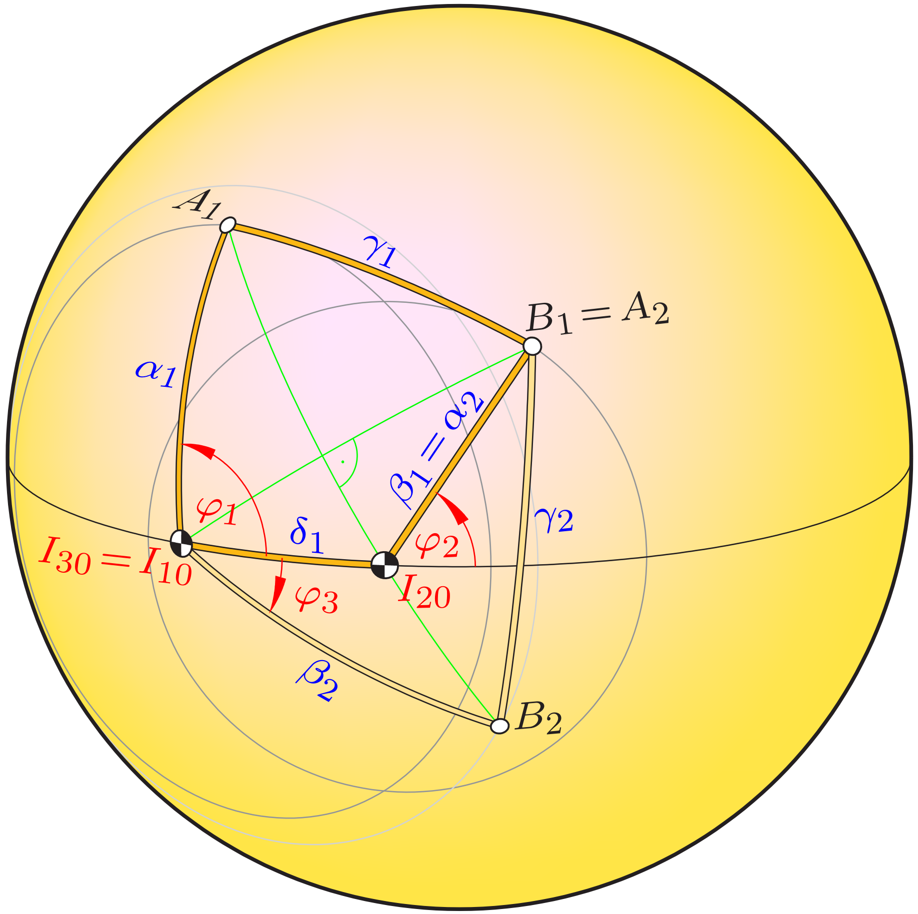
Twofold decomposition of 4-4-correspondance

Under the conditions

$$\cos\alpha_1\ \cos\beta_1 = \cos\gamma_1\ \cos\delta_1\,, \quad \alpha_2 = \beta_1\,,$$
$$\cos\alpha_2\ \cos\beta_2 = \cos\gamma_2\ \cos\delta_2\,, \quad \delta_2 = -\delta_1\,,$$

both four-bars share the orthogonal diagonals.

Due to Graf and Sauer (1931) there is a second decomposition of the same kind; all four-bars share one diagonal (spherical Dixon mechanism).

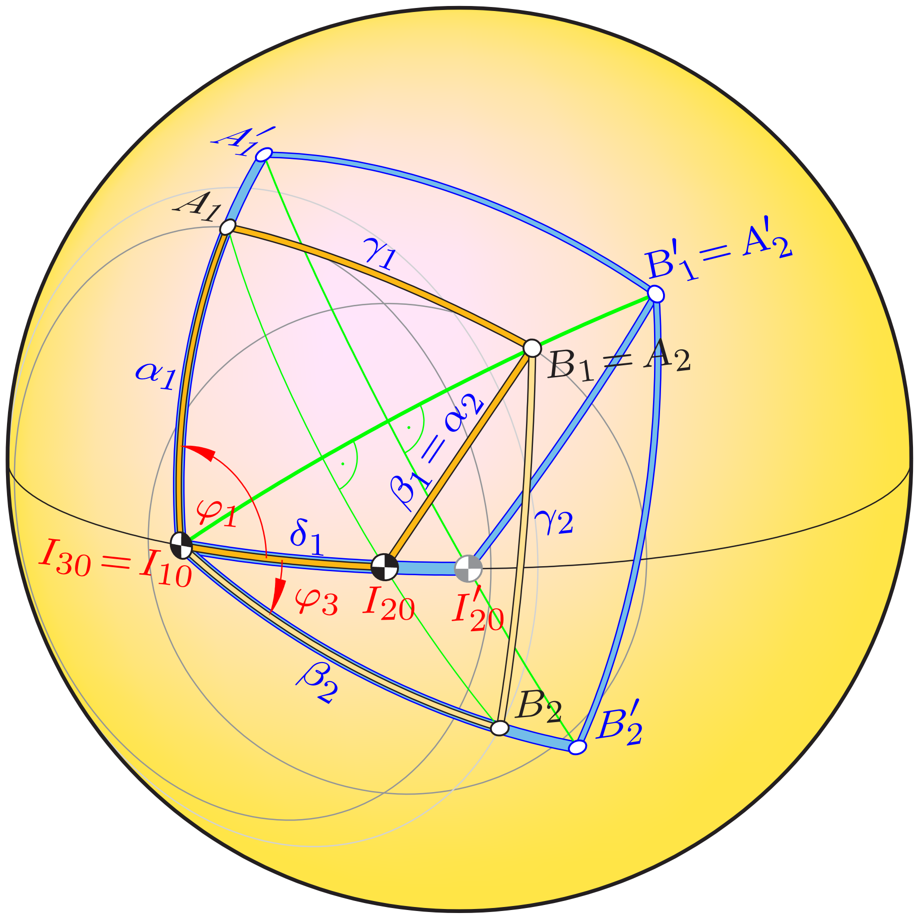


Twofold decomposition of 4-4-correspondance

The 4-4-correspondance is the square of a 2-2-correspondance

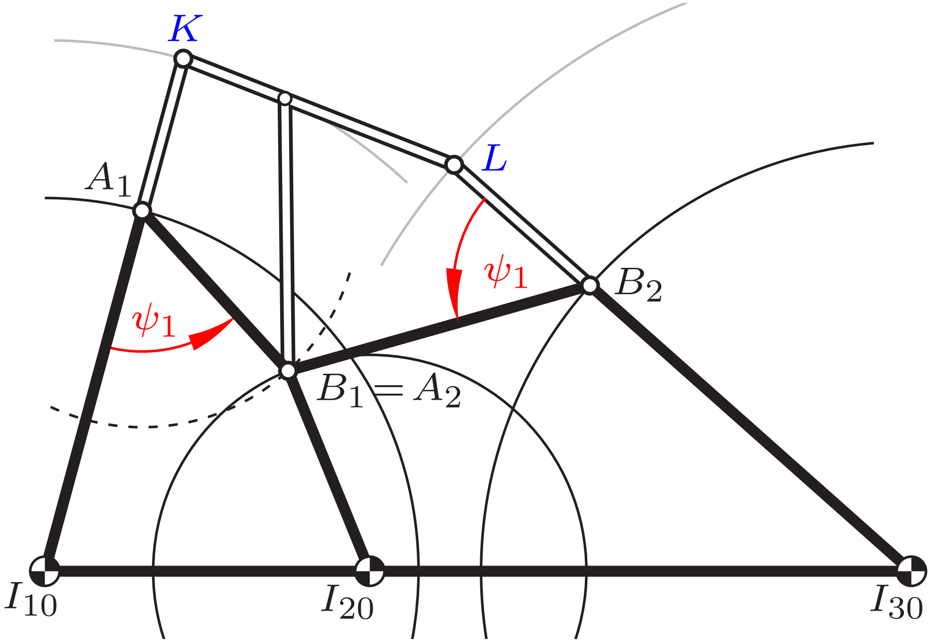
$$c_{21}t_1^2t_3 + c_{12}t_1t_3^2 + c_{10}t_1 + c_{01}t_3 = 0$$

with coefficients depending on $\tan\alpha_1$, $\tan\delta_1$, $\tan\beta_2$, only.



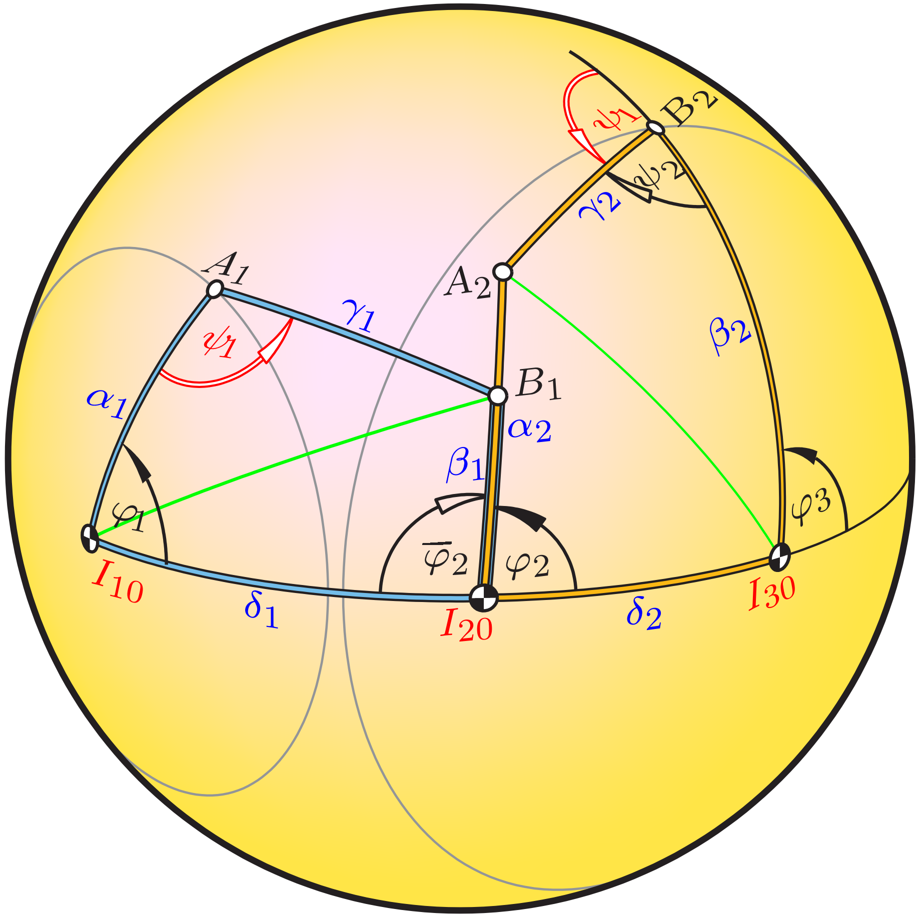
In all known non-trivial examples (III, IV, V) the 4-4-correspondance between t_1 and t_3 is reducible.
There is a new example of a reducible composition:

4. Reducible compositions of spherical four-bars



BURMESTER's focal mechanism

Right hand figure: Reducible spherical composition obeying DIXON's angle condition for ψ_1



4. Reducible compositions of spherical four-bars

For the composition of two spherical four-bars Dixon's angle condition $\sphericalangle\, I_{10}A_1B_1 = \pm \sphericalangle\, \overline{I}_{30}B_2A_2$ is equivalent to the statement that the discriminants

$$D_1 = (c_{11}t_2)^2 - 4(c_{22}t_2^2 + c_{20})(c_{02}t_2^2 + c_{00}) \quad \text{and}$$
$$D_2 = (d_{11}t_2)^2 - 4(d_{22}t_2^2 + d_{02})(d_{20}t_2^2 + d_{00})$$

are proportional.

4. Reducible compositions of spherical four-bars

Theorem: (G. Nawratil, 2010)

There are 4 cases where the biquartic polynomial splits into two biquartic polynomials:

1. Isogonal case: One of the spherical quadrangles is isogonal.

2. Dixon case: The two spherical four-bars obey Dixon's angle condition.

3. Orthogonal case: Identical with the orthogonal type (Graf, Sauer).

4: A new condition; its geometrical meaning hasn't been figured out, yet.

4. Reducible compositions of spherical four-bars

Conjecture:

Apart from the trival translatory and planar-symmetrical types there is no continuously flexible Kokotsakis-mesh with irreducible 4-4-correspondance.

If this is true then the only candidates for flexible Kokotsakis-meshes are the four cases. This would enable a classification of all flexible types.

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